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Finite element approximation of stabilized mixed models in finite strain hyperelasticity involving displacements and stresses and/or pressure—An overview of alternatives

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SUMMARY

This paper presents mixed finite element formulations to approximate the hyperelasticity problem using as unknowns the displacements and either stresses or pressure or both. These mixed formulations require either finite element spaces for the unknowns that satisfy the proper inf-sup conditions to guarantee stability or to employ stabilized finite element formulations that provide freedom for the choice of the interpolating spaces. The latter approach is followed in this work, using the Variational Multiscale concept to derive these formulations. Regarding the tackling of the geometry, we consider both infinitesimal and finite strain problems, considering for the latter both an updated Lagrangian and a total Lagrangian description of the governing equations. The combination of the different geometrical descriptions and the mixed formulations employed provides a good number of alternatives that are all reviewed in this paper.

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KEY WORDS: Hyperelasticity; Finite strains; Mixed formulations; Dual formulations; Stabilized finite element methods

1. INTRODUCTION

Elasticity is probably the most important physical model in engineering, and the basis to understand other models in solid mechanics [1, 2]. Consequently, its approximation using the finite element (FE) method is of foremost importance [3, 4]. To attempt it, the first point to consider is which are the unknowns to be considered. The problem to be solved consists essentially of the equation for the conservation of linear momentum (Cauchy's equation), the geometric equation relating strains and displacements and the constitutive equation relating stresses and strains. The unknowns are the displacements, the stresses and the strains, although in some cases it is also convenient to introduce other intermediate variables, or parts of the stresses or of the strains, mainly their volumetric and deviatoric components; in particular, this is useful in the case of incompressible materials [5, 6, 7, 8].

The geometric and the constitutive equations can be used to write Cauchy's equation in terms of the displacements only [9]. This is the so-called irreducible form of the problem, perhaps the most

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widely used. However, there are situations in which it is convenient to use other unknowns [3]. There are many possibilities for the selection of these unknowns, but in this paper we shall concentrate on three different cases, namely, the displacement-pressure formulation [10, 11, 12], the displacement-stress formulation [13, 14, 15] and the displacement-pressure-stress formulation [16, 17, 18, 19]. Other common possibilities involve the introduction of the strains as unknowns of the problem, which will not be considered here [20], although if these strains replace stresses as variables the resulting formulations are essentially equivalent. In some situations it is also convenient to introduce additional unknowns to deal with complex constitutive laws [21, 22, 23]. All these alternatives fall within the class of *mixed formulations*, i.e., formulations with different unknowns belonging to different functional spaces (see, e.g., [24]). In the stationary case, these formulations can be interpreted from a classical variational viewpoint, the governing equations being the Euler-Lagrange equations for the stationary conditions of certain functionals. Thus, the displacement-stress formulation follows from the Hellinger-Reissner principle, or the displacements-strain-stress formulation from the Hu-Washizu principle, for example. However, we will not consider this energy viewpoint and state directly the governing equations of each formulation, both in strong (differential) and weak (variational) forms.

We shall assume that all formulations considered are well posed at the continuous level, even though this sometimes requires delicate technical conditions (on the loads, boundary conditions or constitutive laws); in particular, buckling will not be considered in the finite strain case. Even in this case, the FE approximation of these mixed problems is by no means straightforward. In particular, if the standard Galerkin method is employed, there are compatibility conditions that need to be met between the interpolating spaces of the different unknowns that can be expressed in the form of inf-sup conditions [24]. Satisfying these conditions leads to complex FE interpolations, difficult to implement in FE codes oriented to applications. Nevertheless, there is an alternative that allows one to use *arbitrary* FE interpolations for all unknown fields, in particular the convenient equal interpolation. This consists in modifying the discrete equations of the Galerkin method by adding stabilization terms that keep consistency but are stable regardless of the interpolating spaces. Methods achieving this goal are termed as *stabilized* FE methods. This is the approach to be followed in this paper; in our case, these stabilized FE formulations will be obtained from the Variational Multiscale (VMS) concept [25, 26].

Apart from the unknowns of the problem, the other critical point to address is the way geometry is described or, more precisely, which are the independent variables in space. The two classical alternatives are the Eulerian and the Lagrangian descriptions [9, 27]. The former is feasible, but inconvenient in solid mechanics, sometimes because of the prescription of boundary conditions and usually because of the difficulty to write the constitutive law. Therefore, we shall restrict ourselves to Lagrangian descriptions, in which the instantaneous balance of conserved quantities written at a time t makes use of the unknowns expressed with the coordinates of the material particles at this time instant or with the coordinates of the particles at the initial time, say $t = 0$. The former is referred to as the *updated Lagrangian* (UL) approach (see, e.g., [28, 29, 30]) and the latter to the *total Lagrangian* (TL) formulation (used for example in [21, 22, 31]). Each of these two alternatives has situations in which it is more convenient than the other, although they both can be used in any situation. For example, the UL formulation is very natural in fluid-structure interaction problems (as in [32], for example), but it may be involved to be used with complex constitutive laws naturally expressed using the TL formulation. In this paper we shall consider both, the UL and the TL descriptions of the geometry. When strains and displacements are very small and the initial and deformed configurations can be considered equal, they both collapse to the infinitesimal strain geometry modeling on which linear Elasticity is based. In fact, this case is obviously the best understood, and the one for which a complete approximation theory is available. Thus, we shall consider three scenarios regarding the geometry description, namely, infinitesimal strains (linear theory), TL and UL.

The combination of the choice of variables indicated (three cases) and the geometry description leads to nine mixed models to be considered, apart from the three models corresponding to the

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irreducible formulation. The purpose of this paper is to review them all, indicating how to proceed in each case to design the stabilized FE formulation and which are the properties we may expect. Most of these formulations are scattered in some of our previous works, and the main objective of the present paper is to gather them and also to fill some gaps not previously published.

This paper is organized as follows. The statement of the problem is presented in Section 2. The starting point is the TL description, from which the UL formulation is derived from a change of coordinates and then the linear problem is obtained in the limit of infinitesimal strains. Section 3 is devoted to summarize the basic concepts of the VMS framework, first described for linear stationary problems, then moving to linear time dependent problems, after which a discussion on the way to deal with nonlinear problems is presented, and concluding with the application to mixed problems. The three mixed formulations we wish to consider are then analyzed in Sections 4, 5 and 6, corresponding respectively to the displacement-pressure, displacement-stress and displacement-pressure-stress formulations. For each case, we present the linearized problem and the final discrete stabilized FE problem in the three geometry descriptions considered (linear Elasticity, TL and UL). In the case of the displacement-pressure and displacement-pressure-stress formulations, the incompressible limit is analyzed, whereas for the displacement-stress formulation the possibility of using a dual formulation, with improved convergence properties for the stress, is highlighted. The paper concludes with some final remarks in Section 7.

2. PROBLEM STATEMENT

2.1. Geometry description

Let $\Omega_0 = \Omega(0) \subset \mathbb{R}^{\rm d}$ be the open domain occupied by the solid to be analyzed at time $t = 0$, with ${\rm d} = 2, 3$, and $(0, t_{\rm fin})$ the time interval of analysis. The material coordinates in Ω_0 are labelled as $\boldsymbol{X}$. Let $\Omega(t)$ be the region occupied by the solid at time $t \in (0, t_{\rm fin})$ and let $\boldsymbol{x} = \boldsymbol{\psi}_t(\boldsymbol{X})$ be the equation of motion from $\Omega(0)$ to $\Omega(t)$. The space-time domain of the problem is $\mathfrak{D} = \{(\boldsymbol{x}, t) \mid \boldsymbol{x} \in \Omega(t), 0 < t < t_{\rm fin}\}$, and we shall call $\mathfrak{D}_0 = \{(\boldsymbol{X}, t) \mid \boldsymbol{X} \in \Omega(0), 0 < t < t_{\rm fin}\}$. The boundary of $\Omega(t)$ is denoted by $\Gamma(t) = \partial\Omega(t)$.

The gradient deformation tensor $\boldsymbol{F}$ is defined as $\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}$, and its determinant as $J = \det(\boldsymbol{F})$. We indicate as $\{\boldsymbol{E}_A\}_{A=1}^{\rm d}$ the basis vectors at $t = 0$ and $\{\boldsymbol{e}_a\}_{a=1}^{\rm d}$ the basis vectors at time t. This distinction is unnecessary using Cartesian coordinates, as we shall do, since both bases coincide. However, it allows one to distinguish to which bases correspond the components of a tensor. In particular, we may write

$$\boldsymbol{F} = F_{aA} \boldsymbol{e}_a \otimes \boldsymbol{E}_A = \frac{\partial x_a}{\partial X_A} \boldsymbol{e}_a \otimes \boldsymbol{E}_A.$$

Einstein summation convention is used here and in what follows, with repeated indexes summing from 1 to ${\rm d}$. Lower case latin letters $a, b, c, \dots$ will be used for indexes of tensors in the deformed configuration $\Omega(t)$, and upper case latin letters $A, B, C, \dots$ for indexes in the initial configuration $\Omega(0)$. We shall also make use of the right Cauchy-Green tensor $\boldsymbol{C}$ and the left Cauchy-Green tensor $\boldsymbol{b}$, defined by

$$C_{AB} = F_{aB} F_{aA}, \qquad b_{ab} = F_{aA} F_{bA}, \tag{1}$$

or, in intrinsic form, $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$, $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$.

For $t' \in [0, t]$ we may also consider the domain $\Omega(t')$, whose particles have coordinates $\boldsymbol{x}' = \boldsymbol{\psi}_{t'}(\boldsymbol{X})$, and define the mapping $\boldsymbol{x} = \boldsymbol{\psi}_{t,t'}(\boldsymbol{x}') = \boldsymbol{\psi}_t \circ \boldsymbol{\psi}_{t'}^{-1}(\boldsymbol{x}')$. For any function $f : \Omega(t) \times (0, t_{\rm fin}) \longrightarrow \mathbb{R}$ we may compute its time derivative keeping $\boldsymbol{x}'$ fixed. Likewise, conservation laws can be imposed for the material particles occupying $\Omega(t')$. If $t' = 0$, the formulation obtained is

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the TL, whereas if $t' = t$ it is the UL. Note that in both cases the formulation can be termed as Lagrangian because the reference coordinates correspond to material particles, since the mapping $\boldsymbol{\psi}_t$ is the solid motion. In fact, it could be replaced by any other mapping $\boldsymbol{\chi}_t$, and again considering $t' = t$ we would obtain the arbitrary Eulerian-Lagrangian equations of motion, which reduce to the UL equations when $\boldsymbol{\chi}_t = \boldsymbol{\psi}_t$ and to the Eulerian equations when $\boldsymbol{\chi}_t$ does not depend on time (it is, for example, the identity) and the reference coordinates are the same for all t.

The case of infinitesimal strains corresponds to taking $\boldsymbol{x} \approx \boldsymbol{X}$ and $\boldsymbol{F} \approx \boldsymbol{I}$, the second order identity tensor, so that $J \approx 1$.

2.2. *Total Lagrangian formulation*

Let us start writing the equations of motion using the TL formulation. This means that all unknowns are expressed in terms of the coordinates $\boldsymbol{X}$ of particles at $t = 0$.

The equation for the conservation of mass reads:

$$\rho J = \rho_0, \tag{2}$$

where $\rho = \rho(\boldsymbol{X}, t)$ is the density at time t and $\rho_0 = \rho_0(\boldsymbol{X})$ at time $t = 0$.

Let $\boldsymbol{\sigma} = \sigma_{ab} \boldsymbol{e}_a \otimes \boldsymbol{e}_b$ be the Cauchy stress tensor, $\boldsymbol{u} = u_a \boldsymbol{e}_a$ the displacement field and $\boldsymbol{f} = f_a \boldsymbol{e}_a$ the vector of external accelerations, all these fields expressed in terms of $\boldsymbol{X}$ and t but referred to the deformed configuration. Abusing of the notation, we shall use the same symbols when writing them in terms of $\boldsymbol{x}$ and t. Let also $\boldsymbol{S} = S_{AB} \boldsymbol{E}_A \otimes \boldsymbol{E}_B$ be the second Piola-Kirchhoff stress tensor, related to $\boldsymbol{\sigma}$ by

$$S_{AB} = J F^{-1}_{Aa} F^{-1}_{Bb} \sigma_{ab}, \tag{3}$$

i.e., $\boldsymbol{S} = J \boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}$ in intrinsic form. Note that $\boldsymbol{S}$ is a symmetric tensor.

The conservation of linear momentum can be expressed as

$$\rho_0 \partial^2_{tt} u_a - \partial_A (F_{aB} S_{BA}) = \rho_0 f_a, \tag{4}$$

or, in intrinsic form $\rho_0 \partial^2_{tt} \boldsymbol{u} - \nabla_{\boldsymbol{X}} \cdot (\boldsymbol{S} \cdot \boldsymbol{F}^T) = \rho_0 \boldsymbol{f}$. The notation that we have used is as follows: $\partial^2_{tt} = \partial_t \partial_t$ is the second order time derivative, ∂_A is the partial derivative with respect to X_A (likewise, ∂_a will denote the partial derivative with respect to x_a) and $\nabla_{\boldsymbol{X}}$ is the operator ∇ expressed in the coordinates $\boldsymbol{X}$ (likewise, $\nabla_{\boldsymbol{x}}$ will be the operator ∇ expressed in the coordinates $\boldsymbol{x}$), so that $\nabla_{\boldsymbol{X}} \cdot$ is the divergence of a vector field expressed in the coordinates $\boldsymbol{X}$.

In the case of incompressible materials, it will be very convenient to introduce the pressure p as a variable, defined as minus the mean Cauchy stress, so that the Cauchy stress tensor can be split into volumetric and deviatoric components as

$$\sigma_{ab} = -p \delta_{ab} + \sigma^{\mathrm{dev}}_{ab}, \tag{5}$$

where δ_{ab} is the Kronecker symbol, i.e., $\boldsymbol{\sigma} = -p \boldsymbol{I} + \boldsymbol{\sigma}^{\mathrm{dev}}$. This induces the following splitting of tensor $\boldsymbol{S}$:

$$\begin{aligned} S_{AB} &= J F^{-1}_{Aa} F^{-1}_{Bb} \sigma^{\mathrm{dev}}_{ab} - p J F^{-1}_{Aa} F^{-1}_{Ba} \\ &=: S'_{AB} - p J C^{-1}_{AB}, \end{aligned} \tag{6}$$

i.e., $\boldsymbol{S} = \boldsymbol{S}' - pJ\boldsymbol{C}^{-1}$. Note that this is not the deviatoric-volumetric split of tensor $\boldsymbol{S}$, but results from the deviatoric-volumetric split of the Cauchy stress. It is easily checked that

$$\boldsymbol{S}' : \boldsymbol{C} = 0, \qquad p = \frac{1}{3J} \boldsymbol{S} : \boldsymbol{C}.$$

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Using split (6), the conservation of linear momentum (4) becomes

$$\rho_0 \partial^2_{tt} u_a - \partial_A (F_{aB} S'_{BA}) + \partial_A (pJF^{-1}_{Aa}) = \rho_0 f_a. \tag{7}$$

To close the problem, it remains to provide the constitutive law. We will consider only hyperelastic models with Helmholtz free energy Ψ depending on tensor $\boldsymbol{C}$, so that

$$\boldsymbol{S} = 2\frac{\partial \Psi}{\partial \boldsymbol{C}}. \tag{8}$$

To deal with incompressible materials we will further assume that Ψ can be split as

$$\Psi(\boldsymbol{C}) = W(\bar{\boldsymbol{C}}) + \kappa G(J), \tag{9}$$

where $\bar{\boldsymbol{C}} := J^{-2/3}\boldsymbol{C}$ is the volume-preserving component of $\boldsymbol{C}$, $W(\bar{\boldsymbol{C}})$ is the deviatoric component of the Helmholtz free energy and $U(J) = \kappa G(J)$ the volumetric one, which is written as a function $G(J)$ scaled by the bulk modulus κ. This allows one to consider the incompressible limit by letting $\kappa \to \infty$ and also permits to obtain a constitutive equation for the components in the split (6):

$$\boldsymbol{S}' = 2\frac{\partial W}{\partial \boldsymbol{C}}, \qquad p = \kappa \frac{\mathrm{d}G}{\mathrm{d}J}, \tag{10}$$

as it turns out that $2\frac{\partial G}{\partial \boldsymbol{C}} = \frac{\mathrm{d}G}{\mathrm{d}J} J \boldsymbol{C}^{-1}$ (see Eq. (6)).

We are now in a position to write the field equations of the TL formulation. If the volumetric-deviatoric split of the constitutive equation is not done, they consist of finding $\boldsymbol{u} : \mathfrak{D}_0 \longrightarrow \mathbb{R}^{\mathrm{d}}$, $\boldsymbol{S} : \mathfrak{D}_0 \longrightarrow \mathbb{R}^{\mathrm{d}} \otimes \mathbb{R}^{\mathrm{d}}$ and $\boldsymbol{F} : \mathfrak{D}_0 \longrightarrow \mathbb{R}^{\mathrm{d}} \otimes \mathbb{R}^{\mathrm{d}}$ such that

$$\rho_0 \partial^2_{tt} \boldsymbol{u} - \nabla_{\boldsymbol{X}} \cdot (\boldsymbol{S} \cdot \boldsymbol{F}^T) = \rho_0 \boldsymbol{f}, \tag{11}$$

$$\boldsymbol{S} - 2\frac{\partial \Psi}{\partial \boldsymbol{C}} = \boldsymbol{0}, \tag{12}$$

$$\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}, \quad \boldsymbol{F} - \nabla_{\boldsymbol{X}} \boldsymbol{u} = \boldsymbol{I}. \tag{13}$$

The unknown $\boldsymbol{F}$ can be replaced by $\boldsymbol{C}$ or by any other tensor measuring deformation, such as the Green-Lagrange strain tensor. Equations (11)-(12)-(13) can be respectively called momentum conservation, constitutive and geometric equations. Since the last two are not differential equations for $\boldsymbol{S}$ and $\boldsymbol{F}$, but algebraic, they permit to express these tensor fields in terms of $\boldsymbol{u}$ and end up with a differential equation in which the only unknown is $\boldsymbol{u}$, which corresponds to the irreducible form of the problem. If both $\boldsymbol{S}$ and $\boldsymbol{u}$ are kept as unknowns, we have a mixed displacement-stress formulation.

If the volumetric-deviatoric split of the constitutive equation is done, the problem consists of finding $\boldsymbol{u} : \mathfrak{D}_0 \longrightarrow \mathbb{R}^{\mathrm{d}}$, $\boldsymbol{S}' : \mathfrak{D}_0 \longrightarrow \mathbb{R}^{\mathrm{d}} \otimes \mathbb{R}^{\mathrm{d}}$, $p : \mathfrak{D}_0 \longrightarrow \mathbb{R}$ and $\boldsymbol{F} : \mathfrak{D}_0 \longrightarrow \mathbb{R}^{\mathrm{d}} \otimes \mathbb{R}^{\mathrm{d}}$ such that

$$\rho_0 \partial^2_{tt} \boldsymbol{u} - \nabla_{\boldsymbol{X}} \cdot (\boldsymbol{S}' \cdot \boldsymbol{F}^T) + \nabla_{\boldsymbol{X}} \cdot (pJ\boldsymbol{F}^{-1}) = \rho_0 \boldsymbol{f}$$

$$\boldsymbol{S}' - 2\frac{\partial W}{\partial \boldsymbol{C}} = \boldsymbol{0}$$

$$\frac{p}{\kappa} + \frac{\mathrm{d}G}{\mathrm{d}J} = 0$$

$$\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}, \quad \boldsymbol{F} - \nabla_{\boldsymbol{X}} \boldsymbol{u} = \boldsymbol{I}$$

Expressing $\boldsymbol{F}$ in terms of $\boldsymbol{u}$ but keeping $\boldsymbol{u}$, $\boldsymbol{S}'$ and p as unknowns leads to the mixed displacement-pressure-stress formulation that we shall also consider. It is also possible to keep p as unknown, yielding the displacement-pressure formulation. This is of special interest, since it allows one to consider the limit of incompressible materials.

Obviously, the problem needs to be completed with initial and boundary conditions. The former are of the form $\boldsymbol{u}(\boldsymbol{X}, 0) = \boldsymbol{u}^0(\boldsymbol{X})$, $\partial_t \boldsymbol{u}(\boldsymbol{X}, 0) = \dot{\boldsymbol{u}}^0(\boldsymbol{X})$, with $\boldsymbol{u}^0(\boldsymbol{X})$ and $\dot{\boldsymbol{u}}^0(\boldsymbol{X})$ given on $\Omega(0)$,

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whereas the latter are

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{u}_D && \text{on } \Gamma_{0,D},\ t \in (0, t_{\text{fin}}), \\ \boldsymbol{n}_0 \cdot (\boldsymbol{S} \cdot \boldsymbol{F}^T) &= \boldsymbol{t}_N && \text{on } \Gamma_{0,N},\ t \in (0, t_{\text{fin}}), \end{aligned}$$

where $\Gamma_{0,D}$ and $\Gamma_{0,N}$ are a partition of $\partial\Omega(0)$, $\boldsymbol{n}_0$ is the unit outward normal to this boundary and $\boldsymbol{u}_D$ and $\boldsymbol{t}_N$ are given.

2.3. *Updated Lagrangian formulation*

The conservation of mass in the UL formulation is obviously Eq. (2), although now we consider that all functions involved depend on the coordinates of the particles in the deformed configuration, $\boldsymbol{x}$, that is to say, $\rho = \rho(\boldsymbol{x}, t)$, $J = J(\boldsymbol{x}, t)$ and $\rho_0 = \rho_0(\boldsymbol{x})$. The conservation of linear momentum now reads

$$\rho \partial^2_{tt} u_a - \partial_b \sigma_{ba} = \rho f_a,$$

now expressing f_a in terms of $\boldsymbol{x}$. In intrinsic form, we may write $\rho \partial^2_{tt} \boldsymbol{u} - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{\sigma} = \rho \boldsymbol{f}$.

Using splitting (5), we have that

$$\rho \partial^2_{tt} u_a - \partial_b \sigma^{\text{dev}}_{ba} + \partial_a p = \rho f_a,$$

or $\rho \partial^2_{tt} \boldsymbol{u} - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{\sigma}^{\text{dev}} + \nabla_{\boldsymbol{x}} p = \rho \boldsymbol{f}$ in intrinsic form.

For a general hyperelastic material, the main difficulty of the UL formulation is writing the constitutive law. Usually, these laws are written in the form given by Eq. (8). Applying this to the UL formulation requires using the inverse of the tensorial transformation (3), i.e.,

$$\sigma_{ab} = J^{-1} F_{aA} S_{AB} F_{Bb},$$

which in intrinsic form reads $\boldsymbol{\sigma} = J^{-1} \boldsymbol{F} \cdot \boldsymbol{S} \cdot \boldsymbol{F}^T$. This equation is expressed in principle in terms of the coordinates $\boldsymbol{X}$, and needs to be transformed to the coordinates $\boldsymbol{x}$ through the mapping of the motion $\boldsymbol{x} = \boldsymbol{\psi}_t(\boldsymbol{X})$. Thus, a permanent change of coordinates and between $\boldsymbol{\sigma}$ and $\boldsymbol{S}$ would be required. This is why very often the UL formulation is applied in combination with constitutive laws that only require tensors evaluated in the deformed configuration $\Omega(t)$, particularly the Saint-Venant and the Neo-Hookean constitutive laws. Thus, for the UL approach we abandon generality and restrict ourselves to the case of the thermodynamically consistent Neo-Hookean materials. In this case, for isotropic materials the constitutive law is

$$\sigma_{ab} = J^{-1} \left[(\lambda \log(J) - \mu) \delta_{ab} + \mu b_{ab} \right],$$

where λ and μ are the so called Lamé's parameters and the left Cauchy-Green tensor $\boldsymbol{b}$ is defined in Eq. (1). The splitting of this expression into deviatoric and volumetric components yields:

$$\begin{aligned} \sigma^{\text{dev}}_{ab} &= J^{-1} \mu \left[b_{ab} - \frac{b_{cc}}{\text{d}} \delta_{ab} \right], \\ p &= -J^{-1} \left[(\lambda \log(J) - \mu) + \mu \frac{b_{cc}}{\text{d}} \right]. \end{aligned}$$

Regarding the geometric equation, if we have $\boldsymbol{u}$ in terms of $\boldsymbol{x}$ we may write $\boldsymbol{x} = \boldsymbol{X} + \boldsymbol{u} = \boldsymbol{\psi}_t^{-1}(\boldsymbol{x}) + \boldsymbol{u}(\boldsymbol{x}, t)$ and introduce $\boldsymbol{F}^{-1} = \nabla_{\boldsymbol{x}} \boldsymbol{\psi}_t^{-1}(\boldsymbol{x})$ as a tensor with both components in the deformed configuration. This allows us to write the field equations in the UL formulation, which read: find $\boldsymbol{u} : \mathfrak{D} \longrightarrow \mathbb{R}^{\text{d}}$, $\boldsymbol{\sigma} : \mathfrak{D} \longrightarrow \mathbb{R}^{\text{d}} \otimes \mathbb{R}^{\text{d}}$ and $\boldsymbol{F} : \mathfrak{D} \longrightarrow \mathbb{R}^{\text{d}} \otimes \mathbb{R}^{\text{d}}$ such that

$$\rho \partial^2_{tt} \boldsymbol{u} - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{\sigma} = \rho \boldsymbol{f}, \tag{14}$$

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$$\boldsymbol{\sigma} - J^{-1}\left[(\lambda \log(J) - \mu)\boldsymbol{I} + \mu \boldsymbol{b}\right] = \boldsymbol{0}, \tag{15}$$

$$\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T, \quad \boldsymbol{F}^{-1} + \nabla_{\boldsymbol{x}} \boldsymbol{u} = \boldsymbol{I}. \tag{16}$$

The unknown $\boldsymbol{F}$ can be replaced by $\boldsymbol{b}$ or by any other tensor measuring deformation, such as the Almansi strain tensor. As for the TL formulation, equations (14)-(15)-(16) can be respectively called momentum conservation, constitutive and geometric equations. The last two allow one to express $\boldsymbol{F}$ and $\boldsymbol{\sigma}$ in terms of $\boldsymbol{u}$ and end up with a differential equation in which the only unknown is $\boldsymbol{u}$, the irreducible form of the problem. If both $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ are kept as unknowns, we have a mixed displacement-stress formulation, now using an UL approach.

If the volumetric-deviatoric split of the constitutive equation is done, the problem consists of finding $\boldsymbol{u} : \mathfrak{D} \longrightarrow \mathbb{R}^{\mathrm{d}}$, $\boldsymbol{\sigma}^{\mathrm{dev}} : \mathfrak{D} \longrightarrow \mathbb{R}^{\mathrm{d}} \otimes \mathbb{R}^{\mathrm{d}}$, $p : \mathfrak{D} \longrightarrow \mathbb{R}$ and $\boldsymbol{F} : \mathfrak{D} \longrightarrow \mathbb{R}^{\mathrm{d}} \otimes \mathbb{R}^{\mathrm{d}}$ such that

$$\begin{aligned}
\rho \partial_{tt}^2 \boldsymbol{u} - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{\sigma} &= \rho \boldsymbol{f}, \\
\boldsymbol{\sigma}^{\mathrm{dev}} - J^{-1}\mu\Big[\boldsymbol{b} - \frac{1}{\mathrm{d}}\mathrm{tr}(\boldsymbol{b})\boldsymbol{I}\Big] &= \boldsymbol{0}, \\
\frac{J}{\lambda}p + \log(J) + \frac{\mu}{\lambda}\Big(\frac{1}{\mathrm{d}}\mathrm{tr}(\boldsymbol{b}) - 1\Big) &= 0, \\
\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T, \quad \boldsymbol{F}^{-1} + \nabla_{\boldsymbol{x}} \boldsymbol{u} &= \boldsymbol{I},
\end{aligned}$$

where $\mathrm{tr}(\boldsymbol{b})$ is the trace of $\boldsymbol{b}$. Once again, expressing $\boldsymbol{F}$ in terms of $\boldsymbol{u}$ but keeping $\boldsymbol{u}$, $\boldsymbol{\sigma}^{\mathrm{dev}}$ and p as unknowns leads to the three-field displacement-pressure-stress formulation. As for the TL formulation, it is possible to eliminate $\boldsymbol{\sigma}^{\mathrm{dev}}$ but keep p, thus obtaining a mixed displacement-pressure formulation required to deal with truly incompressible materials.

As for the TL formulation, initial and boundary conditions are needed to close the problem.

2.4. *Linear Elasticity*

Let us finally write the problem in the case of linear Elasticity, which can be shown to be the limit when the strains tend to zero of both the TL and the UL formulations. Coordinates $\boldsymbol{x}$ and $\boldsymbol{X}$ do not need to be distinguished, and we shall simply write $\nabla_{\boldsymbol{x}} = \nabla_{\boldsymbol{X}} \equiv \nabla$. We will consider only isotropic materials, with Lamé's parameters μ and λ and bulk modulus $\kappa = \lambda + \frac{2}{3}\mu$. In this case, the constitutive law reads

$$\begin{aligned}
\sigma_{ab} &= C_{abcd}\varepsilon_{cd}, \\
C_{abcd} &= \lambda\delta_{ac}\delta_{bd} + 2\mu\delta_{ab}\delta_{cd}, \\
\varepsilon_{ab} &= \frac{1}{2}(\partial_a u_b + \partial_b u_a),
\end{aligned}$$

where C_{abcd} are the components of the fourth order constitutive tensor $\boldsymbol{C}^{\mathrm{cons}}$. The volumetric-deviatoric split now yields:

$$\begin{aligned}
\sigma_{ab}^{\mathrm{dev}} &= C_{abcd}^{\mathrm{dev}}\varepsilon_{cd}, \\
C_{abcd}^{\mathrm{dev}} &= 2\mu\Big(\delta_{ac}\delta_{bd} - \frac{1}{3}\delta_{ab}\delta_{cd}\Big), \\
p &= -\kappa\partial_a u_a.
\end{aligned}$$

The field equations are: find $\boldsymbol{u} : \mathfrak{D} \longrightarrow \mathbb{R}^{\mathrm{d}}$, $\boldsymbol{\sigma} : \mathfrak{D} \longrightarrow \mathbb{R}^{\mathrm{d}} \otimes \mathbb{R}^{\mathrm{d}}$ and $\boldsymbol{\varepsilon} : \mathfrak{D} \longrightarrow \mathbb{R}^{\mathrm{d}} \otimes \mathbb{R}^{\mathrm{d}}$ such that

$$\rho \partial_{tt}^2 \boldsymbol{u} - \nabla \cdot \boldsymbol{\sigma} = \rho \boldsymbol{f}, \tag{17}$$

$$\boldsymbol{\sigma} - \boldsymbol{C}^{\mathrm{cons}} : \boldsymbol{\varepsilon} = \boldsymbol{0}, \tag{18}$$

$$\boldsymbol{\varepsilon} - \nabla^{\mathrm{s}} \boldsymbol{u} = \boldsymbol{0}, \tag{19}$$

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where $\nabla^{\mathrm{s}}\boldsymbol{u}$ is the symmetrical part of $\nabla\boldsymbol{u}$, whose components will be denoted $(\partial_a u_b)^{\mathrm{s}}$. If Eq. (19) is inserted into Eq. (18) and the result into Eq. (17) the irreducible form of the problem is obtained, whereas if the second step is not done the result is a displacement-stress formulation. Obviously, a mixed formulation involving strains is also possible, but we shall not consider this.

Finally, if the volumetric-deviatoric split is done, the equations we obtain are: find $\boldsymbol{u} : \mathfrak{D} \longrightarrow \mathbb{R}^{\mathrm{d}}$, $\boldsymbol{\sigma}^{\mathrm{dev}} : \mathfrak{D} \longrightarrow \mathbb{R}^{\mathrm{d}} \otimes \mathbb{R}^{\mathrm{d}}$, $p : \mathfrak{D} \longrightarrow \mathbb{R}$ and $\boldsymbol{\varepsilon} : \mathfrak{D} \longrightarrow \mathbb{R}^{\mathrm{d}} \otimes \mathbb{R}^{\mathrm{d}}$ such that

$$\begin{aligned}
\rho\partial_{tt}^2\boldsymbol{u} - \nabla\cdot\boldsymbol{\sigma}^{\mathrm{dev}} + \nabla p &= \rho\boldsymbol{f},\\
\boldsymbol{\sigma}^{\mathrm{dev}} - \boldsymbol{C}^{\mathrm{dev}} : \boldsymbol{\varepsilon} &= \boldsymbol{0},\\
\frac{p}{\kappa} + \nabla\cdot\boldsymbol{u} &= 0,\\
\boldsymbol{\varepsilon} - \nabla^{\mathrm{s}}\boldsymbol{u} &= \boldsymbol{0}.
\end{aligned}$$

As for the previous approaches, these equations are the basis of the displacement-pressure and displacement-pressure-stress formulations.

Again, initial and boundary conditions are needed to close the problem. Regarding the latter, since now $\Omega(t) = \Omega(0) \equiv \Omega$, we consider $\partial\Omega = \Gamma_D \cup \Gamma_N$, with Γ_D and Γ_N disjoint, and write them as

$$\boldsymbol{u} = \boldsymbol{u}_D \qquad \text{on } \Gamma_D,\ t \in (0, t_{\mathrm{fin}}), \tag{20}$$

$$\boldsymbol{n}\cdot\boldsymbol{\sigma} = -p\boldsymbol{n} + \boldsymbol{n}\cdot\boldsymbol{\sigma}^{\mathrm{dev}} = \boldsymbol{t}_N \qquad \text{on } \Gamma_N,\ t \in (0, t_{\mathrm{fin}}). \tag{21}$$

2.5. Summary

Table I summarizes all equations introduced so far in the stationary case, for simplicity (i.e., neglecting the inertia term $\rho\partial_{tt}^2\boldsymbol{u}$) and expressed in components.

When the FE approximation of all the mixed formulations presented is attempted, care needs to be paid to the satisfaction of the appropriate inf-sup conditions between the interpolating spaces or, as we shall do in the following sections, a stabilized FE formulation needs to be designed. We also indicate in Table I the references where we have proposed FE methods for the different mixed problems considered. Besides the irreducible formulations, the mixed displacement-pressure formulation is well known, as it corresponds to the classical Stokes problem when $\kappa \to \infty$, and no references are provided for it, since many alternatives exist to deal with this problem. For the rest of models, only the most relevant references are given.

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Unknowns	**Geometry definition**		
	Infinitesimal strains	Updated Lagrangian (Neo-Hookean material)	Total Lagrangian
$\boldsymbol{u}$	$-\partial_b C_{abcb}(\partial_c u_d)^{\mathrm{s}} = \rho f_a$	$-\partial_b \left\{ J^{-1} \left[(\lambda \log(J) - \mu)\delta_{ab} + \mu b_{ab} \right] \right\} = \rho f_a$	$-\partial_A(F_{aB} S_{BA}) = \rho_0 f_a$
$\boldsymbol{u}$ & p	$-\partial_b C^{\mathrm{dev}}_{abcd}(\partial_c u_d)^{\mathrm{s}} + \partial_a p = \rho f_a$ $\frac{1}{\kappa} p + \partial_a u_a = 0$ Ref. [10]	$-\partial_b \left[J^{-1} \mu \left(b_{ab} - \frac{b_{cc}}{\mathrm{d}} \delta_{ab} \right) \right] + \partial_a p = \rho f_a$ $\frac{J}{\lambda} p + \log(J) + \frac{\mu}{\lambda} \left(\frac{b_{cc}}{\mathrm{d}} - 1 \right) = 0$ Ref. [11]	$-\partial_A(F_{aB} S'_{BA}) + \partial_A(pJF^{-1}_{Aa}) = \rho_0 f_a$ $\frac{p}{\kappa} + \frac{\mathrm{d}G}{\mathrm{d}J} = 0$ Ref. [12]
$\boldsymbol{u}$ & stress	$-\partial_b \sigma_{ab} = \rho f_a$ $C^{-1}_{abcd} \sigma_{cd} - (\partial_a u_b)^{\mathrm{s}} = 0$ Ref. [13, 14]	$-\partial_b \sigma_{ab} = \rho f_a$ $\sigma_{ab} - J^{-1} \left[(\lambda \log(J) - \mu)\delta_{ab} + \mu b_{ab} \right] = 0$	$-\partial_A(F_{aB} S_{BA}) = \rho_0 f_a$ $S_{AB} - 2\frac{\partial \Psi}{\partial C_{AB}} = 0$ Ref. [15]
$\boldsymbol{u}$, p & stress	$-\partial_b \sigma^{\mathrm{dev}}_{ab} + \partial_a p = \rho f_a$ $\frac{1}{\kappa} p + \partial_a u_a = 0$ ${C^{\mathrm{dev}}_{abcd}}^{-1} \sigma^{\mathrm{dev}}_{cd} - (\partial_a u_b)^{\mathrm{s}} = 0$ Ref. [16]	$-\partial_b \sigma^{\mathrm{dev}}_{ab} + \partial_a p = \rho f_a$ $\sigma^{\mathrm{dev}}_{ab} - J^{-1} \mu \left(b_{ab} - \frac{b_{cc}}{\mathrm{d}} \delta_{ab} \right) = 0$ $\frac{J}{\lambda} p + \log(J) + \frac{\mu}{\lambda} \left(\frac{b_{cc}}{\mathrm{d}} - 1 \right) = 0$ Ref. [18]	$-\partial_A(F_{aB} S'_{BA}) + \partial_A(pJF^{-1}_{Aa}) = \rho_0 f_a$ $\frac{p}{\kappa} + \frac{\mathrm{d}G}{\mathrm{d}J} = 0$ $S'_{AB} - 2\frac{\partial W}{\partial C_{AB}} = 0$ Ref. [19]

Table I. Mixed formulations in the stationary problem

3. THE VMS FRAMEWORK

To avoid using involved mixed interpolations satisfying the adequate inf-sup conditions, we favor the use of stabilized FE methods that permit to employ equal interpolations for all variables. The formulations we present in this work are based on the VMS framework, and this is why we present here a summary of the formulation, emphasizing the aspects that will be relevant for the problem we wish to consider.

3.1. Stationary and linear problems

Since we wish to consider a significant number of models, it is important to summarize the VMS approach in an abstract framework. To simplify the exposition, we start first with linear stationary problems, and later we will comment on transient and nonlinear problems. Therefore, let us start with the stationary version of the problems in the first column of Table I.

The starting point is the variational form of the equations to be solved. In the following, we denote as V, Q and T the continuous spaces for the velocity, the pressure and the stress, either the complete stress tensor or only its deviatoric part, which depend on the variational form of the problem. In the case of T, it is composed of symmetric tensors. This symmetry can be imposed a priori, in the construction of space T, or weakly, as a result of the equations that need to be solved. In our implementation of the discretized problems we adopt the former option.

As usual, $L^2(\Omega)$ denotes the space of square integrable functions in Ω, $H^1(\Omega)$ the space of functions in $L^2(\Omega)$ with first derivatives in $L^2(\Omega)$, and $H(\mathrm{div},\Omega)$ the space of second order tensors with components in $L^2(\Omega)$ and whose divergence has also components in $L^2(\Omega)$. The $L^2(\Omega)$-inner product is denoted as $(\cdot,\cdot)$ and the integral of the product of two functions, whenever it makes sense, as $\langle\cdot,\cdot\rangle$, with a subscript to indicate the integration domain if it is not Ω. The space of functions in $H^1(\Omega)$ that vanish on Γ_D is denoted as $H^1_D(\Omega)$, and the space of tensors in $H(\mathrm{div},\Omega)$ that vanish on Γ_N as $H_N(\mathrm{div},\Omega)$. The norm in a functional space X will be denoted as $\|\cdot\|_X$, except when $X = L^2(\Omega)$, case in which no subscript will be used.

The unknown of the problem differs according to the problem being considered. In general, we will denote it as u and the functional space where it belongs as X; generic test functions will be denoted by $v \in X$. The different expressions of the unknown u are given in the second row of Table II, in which the order of the unknowns in the case of the mixed problems is relevant. Likewise, the test functions in this table have been denoted as $\boldsymbol{v} \in V$, $q \in Q$ and $\boldsymbol{\tau} \in T$.

For all problems, the momentum equation is tested with $\boldsymbol{v} \in V$, leading to the external work $\langle \boldsymbol{v}, \rho\boldsymbol{f}\rangle$. For all formulations except the dual form of the displacement-stress one, the boundary condition Eq. (21) is natural, and the traction $\boldsymbol{t}_N$ contributes with an external work $\langle \boldsymbol{v}, \boldsymbol{t}_N\rangle_{\Gamma_N}$, so that the total external work is $L(v) = \langle \boldsymbol{v}, \rho\boldsymbol{f}\rangle + \langle \boldsymbol{v}, \boldsymbol{t}_N\rangle_{\Gamma_N}$; for the dual form of the displacement-stress formulation the displacement $\boldsymbol{u}_D$ in Eq. (20) contributes with an external work $\langle \boldsymbol{n}\cdot\boldsymbol{\tau}, \boldsymbol{u}_D\rangle_{\Gamma_D}$, $\boldsymbol{\tau} \in T$ being the stress test function, and thus the total external work is $L(v) = \langle \boldsymbol{v}, \rho\boldsymbol{f}\rangle + \langle \boldsymbol{n}\cdot\boldsymbol{\tau}, \boldsymbol{u}_D\rangle_{\Gamma_D}$. This is a result of the integration-by-parts formula

$$B(u,v) = \langle \mathscr{L}u, v\rangle + \langle \mathscr{F}_n u, \mathscr{D}v\rangle_{\partial\Omega} \tag{22}$$

$$= \langle u, \mathscr{L}^* v\rangle + \langle \mathscr{D}^* u, \mathscr{F}_n^* v\rangle_{\partial\Omega}, \tag{23}$$

that holds for all problems considered, where $\mathscr{L}$ is the differential operator of the problem, which in general we write as $\mathscr{L}u = f$ (with f easily identified), $\langle \mathscr{L}u, v\rangle$ needs to be understood in the distributional sense, $\mathscr{F}_n$ is the normal component of the flux operator associated to $\mathscr{L}$ and $\mathscr{D}$ the Dirichlet operator on the boundaries giving the adequate trace depending again on $\mathscr{L}$. We have also introduced the adjoints of operators $\mathscr{L}$, $\mathscr{F}_n$ and $\mathscr{D}$, respectively denoted as $\mathscr{L}^*$, $\mathscr{F}_n^*$ and $\mathscr{D}^*$. These equations also hold if instead of extending the integrals over the whole domain Ω they are extended to any other domain. All these operators, as well as the bilinear form $B(u,v)$, are given in Table II.

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In all cases, it can be shown that $\mathscr{D}^* v = \mathscr{D} v$. The adjoints $\mathscr{L}^*$ and $\mathscr{F}_n^*$ in Table II are easily obtained using Gauss' theorem. For example, for the displacement-pressure formulation we have that

$$\begin{aligned} B(u,v) &= \int_\Omega \left[\nabla^{\rm s} \boldsymbol{v} \cdot (\boldsymbol{C}^{\rm dev} : \nabla^{\rm s} \boldsymbol{u}) - p \nabla \cdot \boldsymbol{v} + \frac{1}{\kappa} p\, q + q \nabla \cdot \boldsymbol{u} \right] \\ &= \int_\Omega \begin{bmatrix} -\nabla \cdot (\boldsymbol{C}^{\rm dev} : \nabla^{\rm s} \boldsymbol{u}) + \nabla p \\ \frac{1}{\kappa} p + \nabla \cdot \boldsymbol{u} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{v} \\ q \end{bmatrix} + \int_{\partial\Omega} \left[-p \boldsymbol{n} + \boldsymbol{n} \cdot (\boldsymbol{C}^{\rm dev} : \nabla^{\rm s} \boldsymbol{u}) \right] \cdot \boldsymbol{v} \\ &= \int_\Omega \begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix} \cdot \begin{bmatrix} -\nabla \cdot (\boldsymbol{C}^{\rm dev} : \nabla^{\rm s} \boldsymbol{v}) - \nabla q \\ \frac{1}{\kappa} q - \nabla \cdot \boldsymbol{v} \end{bmatrix} + \int_{\partial\Omega} \left[q \boldsymbol{n} + \boldsymbol{n} \cdot (\boldsymbol{C}^{\rm dev} : \nabla^{\rm s} \boldsymbol{v}) \right] \cdot \boldsymbol{u}, \end{aligned}$$

from where operators $\mathscr{L}^*$ and $\mathscr{F}_n^*$ can be identified. These operators for the rest of formulations can be obtained similarly. We shall see that they are relevant when applying the VMS idea to the problems we consider.

The functional spaces where the problems are well posed are given in the last row of Table II, considering homogeneous boundary conditions for simplicity. Except for the dual form of the displacement-stress formulation, $V = H^1_D(\Omega)^{\rm d}$. This implicitly assumes that $\boldsymbol{u}_D = \boldsymbol{0}$; if the displacement prescription is not zero, the usual lifting argument can be used to set the problem. In all these formulations, stresses and pressure are only square integrable. In the case of the dual form of the displacement-stress formulation, the stress divergence is not integrated by parts but the displacement gradient is, leading to milder regularity restrictions for the displacement and stronger regularity for the stress, which needs to belong to $H_N({\rm div}, \Omega)$ to have a well posed variational form. This presumes that $\boldsymbol{t}_N = \boldsymbol{0}$; analogously to the rest of formulations, if the traction prescription is not zero, the usual lifting argument can be used to set the problem.

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u	**Formulation**				
	$\boldsymbol{u}$	$[\boldsymbol{u}, p]$	$[\boldsymbol{u}, \boldsymbol{\sigma}]$		$[\boldsymbol{u}, p, \boldsymbol{\sigma}^{\text{dev}}]$
			Primal	Dual	
$\mathscr{L}u$	$-\nabla \cdot (\boldsymbol{C}^{\text{cons}} : \nabla^{\text{s}}\boldsymbol{u})$	$\begin{bmatrix} -\nabla \cdot (\boldsymbol{C}^{\text{dev}} : \nabla^{\text{s}}\boldsymbol{u}) + \nabla p \\ \frac{1}{\kappa}p + \nabla \cdot \boldsymbol{u} \end{bmatrix}$	$\begin{bmatrix} -\nabla \cdot \boldsymbol{\sigma} \\ \boldsymbol{C}^{-1} : \boldsymbol{\sigma} - \nabla^{\text{s}}\boldsymbol{u} \end{bmatrix}$		$\begin{bmatrix} -\nabla \cdot \boldsymbol{\sigma}^{\text{dev}} + \nabla p \\ \frac{1}{\kappa}p + \nabla \cdot \boldsymbol{u} \\ {\boldsymbol{C}^{\text{dev}}}^{-1} : \boldsymbol{\sigma}^{\text{dev}} - \nabla^{\text{s}}\boldsymbol{u} \end{bmatrix}$
$\mathscr{L}^*v$	$-\nabla \cdot (\boldsymbol{C}^{\text{cons}} : \nabla^{\text{s}}\boldsymbol{v})$	$\begin{bmatrix} -\nabla \cdot (\boldsymbol{C}^{\text{dev}} : \nabla^{\text{s}}\boldsymbol{v}) - \nabla q \\ \frac{1}{\kappa}q - \nabla \cdot \boldsymbol{v} \end{bmatrix}$	$\begin{bmatrix} \nabla \cdot \boldsymbol{\tau} \\ \boldsymbol{C}^{-1} : \boldsymbol{\tau} + \nabla^{\text{s}}\boldsymbol{v} \end{bmatrix}$		$\begin{bmatrix} \nabla \cdot \boldsymbol{\tau}^{\text{dev}} - \nabla q \\ \frac{1}{\kappa}q - \nabla \cdot \boldsymbol{v} \\ {\boldsymbol{C}^{\text{dev}}}^{-1} : \boldsymbol{\tau}^{\text{dev}} + \nabla^{\text{s}}\boldsymbol{v} \end{bmatrix}$
$\mathscr{F}_n u$	$\boldsymbol{n} \cdot (\boldsymbol{C}^{\text{cons}} : \nabla^{\text{s}}\boldsymbol{u})$	$-p\boldsymbol{n} + \boldsymbol{n} \cdot (\boldsymbol{C}^{\text{dev}} : \nabla^{\text{s}}\boldsymbol{u})$	$\boldsymbol{n} \cdot \boldsymbol{\sigma}$	$\boldsymbol{u}$	$-p\boldsymbol{n} + \boldsymbol{n} \cdot \boldsymbol{\sigma}^{\text{dev}}$
$\mathscr{F}_n^* v$	$\boldsymbol{n} \cdot (\boldsymbol{C}^{\text{cons}} : \nabla^{\text{s}}\boldsymbol{v})$	$q\boldsymbol{n} + \boldsymbol{n} \cdot (\boldsymbol{C}^{\text{dev}} : \nabla^{\text{s}}\boldsymbol{v})$	$\boldsymbol{n} \cdot \boldsymbol{\tau}$	$\boldsymbol{v}$	$q\boldsymbol{n} - \boldsymbol{n} \cdot \boldsymbol{\tau}^{\text{dev}}$
$\mathscr{D}u$	$\boldsymbol{u}$	$\boldsymbol{u}$	$\boldsymbol{u}$	$\boldsymbol{n} \cdot \boldsymbol{\sigma}$	$\boldsymbol{u}$
$B(u, v)$	$(\nabla^{\text{s}}\boldsymbol{v}, \boldsymbol{C}^{\text{cons}} : \nabla^{\text{s}}\boldsymbol{u})$	$(\nabla^{\text{s}}\boldsymbol{v}, \boldsymbol{C}^{\text{dev}} : \nabla^{\text{s}}\boldsymbol{u})$ $-(p, \nabla \cdot \boldsymbol{v}) + (q, \nabla \cdot \boldsymbol{u})$ $+\frac{1}{\kappa}(p, q)$	$(\nabla^{\text{s}}\boldsymbol{v}, \boldsymbol{\sigma})$ $+(\boldsymbol{C}^{-1} : \boldsymbol{\sigma}, \boldsymbol{\tau})$ $-(\nabla^{\text{s}}\boldsymbol{u}, \boldsymbol{\tau})$	$-(\boldsymbol{v}, \nabla \cdot \boldsymbol{\sigma})$ $+(\boldsymbol{C}^{-1} : \boldsymbol{\sigma}, \boldsymbol{\tau})$ $+(\boldsymbol{u}, \nabla \cdot \boldsymbol{\tau})$	$(\nabla^{\text{s}}\boldsymbol{v}, \boldsymbol{\sigma}^{\text{dev}}) - (p, \nabla \cdot \boldsymbol{v})$ $+(q, \nabla \cdot \boldsymbol{u}) + \frac{1}{\kappa}(p, q)$ $+({\boldsymbol{C}^{\text{dev}}}^{-1} : \boldsymbol{\sigma}^{\text{dev}}, \boldsymbol{\tau}) - (\nabla^{\text{s}}\boldsymbol{u}, \boldsymbol{\tau})$
X $(\boldsymbol{u}_D = \boldsymbol{0},$ $\boldsymbol{t}_N = \boldsymbol{0})$	$H_D^1(\Omega)^{\text{d}}$	$H_D^1(\Omega)^{\text{d}} \times L^2(\Omega)$	$H_D^1(\Omega)^{\text{d}} \times L^2(\Omega)^{\text{d}\times\text{d}}$	$L^2(\Omega)^{\text{d}} \times H_N(\text{div}, \Omega)$	$H_D^1(\Omega)^{\text{d}} \times L^2(\Omega) \times L^2(\Omega)^{\text{d}\times\text{d}}$

Table II. Linear problems

With the notation introduced, all problems to be considered can be written in the following abstract variational form: find $u \in X$ such that

$$B(u, v) = L(v) \qquad \text{for all } v \in X. \tag{24}$$

We wish to consider now the FE approximation to this problem. For the sake of simplicity, let us assume the Ω is a polyhedral domain for which we can consider a family of FE partitions $\mathscr{T}_h = \{K\}$, where K denotes a generic FE domain, h_K will be its diameter and $h = \max_K\{h_K\}$ the mesh size. All FE functions will be identified with the subscript h. We will consider the same interpolation order for all the elements, either simplicial or quadrilateral/hexahedral (for $\mathrm{d} = 2, 3$). We may however use different interpolation orders for the different components of u. All approximations we shall consider are conforming, i.e., the FE spaces will be chosen as finite dimensional subspaces of the functional spaces where the problem is posed; the important case of discontinuous Galerkin approximations is thus excluded. Thus, from $\mathscr{T}_h$ we may construct approximation spaces for the displacement $V_h \subset V$, the pressure $Q_h \subset Q$ and the stress $T_h \subset T$, and from them we can construct $X_h \subset X$, that depends on the problem being analyzed. We will also be interested in cases in which the test functions belong to the same space as the unknown.

Once the the FE approximation is set, the Galerkin approximation to problem (24) consists of finding $u_h \in X_h$ such that

$$B(u_h, v_h) = L(v_h) \quad \text{for all } v_h \in X_h. \tag{25}$$

This problem is well posed, i.e., it has a unique solution bounded in the norm of X independently of h, if the following condition holds (see, e.g., [24]):

$$\inf_{u_h \in X_h \setminus \{0\}} \sup_{v_h \in X_h \setminus \{0\}} \frac{B(u_h, v_h)}{\|u_h\|_X \|v_h\|_X} \geq K_B > 0, \tag{26}$$

for a certain constant K_B. In the case of the irreducible formulation, $X_h = V_h$ and verifying this condition is trivial (the bilinear form $B(u, v)$ is coercive in this case). However, for the rest of problems X_h is obtained from the Cartesian product of two or three spaces, and the previous inf-sup condition poses stringent compatibility conditions on the choice of these FE spaces. Note that the continuous counterpart of (26), replacing X_h by X, is known to hold for all problems considered.

Perhaps the most well known case is the mixed displacement-pressure formulation. In this case, $X_h = V_h \times Q_h$. If κ is small, the problem is well posed for any combination of FE pairs V_h-Q_h. However, when $\kappa \to \infty$, i.e., in the incompressible limit, not any combination of V_h and Q_h is possible. Nevertheless, the way to design such stable FE pairs is well understood and has a vast literature (see, e.g., [24]). In any case, the resulting mixed interpolations are always more involved to implement that the equal interpolation to construct V_h and Q_h, despite it requires a modification of the Galerkin approach and the introduction of stabilization terms, which we will design using the VMS concept introduced in the following.

Similar comments can be applied to the mixed displacement-stress formulation, in which $X_h = V_h \times T_h$. In the primal form, it is relatively simple to construct inf-sup stable interpolations, simply by taking stress spaces contained in the gradient of the displacement space. However, the dual form is by far more intricate. In essence, the difficulty is the same as for the approximation of Darcy's problem, and the inf-sup stable pairs are extensions of those known to be stable for it, such as the Raviart-Thomas or the Brezzi-Douglas-Marini FE spaces (see [33]).

Finally, the displacement-pressure-stress interpolation is definitely the one in which it is most difficult to design inf-sup stable interpolations. In this case, $X_h = V_h \times Q_h \times T_h$, and the inf-sup condition (26) holds if an inf-sup condition between V_h and Q_h and another one between V_h and T_h hold; the former, though, is only needed in the incompressible limit. FE interpolations satisfying

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these two conditions are rare and difficult to implement (see the discussion in [16] and references therein). As already mentioned, both in this case and in the displacement-stress formulation, there is the possibility to prescribe the symmetry of the stress tensor strongly in space T_h or weakly; we use the former approach in our implementations, but we shall not state it explicitly.

These comments for different formulations motivate the need to introduce stabilized FE methods. Here we will present the VMS framework in an abstract context, leaving for the forthcoming sections its realization for each model. Although it is not our purpose to undertake any numerical analysis of the resulting discrete problems, let us stress that it can be shown, at least for the linear problems, that the formulations proposed accomplish the target of being stable in appropriate norms.

The key idea of the VMS approach is to split space X as [25, 26]:

$$X = X_h \oplus X', \tag{27}$$

where X' is any complement to X_h in X (not to be confused with the dual of X). Each VMS-type method will depend precisely on the way X' is approximated. We will still denote the approximation as X', since no confusion will be possible.

Splitting (27) will have the associated splitting of the unknowns and tests functions $u = u_h + u'$ and $v = v_h + v'$, with $u_h, v_h \in X_h$ and $u', v' \in X'$. We call u' the sub-grid scale (SGS) and X' the space of SGSs. Because of the linearity of the problem considered so far, we may write the continuous problem (24) as: find $u_h \in X_h$ and $u' \in X'$ such that

$$B(u_h, v_h) + B(u', v_h) = L(v_h) \quad \text{for all } v_h \in X_h, \tag{28}$$

$$B(u_h, v') + B(u', v') = L(v') \quad \text{for all } v' \in X'. \tag{29}$$

No approximation has been done, yet. In fact, the approximation to X' will be a consequence of the approximation to u' (still denoted u'). In order to avoid approximating derivatives of u', and require only the unknown u' itself, making use of the additivity of the integral and Eq. (23) we may write Eq. (28) as

$$B(u_h, v_h) + \sum_K \left[\langle u', \mathscr{L}^* v_h \rangle_K + \langle \mathscr{D} u', \mathscr{F}_n^* v_h \rangle_{\partial K} \right] = L(v_h) \quad \text{for all } v_h \in X_h. \tag{30}$$

Since in the FE approximation v_h is piecewise polynomial, $\mathscr{L}^* v_h$ and $\mathscr{F}_n^* v_h$ are well defined element-wise.

Eq. (30) will be our FE problem once u' is approximated in terms of u_h. The description of how to attempt this will be omitted, and only the final result will be stated. In any case, this approximation must be obtained from Eq. (29), which using Eq. (22) may be written as

$$\begin{aligned} B(u', v') &= \sum_K \left[\langle \mathscr{L} u', v' \rangle_K + \langle \mathscr{F}_n u', \mathscr{D} v' \rangle_{\partial K} \right] \\ &= L(v') - B(u_h, v') \\ &= L(v') - \sum_K \left[\langle \mathscr{L} u_h, v' \rangle_K + \langle \mathscr{F}_n u_h, \mathscr{D} v' \rangle_{\partial K} \right] \quad \text{for all } v' \in X'. \end{aligned} \tag{31}$$

All VMS-type methods consist in approximating u' from Eq. (31). Because the SGS problem is infinite dimensional, some approximation needs to be introduced to make the method computationally feasible.

Taking v' such that $L(v') = \langle v', f \rangle$, the fine scale problem (31) can be written as

$$\sum_K \left[\langle v', \mathscr{L} u' \rangle_K + \langle \mathscr{D} v', \mathscr{F}_n u' \rangle_{\partial K} \right] = \sum_K \left[\langle v', \mathscr{R} u_h \rangle_K - \langle \mathscr{D} v', \mathscr{F}_n u_h \rangle_{\partial K} \right],$$

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where $\mathscr{R}u_h = f - \mathscr{L}u_h$ is the (strong) FE residual.

Let us denote by $\mathscr{E}_h$ the set of element edges (faces when $\mathrm{d} = 3$) including those on the boundary of the domain $\partial\Omega$, denoted by $\mathscr{E}_h^\Gamma$, and the set of internal edges, denoted by $\mathscr{E}_h^0$. Using this notation the boundary terms can be grouped as

$$\begin{aligned}\sum_K [\langle \mathscr{D}v', \mathscr{F}_n u_h\rangle_{\partial K} + \langle \mathscr{D}v', \mathscr{F}_n u'\rangle_{\partial K}] &= \sum_K \langle \mathscr{D}v', \mathscr{F}_n u\rangle_{\partial K} \\ &= \sum_{E\in\mathscr{E}_h^0} \langle \mathscr{D}v', [\![\mathscr{F}_n u]\!]\rangle_E + \sum_{E\in\mathscr{E}_h^\Gamma} \langle \mathscr{D}v', \mathscr{F}_n u\rangle_E\,, \end{aligned} \tag{32}$$

where $[\![\mathscr{F}_n u]\!]$ denotes the jump of $\mathscr{F}_n u$, i.e. $\mathscr{F}_{n^+}u + \mathscr{F}_{n^-}u$, where $\boldsymbol{n}^\pm$ denotes the external normal to each of the elements $K^\pm$ that share edge E. Because the normal fluxes of the total unknown are continuous across the interelement boundaries the first term in the right-hand side (RHS) vanishes and we arrive to

$$\sum_K \langle v', \mathscr{L}u'\rangle_K + \sum_{E\in\mathscr{E}_h^\Gamma} \langle \mathscr{D}v', \mathscr{F}_n u'\rangle_E = \sum_K \langle v', \mathscr{R}u_h\rangle_K - \sum_{E\in\mathscr{E}_h^\Gamma} \langle \mathscr{D}v', \mathscr{F}_n u_h\rangle_E \tag{33}$$

The boundary terms are also zero if we prescribe $\mathscr{D}u'$ on each edge E in an essential way, so that the SGS test function also satisfies $\mathscr{D}v' = 0$ on Γ. Therefore, the equation for the SGSs reads:

$$\sum_K \langle v', \mathscr{L}u'\rangle_K = \sum_K \langle v', \mathscr{R}u_h\rangle_K, \tag{34}$$

with the condition that $\mathscr{D}u'$ is given on the element edges. This is the problem that needs to be approximated, and several options are explained in [34]. The one we favor leverages a Fourier analysis of the problem, which in particular justifies that the SGSs can be computed first in the element interiors regardless of their values on the edges, and then the SGSs on the edges can be computed. This approximate Fourier analysis of the SGS problem was proposed first in [35] and later extended in [36], for example, with the objective of determining the functional form the stabilization parameters (see below), i.e., their dependence on the equation coefficients and on the mesh size up to algorithmic constants. The main heuristic assumption is that u' is highly fluctuating, and therefore dominated by high wave numbers.

The SGSs on the element edges are required when discontinuous interpolations are used for any of the fields involved in u. For example, they are needed if one wishes to consider arbitrary discontinuous interpolations of pressures and/or stresses. However, to simplify the exposition we will restrict ourselves to (possibly equal) continuous interpolations for all variables, case in which stable FE formulations are obtained setting $\mathscr{D}u' = 0$ on all edges of the FE partition, although this condition can be relaxed [37].

Regarding the approximation of the SGSs in the interior of the element domains, from (34) it turns out that we may approximate it within each element K as (see [34]):

$$u'|_K = \tau_K \mathscr{P}'[\mathscr{R}u_h]|_K, \tag{35}$$

where $\mathscr{P}'$ is the $L^2(\Omega)$ projection onto the space of SGSs, which still needs to be chosen, and τ_K is a matrix of stabilization parameters that tries to approximate (in an integral sense) operator $\mathscr{L}^{-1}$. We shall come back later on to its expression for the problems we are considering. Regarding the space of SGSs and the associated projection $\mathscr{P}'$, we consider two options. The first is to take X' as the space of FE residuals, case in which $u'|_K = \tau_K \mathscr{R}u_h|_K$. This approach was called Algebraic Sug-Grid Scale (ASGS) in [35]. The second option is to take $X' = X_h^\perp$, i.e., the $L^2(\Omega)$-orthogonal to the FE space. This leads to the Orthogonal Sug-Grid Scale (OSGS) approach [35], case in which $\mathscr{P}' = I - \mathscr{P}_h$, where $\mathscr{P}_h$ is the $L^2(\Omega)$-projection onto the FE space. Whichever the option is, the problem to be solved is: find $u_h \in X_h$ such that

$$B(u_h, v_h) + \sum_K \langle u', \mathscr{L}^* v_h\rangle_K = L(v_h) \quad \text{for all } v_h \in X_h. \tag{36}$$

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3.2. *Second order problems in time*

When the elastodynamics problem needs to be considered, the differential equation is not stationary, but second order in time for the displacement, although not for the rest of unknowns in the case of mixed formulations. This leads to an differential-algebraic system of equations. Nevertheless, to explain the extension of the previous methodology to time dependent problems we shall consider the following abstract variational equation: find $u : (0, t_{\text{fin}}) \longrightarrow X$ such that

$$(\partial^2_{tt} u, v) + B(u, v) = L(v) \qquad \text{for all } v \in X, \tag{37}$$

instead of Eq. (24), with adequate initial conditions for u. The particular application of the approach to be described to Elasticity is deferred to the following sections.

If the splitting (27) is applied to problem (37) we obtain, instead of (28)-(29):

$$(\partial^2_{tt} u_h, v_h) + (\partial^2_{tt} u', v_h) + B(u_h, v_h) + B(u', v_h) = L(v_h) \quad \text{for all } v_h \in X_h, \tag{38}$$

$$(\partial^2_{tt} u_h, v') + (\partial^2_{tt} u', v') + B(u_h, v') + B(u', v') = L(v') \quad \text{for all } v' \in X'. \tag{39}$$

Two alternatives are now possible, either to consider $\partial^2_{tt} u'$ negligible or not. The former was termed as *quasi-static* SGSs in [35], whereas the latter was termed as *dynamic* SGSs. While quasi-static SGSs yield stable and accurate schemes if the time step size of the temporal discretization is relatively large with respect to the mesh size, dynamic SGSs are needed when this time step size is very small. This fact has been extensively verified for first order problem in time [38, 39], and to a lesser extend for the second order problem of elastodynamics, particularly in the context of fluid-structure interaction [18, 11].

If quasi-static SGSs are considered, the same arguments as for the stationary problem lead to the approximation:

$$u'|_K = \tau_K \mathscr{P}'[\mathscr{R} u_h - \partial^2_{tt} u_h]|_K, \tag{40}$$

that replaces (35). This is then inserted into Eq. (38) to obtain a problem posed in terms of $u_h \in X_h$ alone (assuming $\partial^2_{tt} u' = 0$).

When using dynamic SGSs, the algebraic expression Eq. (40) needs to be replaced by the ordinary differential equation

$$\partial^2_{tt} u' + \tau_K^{-1} u'|_K = \mathscr{P}'[\mathscr{R} u_h - \partial^2_{tt} u_h]|_K, \tag{41}$$

with appropriate initial conditions for the SGSs. In the computer implementation of a FE code, this equation needs to be solved at each numerical integration point. Again, once the SGSs (and their second derivatives in time) are obtained, they are inserted into Eq. (38) to obtain a problem posed in terms of $u_h \in X_h$ alone. The algorithmic process is explained in [18, 11] for the elastodynamics problem.

Both if quasi-static or dynamic SGSs are used, in time dependent problems in general and in elastodynamics in particular, it is very convenient to use orthogonal SGSs. In this case, $(\partial^2_{tt} u', v_h) = (\partial^2_{tt} u_h, v') = 0$ and $\mathscr{P}'[\partial^2_{tt} u_h] = 0$. Thus, the only term where $\partial^2_{tt} u_h$ appears is in $(\partial^2_{tt} u_h, v_h)$, yielding the symmetric and positive-definite mass-matrix when FE basis functions are introduced. However, if the SGSs are not orthogonal, $\partial^2_{tt} u_h$ appears in other places and the final mass matrix is not even symmetric.

Any time integration scheme can be employed to approximate (41). The natural option is to use the same as for the FE unknown, but in fact it can be shown that one can use an approximation of one order less and still obtain the same accuracy for the FE solution.

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3.3. Nonlinear problems

Another issue to consider when applying the VMS ideas to finite strain problems is the treatment of the nonlinearity. We slightly reformulate the ideas presented in [40] in this subsection.

Let us consider a nonlinear stationary problem of the form $\mathscr{A}(u) = f$, where $\mathscr{A}(u)$ is now a nonlinear operator. Suppose that this equation is solved by an iterative scheme, so that given a guess for the solution, denoted $\bar{u}$, the correction δu is computed from the equation

$$\mathscr{L}(\bar{u}; \delta u) = f - \mathscr{A}(\bar{u}),$$

where $\mathscr{L}(\bar{u}; \delta u)$ is linear in δu. This is now a linear problem to which the general VMS ideas presented earlier can be applied. In particular, assuming for simplicity homogeneous boundary conditions, Eq. (36) becomes

$$B(\bar{u}; \delta u_h, v_h) + \sum_K \langle \delta u', \mathscr{L}^*(\bar{u}; v_h) \rangle_K = \langle v_h, f - \mathscr{A}(\bar{u}_h + \bar{u}') \rangle \quad \text{for all } v_h \in X_h, \tag{42}$$

where the initial guess is split as $\bar{u} = \bar{u}_h + \bar{u}'$. Since $\bar{u}'$ is expected to be small, we may approximate

$$\mathscr{A}(\bar{u}_h + \bar{u}') \approx \mathscr{A}(\bar{u}_h) + \mathscr{L}(\bar{u}; \bar{u}'),$$

and integrating by parts $\langle v_h, \mathscr{L}(\bar{u}; \bar{u}') \rangle$ in (42) we arrive to

$$B(\bar{u}; \delta u_h, v_h) + \sum_K \langle u', \mathscr{L}^*(\bar{u}; v_h) \rangle_K = \langle v_h, f - \mathscr{A}(\bar{u}_h) \rangle \quad \text{for all } v_h \in X_h. \tag{43}$$

Note that the unknown SGS is the total one, $u' = \bar{u}' + \delta u'$, not only its increment $\delta u'$. Let us also remark that the duality in the second term of the RHS implies that some terms have to be integrated by parts (recall that we assume homogeneous boundary conditions). The total SGS to be used in Eq. (43) is

$$u'|_K = \tau_K \mathscr{P}'[f - \mathscr{A}(\bar{u}_h) - \mathscr{L}(\bar{u}; \delta u_h)]|_K. \tag{44}$$

At convergence, $\delta u_h = 0$, so that the problem that is finally solved is

$$\langle v_h, \mathscr{A}(u_h) \rangle \; + \sum_K \langle \tau_K \mathscr{P}'[f - \mathscr{A}(u_h)], \mathscr{L}^*(u; v_h) \rangle_K = \langle v_h, f \rangle \quad \text{for all } v_h \in X_h.$$

Note that this equation reduces to Eq. (36) using Eq. (35) when $\mathscr{A}(u) = \mathscr{L}u$.

In Eqs. (43)-(44) there are two points to consider. First, in principle we have that $\mathscr{L}(\bar{u}; \delta u_h) = \mathscr{L}(\bar{u}_h + \bar{u}'; \delta u_h)$, so that the effect of the SGS on the first argument of $\mathscr{L}$ needs to be approximated, and likewise for $B(\bar{u}; \delta u_h, v_h)$. This offers no difficulty, except for the fact that no derivatives of u' are available. If they are not needed, $\mathscr{L}(\bar{u}_h + \bar{u}'; \delta u_h)$ is computable. Since the effect of the SGS in this case is also taken into account in the nonlinear terms, we call this approach *nonlinear SGSs*. However, very often similar results are obtained approximating $\mathscr{L}(\bar{u}_h + \bar{u}'; \delta u_h) \approx \mathscr{L}(\bar{u}_h; \delta u_h)$ and also $\mathscr{L}^*(\bar{u}_h + \bar{u}'; v_h) \approx \mathscr{L}^*(\bar{u}_h; v_h)$ and $B(\bar{u}_h + \bar{u}'; \delta u_h, v_h) \approx B(\bar{u}_h; \delta u_h, v_h)$; for a thorough discussion in the context of the Navier-Stokes equations, see [41]. In nonlinear Elasticity, we have always used this approximation in our previous works.

The second point to consider is more subtile. The SGS u' is obtained from Eq. (44), and in this equation, like for the linear case, τ_K is an approximation to $\mathscr{L}^{-1}$ on each element domain K. Therefore, the SGS depends through τ_K on the iterative scheme chosen, and consequently the approximated solution u_h depends on the linearization of the nonlinear problem. In the particular problems we consider we will give in each case the expression of τ_K and the inherent linearization from which it is computed.

Finally, in the nonlinear problems we shall consider, obtaining the adjoint $\mathscr{L}^*(\bar{u}_h; v_h)$ of $\mathscr{L}(\bar{u}_h; \delta u_h)$ arising from the linearization of $\mathscr{A}(u_h)$ may be quite involved. Should we consider another (simpler) linearization $\mathscr{A}(\bar{u} + \delta u) \approx \mathscr{A}(\bar{u}) + \hat{\mathscr{L}}(\bar{u}; \delta u)$, we would need the adjoint $\hat{\mathscr{L}}^*(\bar{u}_h; v_h)$, and Eq. (43) would become:

$$B(\bar{u}; \delta u_h, v_h) + \sum_K \langle u', \hat{\mathscr{L}}^*(\bar{u}; v_h)\rangle_K = \langle v_h, f - \mathscr{A}(\bar{u}_h)\rangle \quad \text{for all } v_h \in X_h. \tag{45}$$

The inconvenience of this strategy is that we may loose some symmetries of the formulation, but the calculation of $\hat{\mathscr{L}}^*(\bar{u}; v_h)$ is sometimes much easier that that of $\mathscr{L}^*(\bar{u}; v_h)$. In fact, we could use $\hat{\mathscr{L}}$ instead of $\mathscr{L}$ in Eq. (44), recovering symmetries but perhaps spoiling the rate of convergence of the iterative scheme.

3.4. Stabilization of mixed problems

Suppose that we are dealing with a generic mixed problem, in which the unknown is $u = [u_1, u_2] \in X_1 \times X_2 = X$. For simplicity, let us assume that this problem is stationary and linear. Its differential form can be written as

$$\mathscr{L}_1[u_1, u_2] = \mathscr{L}_{11}u_1 + \mathscr{L}_{12}u_2 = f_1,$$
$$\mathscr{L}_2[u_1, u_2] = \mathscr{L}_{21}u_1 + \mathscr{L}_{22}u_2 = f_2,$$

with operators $\mathscr{L}_{ij}$ linear, $i, j = 1, 2$. The extension of what follows to mixed problems of several unknowns is straightforward.

The application of the VMS concepts will lead to the FE problem defined by Eqs. (36)-(35), but now τ_K is a matrix that approximates $\mathscr{L}^{-1}$. The procedure we propose to obtain τ_K is explained in [40] for a problem whose unknowns are a velocity field $\boldsymbol{u}$ and a pressure p, and in [16] for the three field formulation of the Stokes problem.

The first step is to scale the equations properly, so that the sum $v_1 f_1 + v_2 f_2$, with $[v_1, v_2] \in X_1 \times X_2$, is well defined. In our case, this is already accomplished for linear problems using the expression of the differential operators given in Table II. Next, let $f = [f_1, f_2]$ and consider a matrix M such that $f^T M f$ and $u^T M^{-1} u$ is dimensionally consistent. The L^2-norm in a domain weighted by M is denoted by L^2_M.

The main design criterion for τ_K is that for each element domain K, the $L^2_M(K)$-norm of $\mathscr{L}$ be bounded by the $L^2_M(K)$-norm of τ_K^{-1}. The $L^2_M(K)$-norm of $\mathscr{L}$ can be approximated by the $L^2_M(K)$-norm of its Fourier transform and this, in turn, by its spectral radius, $\lambda_{\max}$. The condition indicated yields $\tau_K = \lambda_{\max}^{-1/2} M$. Details of how to put into practice this idea can be found in the cited references (see also [34]).

The question now is how to choose M. It is only a scaling matrix, and in particular it can be taken as diagonal. This leads to a diagonal expression for τ_K, which is what we use in all cases, i.e., $\tau_K = \mathrm{diag}(\tau_{1,K}, \tau_{2,K})$.

The expression of τ_K for the mixed linear problems considered in this work is given in Table III. Some remarks are in order. First, c_1 and c_2 denote algorithmic constants (different at different appearances), possibly dependent on the polynomial order of the FE interpolation, but not on the element size h or on the equation coefficients. In the case of the dual form of the $[\boldsymbol{u}, \boldsymbol{\sigma}]$ formulation, L_0 is a fixed length scale of the problem. Second, any expression with the same *asymptotic* behavior in terms of h and the equation coefficients yields similar numerical results and the same stability and error estimates. Third, all these expressions are computed locally, with $h \equiv h_K$ the diameter of element K and the physical parameters evaluated point-wise if they are variable. Finally, no anisotropy of the FE mesh is considered, so that the definition of h_K is unambiguous (see [36] for a discussion about this point). The justification of the expressions in Table III can be found in

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$[\boldsymbol{u}, p]$	$[\boldsymbol{u}, \boldsymbol{\sigma}]$		$[\boldsymbol{u}, p, \boldsymbol{\sigma}^{\text{dev}}]$
	Primal	Dual	
$\begin{bmatrix} c_1 \frac{h^2}{\mu} & 0 \\ 0 & c_2\mu \end{bmatrix}$	$\begin{bmatrix} c_1 \frac{h^2}{\mu} & 0 \\ 0 & c_2\mu \end{bmatrix}$	$\begin{bmatrix} c_1 \frac{L_0^2}{\mu} & 0 \\ 0 & c_2 \frac{h^2}{L_0^2}\mu \end{bmatrix}$	$\begin{bmatrix} c_1 \frac{h^2}{\mu} & 0 & 0 \\ 0 & c_2\mu & 0 \\ 0 & 0 & c_2\mu \end{bmatrix}$

Table III. Stabilization matrix for the linear mixed problems considered

[35, 13, 14, 16]. We will discuss its implications in the following section devoted to each individual mixed formulation.

Once τ_K has been designed, the formulation for the mixed problem given by Eqs. (36)-(35) is complete. However, it remains to answer a fundamental question, namely, what do we gain using a stabilized formulation? Since we wish to use arbitrary interpolations for the functional spaces X_1 and X_2, no discrete inf-sup condition can be guaranteed. However, even without this, one can obtain stability and convergence in a norm that, in general, depends on τ_K and to which one often refers to as the *stabilized* norm. While in the stabilization of singularly perturbed problems (such as convection diffusion with dominant convection or plates in the limit of zero thickness) this is the best that can be obtained, in the mixed problems we are studying one can prove stability and convergence in *natural* norms, i.e., in the norm $\|u\|_X = \|u_1\|_{X_1} + \|u_2\|_{X_2}$, with an adequate scaling for the two terms. In the following sections, we will give these norms and the convergence estimates that can be proven for the linear version of the problems we are considering. We are not aware of stability and convergence results using stabilization for the nonlinear counterparts of these problems.

4. MIXED DISPLACEMENT-PRESSURE FORMULATION—THE INCOMPRESSIBLE LIMIT

The first mixed formulation we consider is the displacement-pressure one. For compressible materials, its Galerkin FE approximation is stable and convergent using any interpolation for displacement and pressure. The interest of this approach relies on the possibility to deal with quasi-incompressible and fully incompressible materials, i.e., in the case $\kappa \to \infty$, which is the case we consider in what follows.

The following comments apply both to this section and to Sections 5 and 6. For the three geometric approximations considered, we provide the final linearized and discrete problem in space using the concepts introduced above. Since any time discretization can be employed for the acceleration, we simply assume that a finite difference approximation in time is used, with the second order time derivative approximated by a difference operator δ_{tt}^2 and all terms evaluated at the time step associated to the time integration scheme.

In all cases, we assume that the OSGS formulation is used, i.e., the projection $\mathscr{P}'$ in Eq. (35) is the $L^2(\Omega)$-orthogonal to the FE space, denoted $\mathscr{P}_h^\perp$. The interest of this choice in the nonlinear problems will be highlighted. Likewise, quasi-static SGSs are assumed for the sake of simplicity, although the extension to dynamic SGSs is straightforward; in fact, dynamic SGSs would be required if the time step size is small. Note that, since $\mathscr{P}_h^\perp[\delta_{tt}^2 u_h] = 0$, the expression for the SGSs is that given in Eq. (35) even in the transient case.

Finally, recall that we consider continuous interpolations for all variables, and thus SGSs on the element boundaries are not introduced in the formulations presented.

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4.1. Linear Elasticity: $\boldsymbol{u}$-p approach

We already have all the ingredients to write the fully discrete linear elastic problem, for simplicity considering $\boldsymbol{u}_D = \boldsymbol{0}$. The bilinear form of the continuous problem is given in Table II, and the formulation we propose is given by Eqs. (36)-(35) extended to the transient case, with the stabilization matrix given in Table III. Thus, the discrete problem we propose to solve is the following: for each time step, find $[\boldsymbol{u}_h, p_h] \in V_h \times Q_h$ such that

$$
\begin{aligned}
(\boldsymbol{v}_h, \rho\delta^2_{tt}\boldsymbol{u}_h) + (\nabla^{\mathrm{s}}\boldsymbol{v}_h, \boldsymbol{C}^{\mathrm{dev}} : \nabla^{\mathrm{s}}\boldsymbol{u}_h) - (p_h, \nabla\cdot\boldsymbol{v}_h) + (q_h, \nabla\cdot\boldsymbol{u}_h)& \\
+ \sum_K c_1 \frac{h^2}{\mu} \langle \nabla\cdot\boldsymbol{C}^{\mathrm{dev}} : \nabla^{\mathrm{s}}\boldsymbol{v}_h + \nabla q_h, \mathscr{P}_h^{\perp}[-\nabla\cdot\boldsymbol{C}^{\mathrm{dev}} : \nabla^{\mathrm{s}}\boldsymbol{u}_h + \nabla p_h - \boldsymbol{f}]\rangle_K& \\
+ \sum_K c_2 \mu \langle \nabla\cdot\boldsymbol{v}_h, \mathscr{P}_h^{\perp}[\nabla\cdot\boldsymbol{u}_h]\rangle_K& \\
= \langle \boldsymbol{v}_h, \boldsymbol{f}\rangle + \langle \boldsymbol{v}_h, \boldsymbol{t}_N\rangle_{\Gamma_N} \qquad \text{for all } [\boldsymbol{v}_h, q_h] \in V_h \times Q_h.&
\end{aligned}
$$

Note that $\mathscr{P}_h^{\perp}$ may be applied to vectors or to scalars, we do not distinguish between both cases. This orthogonal projection allows one to simplify the formulation, which can be very convenient in the nonlinear case. The observation is that for any function f, smooth enough, the norm of $\mathscr{P}_h^{\perp}[f]$ converges as $h \to 0$ (or as the polynomial order increases) as the interpolation error. Therefore, for accuracy reasons we may omit the orthogonal projection of any term, and keep only those terms that enhance stability. This is the basis of the split-OSGS formulation [42, 43, 44]. In the problem we consider now, the simplification we could consider is:

$$
\begin{aligned}
&\sum_K c_1 \frac{h^2}{\mu} \langle \nabla\cdot\boldsymbol{C}^{\mathrm{dev}} : \nabla^{\mathrm{s}}\boldsymbol{v}_h + \nabla q_h, \mathscr{P}_h^{\perp}[-\nabla\cdot\boldsymbol{C}^{\mathrm{dev}} : \nabla^{\mathrm{s}}\boldsymbol{u}_h + \nabla p_h - \boldsymbol{f}]\rangle_K \\
&\quad\approx \sum_K c_1 \frac{h^2}{\mu} \langle \nabla q_h, \mathscr{P}_h^{\perp}[\nabla p_h]\rangle_K.
\end{aligned}
$$

In any case, in the stationary problem the formulation is stable and optimally convergent in the norm [43]:

$$
|||[\boldsymbol{v}_h, q_h]|||^2 = \mu\|\nabla\boldsymbol{v}_h\|^2 + \frac{1}{\mu}\|q_h\|^2.
$$

In particular, if k_u and k_p are the interpolation orders in V_h and Q_h, respectively, and r_u and r_p the Sobolev regularities of the continuous solution $\boldsymbol{u}$ and p, respectively, one can prove that

$$
|||[\boldsymbol{u} - \boldsymbol{u}_h, p - p_h]|||^2 \lesssim h^{s_u - 1}\mu\|\boldsymbol{u}\|^2_{H^{s_u}(\Omega)} + h^{s_p}\frac{1}{\mu}\|p\|^2_{H^{s_p}(\Omega)},
$$

where $s_u = \min\{k_u + 1, r_u\}$ and $s_p = \min\{k_p + 1, r_p\}$ and $\lesssim$ stands for $\leq$ up to dimensionless constants. In particular, for smooth solutions and equal order interpolation $k_u = k_p = k$, we have that

$$
|||[\boldsymbol{u} - \boldsymbol{u}_h, p - p_h]|||^2 \lesssim h^k\Big(\mu\|\boldsymbol{u}\|^2_{H^{k+1}(\Omega)} + \frac{1}{\mu}\|p\|^2_{H^k(\Omega)}\Big).
$$

4.2. Total Lagrangian formulation: $\boldsymbol{u}$-p approach

Let us consider now the finite strain problem using the TL approach. The differential equations are written in Table I, and once extended to the transient case and complemented with initial and boundary conditions lead in the incompressible limit to the initial and boundary value problem (IBVP) of finding $\boldsymbol{u} : \mathfrak{D}_0 \longrightarrow \mathbb{R}^{\mathrm{d}}$ and $p : \mathfrak{D}_0 \longrightarrow \mathbb{R}$ such that

$$
\rho_0 \partial^2_{tt} u_a - \partial_A(F_{aB}S'_{BA}) + \partial_A(pJF^{-1}_{Aa}) = \rho_0 f_a \quad \text{in } \mathfrak{D}_0, \tag{46}
$$

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$$\begin{aligned}
\frac{\mathrm{d}G}{\mathrm{d}J} &= 0 && \text{in } \mathfrak{D}_0, && (47)\\
u_a = u_a^0,\ \partial_t u_a &= \dot{u}_a^0 && \text{in } \Omega(0),\ t = 0,\\
u_a &= u_{D,a} && \text{in } \Gamma_{0,D},\ t \in (0, t_{\mathrm{fin}}),\\
n_{0,A}(F_{aB}S'_{BA}) - n_{0,A}pJF^{-1}_{Aa} &= t_{N,a} && \text{in } \Gamma_{0,N},\ t \in (0, t_{\mathrm{fin}}).
\end{aligned}$$

The notation employed here has been introduced previously. It is important to stress that tensors $\boldsymbol{F}$ and $\boldsymbol{S}'$ and the Jacobian J are understood to be written in terms of the displacement $\boldsymbol{u}$, which together with p are the unknowns of the problem. This, in particular, needs to be taken into account when linearizing the problem.

Let V and Q be, respectively, the proper functional spaces where displacement and pressure solutions are well-defined for each fixed time $t \in (0, t_{\mathrm{fin}})$. The regularity of these spaces depends on the constitutive law. In general, they will be subspaces of $H^1(\Omega)^d$ and $L^2(\Omega)$, incorporating the Dirichlet boundary conditions. Let $V_0 \subset V$ be the space of functions in V that vanish on $\Gamma_{0,D}$. In this case, $X = V \times Q$, and we shall also need $X_0 = V_0 \times Q$. The variational statement of the problem is derived by testing system (46)-(47) against arbitrary test functions, $\boldsymbol{v} \in V_0$ and $q \in Q$. The weak form of the problem reads: find $u = [\boldsymbol{u}, p] : (0, t_{\mathrm{fin}}) \longrightarrow X$ such that the initial conditions are satisfied and

$$\langle v, \mathscr{D}_t u \rangle + B_{\mathrm{uptl}}(u, v) = L(v) \quad \text{for all } v = [\boldsymbol{v}, q] \in X_0, \tag{48}$$

where $B_{\mathrm{uptl}}(u, v)$ is a semilinear form defined on $X \times X_0$ as

$$B_{\mathrm{uptl}}(u, v) := \langle \partial_A v_a, F_{aB}S'_{BA} \rangle - \left\langle \partial_A v_a, pJF^{-1}_{Aa} \right\rangle + \left\langle q, \frac{\mathrm{d}G}{\mathrm{d}J} \right\rangle. \tag{49}$$

$L(v)$ is a linear form defined on X_0 as

$$L(v) := \langle v_a, \rho_0 b_a \rangle + \langle v_a, t_a \rangle_{\Gamma_{0_N}},$$

and

$$\mathscr{D}_t(u) := [\rho_0 \partial^2_{tt}\boldsymbol{u}, 0].$$

As usual, integration by parts has been used in order to decrease the continuity requirements of unknowns $\boldsymbol{u}$ and p. Let us also remark that the problem can be symmetrized by multiplying Eq. (47) by an adequate function that depends on G. In that case the semilinear form $B_{\mathrm{uptl}}(u, v)$ would derive from seeking the stationary point of a functional.

Problem (48) needs to be discretized in time and linearized. Let $\bar{u}$ be the solution at a certain time step and a given iteration. The next iterate is written as $\bar{u} + \delta u$, and the approximation to the temporal derivative $\mathscr{D}_t$ as $\mathscr{D}_{\delta t}$. In this and the following nonlinear problems to analyze we will use Newton-Raphson's linearization. Thus, the time discrete and linearized version of Eq. (48) consists of finding $\delta u \in X_0$ such that

$$\langle v, \mathscr{D}_{\delta t}(\delta u) \rangle + B_{\mathrm{uptl,lin}}(\bar{u}; \delta u, v) = L(v) - B_{\mathrm{uptl}}(\bar{u}, v) - \langle v, \mathscr{D}_{\delta t}(\bar{u}) \rangle \quad \text{for all } v \in X_0,$$

where the linear form $B_{\mathrm{uptl,lin}}(\bar{u}; \delta u, v)$ is given by

$$\begin{aligned}
B_{\mathrm{uptl,lin}}(\bar{u}; \delta u, v) &= \left\langle \partial_A v_a, \partial_B \delta u_a \bar{S}'_{BA} \right\rangle + \left\langle \partial_A v_a, \bar{F}_{aB}\delta S'_{BA} \right\rangle - \left\langle \partial_A v_a, \bar{J}\bar{p}\bar{F}^{-1}_{Bb}\partial_B \delta u_b \bar{F}^{-1}_{Aa} \right\rangle \\
&+ \left\langle \partial_A v_a, \bar{J}\bar{p}\bar{F}^{-1}_{Ab}\partial_B \delta u_b \bar{F}^{-1}_{Ba} \right\rangle - \left\langle \partial_A v_a, \bar{J}\delta p \bar{F}^{-1}_{Aa} \right\rangle + \left\langle q, \gamma(\bar{J})\bar{F}^{-1}_{Aa}\partial_A \delta u_a \right\rangle,
\end{aligned} \tag{50}$$

where $\gamma(J)$ is a function coming from the linearization of $\frac{\mathrm{d}G}{\mathrm{d}J}$ and depends upon the volumetric strain energy function into consideration, and $\delta S'_{BA}$ is the first term in the decomposition of the second Piola-Kirchhoff stress tensor computed with $\delta\boldsymbol{u}$. Quantities with an overbar are computed using the known iterate $\bar{u}$.

To write the discrete stabilized FE approximation to the problem for the $\boldsymbol{u}$-p formulation using a TL description we need to apply Eqs. (45)-(44) in our case (extended to the transient problem). The FE

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spaces to consider are $V_h \subset V$, $V_{0,h} \subset V_0$, $Q_h \subset Q$, $X_h = V_h \times Q_h$ and $X_{0,h} = V_{0,h} \times Q_h$. Using quasi-static and linear SGS (i.e., neglecting their time derivatives and their nonlinear effects), the resulting problem reads: find $\delta u_h = [\delta \boldsymbol{u}_h, \delta p_h] \in X_{0,h}$ such that

$$\begin{aligned}
&\langle v_h, \mathscr{D}_{\delta t}(\delta u_h)\rangle + B_{\rm uptl,lin}(\bar{u}_h; \delta u_h, v_h) \\
&\quad + \sum_K \tau_K \langle \mathscr{P}_h^{\perp}[f - \mathscr{A}_{\rm uptl}(\bar{u}_h) - \mathscr{L}_{\rm uptl}(\bar{u}_h; \delta u_h)], \hat{\mathscr{L}}^*_{\rm uptl}(\bar{u}_h; v_h)\rangle_K \\
&\quad = L(v_h) - B_{\rm uptl}(\bar{u}_h, v_h) - \langle v_h, \mathscr{D}_{\delta t}(\bar{u}_h)\rangle \quad \text{for all } v_h \in X_{0,h}.
\end{aligned}$$

We need to identify all the terms in this expression. From Eqs. (49) and (50) we can obtain $B_{\rm uptl}(\bar{u}_h, v_h)$ and $B_{\rm uptl,lin}(\bar{u}_h; \delta u_h, v_h)$, respectively, and the rest of terms required are:

$$\begin{aligned}
\mathscr{A}_{\rm uptl}(u) &= \Big[-\partial_A(F_{aB}S'_{BA}) + \partial_A(pJF^{-1}_{Aa}), \frac{{\rm d}G}{{\rm d}J}\Big], \qquad (51)\\
f &= [\rho_0 f_a, 0],
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}_{\rm uptl}(\bar{u}; \delta u) &= [-\partial_A(\partial_B \delta u_a \bar{S}'_{BA}) - \partial_A(\bar{F}_{aB}\delta S'_{BA}) + \partial_A(\bar{J}\bar{p}\bar{F}^{-1}_{Bb}\partial_B \delta u_b \bar{F}^{-1}_{Aa}) \\
&\quad + \partial_A(\bar{J}\delta p \bar{F}^{-1}_{Aa}) - \partial_A(\bar{J}\bar{p}\bar{F}^{-1}_{Ab}\partial_B \delta u_b \bar{F}^{-1}_{Ba}), \gamma(\bar{J})\bar{F}^{-1}_{Aa}\partial_A \delta u_a], \qquad (52)\\
\hat{\mathscr{L}}^*_{\rm uptl}(\bar{u}; v) &= [-\partial_A(\partial_B v_a \bar{S}'_{AB}) + \partial_A(\bar{J}\bar{p}\bar{F}^{-1}_{Aa}\partial_B v_b \bar{F}^{-1}_{Bb}) \\
&\quad - \partial_A(\gamma(\bar{J})q\bar{F}^{-1}_{Aa}) - \partial_A(\bar{J}\bar{p}\bar{F}^{-1}_{Ba}\partial_B v_b \bar{F}^{-1}_{Ab}), -\bar{J}\bar{F}^{-1}_{Aa}\partial_A v_a]. \qquad (53)
\end{aligned}$$

It is understood in these expressions that the free index a in the first component of these vector operators runs from 1 to $\rm d$. Observe that, at convergence, we expect to obtain $J = 1$, but nevertheless J has been consistently linearized. Note also that $\hat{\mathscr{L}}^*_{\rm uptl}(\bar{u}; v)$ is not exactly the adjoint of $\mathscr{L}_{\rm uptl}(\bar{u}; u)$, but the adjoint of the linearization obtained using a fixed point scheme for tensor $\boldsymbol{S}'$, not for $\boldsymbol{u}$.

To complete the definition of the method, we need an expression for the matrix of stabilization parameters, τ_K. As explained before, its norm has to approximate the norm of operator $\mathscr{L}^{-1}_{\rm uptl}$ within each element, and $\mathscr{L}_{\rm uptl}$ involves a linearization of $\boldsymbol{S}'$ that we have not explicitly written because it depends on the constitutive model employed. In general, if $\bar{\mu}$ is a characteristic value of the tangent deviatoric part of the constitutive tensor in the linearized constitutive law, we may use the same expression for τ_K as for the linear case (given in Table III) using $\bar{\mu}$ instead of μ. For the most common hyperelastic constitutive laws, it is not difficult to identify $\bar{\mu}$.

The formulation presented here was proposed in [12]. A stabilized $\boldsymbol{u}$-p formulation also based on the VMS framework was proposed in [31], in this case modeling the SGSs through bubble functions and putting special emphasis on the inherent approximation to tensor $\boldsymbol{F}$. An ASGS approach was proposed in [45] for linear elements, and extended to higher order interpolation in [46] using a recovery technique for the derivatives. For a stabilization based on a pressure projection, not on the VMS framework, see for example [47].

This is the first nonlinear mixed problem for which the stabilization procedure is described. The same notation introduced here will be used for the rest of nonlinear problems to be considered in the following sections. In all cases, the target is to provide the statement of the IVBP (the analogous to Eqs. (46)-(47) plus boundary and initial conditions), the semilinear form of the problem (see Eq. (49)), its linearized version (see Eq. (50)) and the counterpart of Eqs (51)-(53) that are needed to write the stabilized formulation proposed.

4.3. *Updated Lagrangian formulation: $\boldsymbol{u}$-p approach*

Let us consider now the UL description. The IVBP to be solved considering a Neo-Hookean material in the incompressible limit ($\lambda \to \infty$) is the following: find $\boldsymbol{u} : \mathfrak{D} \longrightarrow \mathbb{R}^{\rm d}$ and $p : \mathfrak{D} \longrightarrow \mathbb{R}$ such

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that

$$\rho\partial_{tt}^2 u_a - \partial_b\left[J^{-1}\mu\left(b_{ab} - \frac{b_{cc}}{\mathrm{d}}\delta_{ab}\right)\right] + \partial_a p = \rho f_a \quad \text{in } \mathfrak{D}, \tag{54}$$

$$\log(J) = 0 \quad \text{in } \mathfrak{D}, \tag{55}$$

$$u_a = u_a^0,\ \partial_t u_a = \dot{u}_a^0 \quad \text{in } \Omega(0),\ t = 0$$

$$u_a = u_{D,a} \quad \text{in } \Gamma_D,\ t \in (0, t_{\text{fin}}),$$

$$n_b\sigma_{ab}^{\text{dev}} - n_a p = t_{N,a} \quad \text{in } \Gamma_N,\ t \in (0, t_{\text{fin}}),$$

where σ_{ab}^{dev} is computed in terms of the displacement.

Let V and Q be, respectively, the proper functional spaces where displacement and pressure solutions are well-defined for each fixed time $t \in (0, t_{\text{fin}})$. The same comments as for the TL formulation apply in this case, and $V_0 \subset V$ is constructed likewise. The variational statement of the problem is derived by testing system (54)-(55) against arbitrary test functions, $\boldsymbol{v} \in V_0$ and $q \in Q$. The weak form of the problem reads: find $u = [\boldsymbol{u}, p] : (0, t_{\text{fin}}) \longrightarrow X$ such that the initial conditions are satisfied and

$$\langle v, \mathscr{D}_t u\rangle + B_{\text{upul}}(u, v) = L(v) \quad \text{for all } v = [\boldsymbol{v}, q] \in X_0, \tag{56}$$

where $B_{\text{upul}}(u, v)$ is a semilinear form defined on $X \times X_0$ as

$$B_{\text{upul}}(u, v) := \left\langle \partial_b v_a, J^{-1}\mu\Big(b_{ab} - \frac{b_{cc}}{\mathrm{d}}\delta_{ab}\Big)\right\rangle - \langle\partial_a v_a, p\rangle + \langle q, \log(J)\rangle. \tag{57}$$

$L(v)$ and $\mathscr{D}_t(u)$ are defined similarly as for the TL case.

After time discretization and linearization using Newton-Raphson's scheme, the linear variational problem to solve is to find $\delta u \in X_0$ such that

$$\langle v, \mathscr{D}_{\delta t}(\delta u)\rangle + B_{\text{upul,lin}}(\bar{u}; \delta u, v) = L(v) - B_{\text{upul}}(\bar{u}, v) - \langle v, \mathscr{D}_{\delta t}(\bar{u})\rangle \quad \text{for all } v \in X_0,$$

where the linear form $B_{\text{upul,lin}}(\bar{u}; \delta u, v)$ is given by

$$\begin{aligned} B_{\text{upul,lin}}(\bar{u}; \delta u, v) &= \left\langle \partial_b v_a, \frac{\mu}{\bar{J}}(\partial_A \delta u_a \bar{F}_{bA} + \bar{F}_{aA}\partial_A \delta u_b)\right\rangle - \left\langle \partial_a v_a, 2\frac{\mu}{\bar{J}\mathrm{d}}\bar{F}_{cA}\partial_A\delta u_c\right\rangle \\ &\quad - \left\langle \partial_b v_a, \frac{\mu}{\bar{J}}(\bar{F}_{Ac}^{-1}\partial_A\delta u_c)\Big(\bar{b}_{ab} - \frac{\bar{b}_{cc}}{\mathrm{d}}\delta_{ab}\Big)\right\rangle \\ &\quad - \langle\partial_a v_a, \delta p\rangle + \langle q, \bar{F}_{Aa}^{-1}\partial_A\delta u_a\rangle. \end{aligned} \tag{58}$$

In this linearization we have used that (see [18]):

$$\begin{aligned} J^{-1} &= \bar{J}^{-1} - \bar{J}^{-1}\bar{F}_{Aa}^{-1}\partial_A\delta u_a + \mathscr{O}(\|\delta\boldsymbol{u}\|^2), \\ \log(J) &= \log(\bar{J}) + \bar{F}_{Aa}^{-1}\partial_A\delta u_a + \mathscr{O}(\|\delta\boldsymbol{u}\|^2). \end{aligned}$$

Let us also note that, since $\rho = \rho_0 J$, the density in the inertia term and in the external forces needs to be also linearized. However, this is not a special feature of mixed formulations, but it is also found in the irreducible one. Here and in the following section when dealing with the UL formulation it will be implicitly assumed that this linearization is done.

Let us apply now Eqs. (45)-(44) to the problem at hand. The resulting problem is: find $\delta u_h = [\delta\boldsymbol{u}_h, \delta p_h] \in X_{0,h}$ such that

$$\begin{aligned} &\langle v_h, \mathscr{D}_{\delta t}(\delta u_h)\rangle + B_{\text{upul,lin}}(\bar{u}_h; \delta u_h, v_h) \\ &\quad + \sum_K \tau_K \langle \mathscr{P}_h^{\perp}[f - \mathscr{A}_{\text{upul}}(\bar{u}_h) - \mathscr{L}_{\text{upul}}(\bar{u}_h; \delta u_h)], \hat{\mathscr{L}}_{\text{upul}}^{*}(\bar{u}_h; v_h)\rangle_K \\ &\quad = L(v_h) - B_{\text{upul}}(\bar{u}_h, v_h) - \langle v_h, \mathscr{D}_{\delta t}(\bar{u}_h)\rangle \quad \text{for all } v_h \in X_{0,h}. \end{aligned}$$

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From Eqs. (57) and (58) we can obtain $B_{\text{upul}}(\bar{u}_h, v_h)$ and $B_{\text{upul,lin}}(\bar{u}_h; \delta u_h, v_h)$, respectively, and the rest of terms in the expression above are:

$$
\begin{aligned}
\mathscr{A}_{\text{upul}}(u) &= \Big[-\partial_b \Big[J^{-1}\mu\Big(b_{ab} - \frac{b_{cc}}{\mathrm{d}}\delta_{ab}\Big)\Big] + \partial_a p, \log(J)\Big], \\
f &= [\rho f_a, 0], \\
\mathscr{L}_{\text{upul}}(\bar{u}; \delta u) &= \Big[-\partial_b\Big(\frac{\mu}{\bar{J}}(\partial_A \delta u_a \bar{F}_{bA} + \bar{F}_{aA}\partial_A \delta u_b)\Big) + 2\partial_a\Big(\frac{\mu}{\bar{J}\mathrm{d}}\bar{F}_{cA}\partial_A \delta u_c\Big) \\
&\quad + \partial_b\Big[\frac{\mu}{\bar{J}}\bar{F}^{-1}_{Ac}\partial_A \delta u_c\Big(\bar{b}_{ab} - \frac{\bar{b}_{cc}}{\mathrm{d}}\delta_{ab}\Big)\Big] \\
&\quad + \partial_a \delta p\ ,\ \bar{F}^{-1}_{Aa}\partial_A \delta u_a)\Big], \\
\mathscr{L}^*_{\text{upul}}(\bar{u}; v) &= \Big[-\partial_A\Big(\frac{\mu}{\bar{J}}(\partial_b v_a \bar{F}_{bA} + \bar{F}_{bA}\partial_a v_b)\Big) + 2\partial_A\Big(\frac{\mu}{\bar{J}\mathrm{d}}\bar{F}_{aA}\partial_c v_c\Big) \\
&\quad + \partial_A\Big[\frac{\mu}{\bar{J}}\bar{F}^{-1}_{Aa}\partial_b v_c\Big(\bar{b}_{cb} - \frac{\bar{b}_{dd}}{\mathrm{d}}\delta_{cb}\Big)\Big] \\
&\quad + \partial_A(\bar{F}^{-1}_{Aa} q)\ ,\ -\partial_a v_a)\Big].
\end{aligned}
$$

Similar remarks as for the TL formulation apply. Regarding the notation, recall that it is understood that the free index a in the first component of these vector operators runs from 1 to d. Also in this case, at convergence we expect to obtain $J = 1$, but J has been consistently linearized. Finally, in this case there is no difficulty in computing the formal adjoint of the linearized differential operator, so that we can use $\hat{\mathscr{L}}^*_{\text{upul}}(\bar{u}; v) = \mathscr{L}^*_{\text{upul}}(\bar{u}; v)$.

To complete the definition of the formulation, we need an expression for the matrix of stabilization parameters, τ_K. In this case, and with the Newton-Raphson linearization employed, we may again use the same expression for τ_K as for the linear case (given in Table III), μ being the first Lamé's parameter of the Neo-Hookean model.

Details of how to extend this formulation to consider dynamic SGSs are presented in [11]. A similar VMS-based formulation, using quasi-static SGSs and the ASGS approach, was proposed in [30].

5. MIXED DISPLACEMENT-STRESS FORMULATION

The second mixed formulation to consider is the displacement-stress one. The Galerkin FE approximation requires compatibility conditions between the interpolating spaces that can be circumvented using stabilization, even in the case of compressible materials. In fact, the incompressible case is the motivation for using the displacement-pressure-stress formulation analyzed in the following section. Thus, compressible materials are considered in this one.

A very particular aspect of the displacement-stress formulation in the linear case is that it can be analyzed in two different functional frameworks, namely, the primal and the dual form, with different approximation properties for the displacement and the stresses. While in principle the formal strategy used to derive them could be extended to the nonlinear cases, we are not aware of any analysis of the dual formulation in the finite strain case, and we will not study its FE approximation.

5.1. Linear Elasticity—Primal and dual formulations for the $\boldsymbol{u}$-$\boldsymbol{\sigma}$ approach

The bilinear forms of the primal and dual forms of the linear Elasticity problem are given in Table II. Again, the formulation we propose is given in Eqs. (36)-(35) extended to the transient case, with the stabilization matrix given in Table III. The FE spaces we consider are constructed with continuous FE functions, but for the primal formulation they have to be viewed as $V_h \times T_h \subset H_D^1(\Omega)^{\rm d} \times L^2(\Omega)^{\rm d\times d}$, whereas for the dual formulation $V_h \times T_h \subset L^2(\Omega)^{\rm d} \times H_N(\mathrm{div}, \Omega)$; here and in the rest of the subsection, we consider $\boldsymbol{u}_D = \boldsymbol{0}$ and $\boldsymbol{t}_N = \boldsymbol{0}$ to simplify the exposition. The discrete problem we propose to solve is then the following: for each time step, find $[\boldsymbol{u}_h, p_h] \in V_h \times Q_h$ such that

$$
\begin{aligned}
(\boldsymbol{v}_h, \rho\delta_{tt}^2\boldsymbol{u}_h) &+ (\nabla^{\rm s}\boldsymbol{v}_h, \boldsymbol{\sigma}_h) + (\boldsymbol{C}^{-1} : \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) - (\nabla^{\rm s}\boldsymbol{u}_h, \boldsymbol{\tau}_h) \\
&+ \sum_K c_1 \frac{h^2}{\mu} \langle \nabla \cdot \boldsymbol{\tau}_h, \mathscr{P}_h^\perp[\nabla \cdot \boldsymbol{\sigma}_h + \boldsymbol{f}] \rangle_K \\
&+ \sum_K c_2 \mu \langle \nabla^{\rm s}\boldsymbol{v}_h, \mathscr{P}_h^\perp[\nabla^{\rm s}\boldsymbol{u}_h] \rangle_K \\
&= \langle \boldsymbol{v}_h, \boldsymbol{f} \rangle + \langle \boldsymbol{v}_h, \boldsymbol{t}_N \rangle_{\Gamma_N} \qquad \text{for all } [\boldsymbol{v}_h, \boldsymbol{\tau}_h] \in V_h \times T_h,
\end{aligned}
$$

for the primal formulation and

$$
\begin{aligned}
(\boldsymbol{v}_h, \rho\delta_{tt}^2\boldsymbol{u}_h) &- (\boldsymbol{v}_h, \nabla \cdot \boldsymbol{\sigma}_h) + (\boldsymbol{C}^{-1} : \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (\boldsymbol{u}_h, \nabla \cdot \boldsymbol{\tau}_h) \\
&+ \sum_K c_1 \frac{L_0^2}{\mu} \langle \nabla \cdot \boldsymbol{\tau}_h, \mathscr{P}_h^\perp[\nabla \cdot \boldsymbol{\sigma}_h + \boldsymbol{f}] \rangle_K \\
&+ \sum_K c_2 \frac{h^2}{L_0^2} \mu \langle \nabla^{\rm s}\boldsymbol{v}_h, \mathscr{P}_h^\perp[\nabla^{\rm s}\boldsymbol{u}_h] \rangle_K \\
&= \langle \boldsymbol{v}_h, \boldsymbol{f} \rangle + \langle \boldsymbol{n} \cdot \boldsymbol{\tau}, \boldsymbol{u}_D \rangle_{\Gamma_D} \qquad \text{for all } [\boldsymbol{v}_h, \boldsymbol{\tau}_h] \in V_h \times T_h,
\end{aligned}
$$

for the dual one. In both cases we have used that $\mathscr{P}_h^\perp[\boldsymbol{C}^{-1} : \boldsymbol{\sigma}_h] = \boldsymbol{0}$. Note that for continuous interpolations both expressions are identical except for the stabilization parameters. Therefore, these stabilization parameters allow one to switch from one formulation to the other.

In the stationary problem, the norms in which stability and convergence can be proved are $|\!|\!|\cdot|\!|\!|_{\rm P}$ and $|\!|\!|\cdot|\!|\!|_{\rm D}$ for the primal and the dual formulations, respectively, given by:

$$
\begin{aligned}
|\!|\!|[\boldsymbol{v}_h, \boldsymbol{\tau}_h]|\!|\!|_{\rm P}^2 &= \frac{\mu}{L_0^2}\|\boldsymbol{v}_h\|^2 + \mu\|\nabla^{\rm s}\boldsymbol{v}_h\|^2 + \frac{1}{\mu}\|\boldsymbol{\tau}_h\|^2 + \frac{h^2}{\mu}\|\nabla \cdot \boldsymbol{\tau}_h\|^2, \\
|\!|\!|[\boldsymbol{v}_h, \boldsymbol{\tau}_h]|\!|\!|_{\rm D}^2 &= \frac{\mu}{L_0^2}\|\boldsymbol{v}_h\|^2 + \mu\frac{h^2}{L_0^2}\|\nabla^{\rm s}\boldsymbol{v}_h\|^2 + \frac{1}{\mu}\|\boldsymbol{\tau}_h\|^2 + \frac{L_0^2}{\mu}\|\nabla \cdot \boldsymbol{\tau}_h\|^2.
\end{aligned}
$$

If k_u and k_σ are the interpolation orders in V_h and T_h, respectively, and r_u and r_σ the Sobolev regularities of the continuous solution $\boldsymbol{u}$ and $\boldsymbol{\sigma}$, respectively, one can prove that

$$
\begin{aligned}
|\!|\!|[\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h]|\!|\!|_{\rm P}^2 &\lesssim h^{s_u - 1}\mu\|\boldsymbol{u}\|_{H^{s_u}(\Omega)}^2 + h^{s_\sigma}\frac{1}{\mu}\|\boldsymbol{\sigma}\|_{H^{s_\sigma}(\Omega)}^2 \quad \text{for the primal formulation,} \\
|\!|\!|[\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h]|\!|\!|_{\rm D}^2 &\lesssim h^{s_u}\mu\|\boldsymbol{u}\|_{H^{s_u}(\Omega)}^2 + h^{s_\sigma - 1}\frac{1}{\mu}\|\boldsymbol{\sigma}\|_{H^{s_\sigma}(\Omega)}^2 \quad \text{for the dual formulation,}
\end{aligned}
$$

where $s_u = \min\{k_u + 1, r_u\}$ and $s_\sigma = \min\{k_\sigma + 1, r_\sigma\}$. In particular, for smooth solutions and equal order interpolation $k_u = k_\sigma = k$, we have that

$$
\begin{aligned}
|\!|\!|[\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h]|\!|\!|_{\rm P}^2 &\lesssim h^k\Big(\mu\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}^2 + \frac{1}{\mu}\|\boldsymbol{\sigma}\|_{H^k(\Omega)}^2\Big) \quad \text{for the primal formulation,} \\
|\!|\!|[\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h]|\!|\!|_{\rm D}^2 &\lesssim h^k\Big(\mu\|\boldsymbol{u}\|_{H^k(\Omega)}^2 + \frac{1}{\mu}\|\boldsymbol{\sigma}\|_{H^{k+1}(\Omega)}^2\Big) \quad \text{for the primal formulation.}
\end{aligned}
$$

However, apart from the classical primal and dual formulations, the stabilization parameters can be chosen such that for equal order interpolation $k_u = k_\sigma = k$ one can prove particularly useful

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estimates. Indeed, if one takes

$$\tau_K = \begin{bmatrix} c_1 \frac{L_0 h}{\mu} & 0 \\ 0 & c_2 \frac{h}{L_0} \end{bmatrix},$$

it can be shown that (see [13]):

$$\|\boldsymbol{u} - \boldsymbol{u}_h\| \lesssim L_0^{1/2} h^{k+1/2} \|\boldsymbol{u}\|_{H^{k+1}(\Omega)}, \quad \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \lesssim L_0^{1/2} h^{k+1/2} \|\boldsymbol{\sigma}\|_{H^{k+1}(\Omega)},$$

and, if the conditions of some duality arguments hold,

$$\|\boldsymbol{u} - \boldsymbol{u}_h\| \lesssim h^{k+1} \|\boldsymbol{u}\|_{H^{k+1}(\Omega)}, \quad \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \lesssim h^{k+1} \|\boldsymbol{\sigma}\|_{H^{k+1}(\Omega)}.$$

This improved convergence in stresses with respect to the primal formulation, in which only $\mathscr{O}(h^k)$ can be expected, has been used in nonlinear constitutive models in which the constitutive law depends on the stresses [48, 49]. However, we are not aware of the application of this or similar strategies to geometrically nonlinear solids. Thus, in the following we restrict ourselves to the extension of the primal formulation to finite strain hyperelasticity.

5.2. *Total Lagrangian formulation:* $\boldsymbol{u}$-$\boldsymbol{S}$ *approach*

Let us consider now the finite strain displacement-stress formulation using the TL approach. The spatial differential operator is given in Table I. The IVBP to be solved consists of finding $\boldsymbol{u} : \mathfrak{D}_0 \longrightarrow \mathbb{R}^{\mathrm{d}}$ and $\boldsymbol{S} : \mathfrak{D}_0 \longrightarrow \mathbb{R}^{\mathrm{d}} \otimes \mathbb{R}^{\mathrm{d}}$ such that

$$\begin{aligned} \rho_0 \partial^2_{tt} u_a - \partial_A (F_{aB} S_{BA}) &= \rho_0 f_a && \text{in } \mathfrak{D}_0, && (59) \\ S_{AB} - 2\frac{\partial \Psi}{\partial C_{AB}} &= 0 && \text{in } \mathfrak{D}_0, && (60) \\ u_a = u_a^0,\ \partial_t u_a &= \dot{u}_a^0 && \text{in } \Omega(0),\ t = 0 \\ u_a &= u_{D,a} && \text{in } \Gamma_{0,D},\ t \in (0, t_{\text{fin}}), \\ n_{0,A}(F_{aB} S_{BA}) &= t_{N,a} && \text{in } \Gamma_{0,N},\ t \in (0, t_{\text{fin}}). \end{aligned}$$

It is understood that tensors $\boldsymbol{C}$ and $\boldsymbol{F}$ are expressed in terms of $\boldsymbol{u}$, so that the unknowns of the problem are $\boldsymbol{u}$ and $\boldsymbol{S}$.

Let V and T be, respectively, the proper functional spaces where displacement and stress solutions are well-defined for each fixed time $t \in (0, t_{\text{fin}})$. As for the displacement-pressure approach, the regularity of these spaces depends on the constitutive law; they will be subspaces of $H^1(\Omega)^{\mathrm{d}}$ and $L^2(\Omega)^{\mathrm{d}\times\mathrm{d}}$, incorporating the Dirichlet boundary conditions. Let $V_0 \subset V$ be the space of functions in V that vanish on $\Gamma_{0,D}$. Now $X = V \times T$, and we shall also need $X_0 = V_0 \times T$. The variational statement of the problem is a result of testing system (59)-(60) against arbitrary test functions, $\boldsymbol{v} \in V_0$ and $\boldsymbol{T} \in T$. The weak form of the problem reads: find $u = [\boldsymbol{u}, \boldsymbol{S}] : (0, t_{\text{fin}}) \longrightarrow X$ such that the initial conditions are satisfied and

$$\langle v, \mathscr{D}_t u \rangle + B_{\text{ustl}}(u, v) = L(v) \quad \text{for all } v = [\boldsymbol{v}, q] \in X_0, \tag{61}$$

where $B_{\text{ustl}}(u, v)$ is a semilinear form defined on $X \times X_0$ as

$$B_{\text{ustl}}(u, v) := \langle \partial_A v_a, F_{aB} S_{BA} \rangle + \langle T_{AB}, S_{AB} \rangle - \left\langle T_{AB}, 2\frac{\partial \Psi}{\partial C_{AB}} \right\rangle. \tag{62}$$

Note that the problem cannot be derived from the stationary conditions of a functional because of the contribution to $B_{\text{ustl}}(u, v)$ of Eq. (60). This could be fixed by multiplying this equation by the inverse of the tangent constitutive tensor arising from the constitutive law this equation represents. If fact, in the way we have written the problem, $\boldsymbol{T}$ is not even a stress, but a strain, and it could

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belong to a tensor space different from that of $\boldsymbol{S}$. Nevertheless, this discussion is outside the scope of this paper, and we will consider that both $\boldsymbol{S}$ and $\boldsymbol{T}$ belong to T. $L(v)$ and $\mathscr{D}_t(u)$ are defined as for the displacement-pressure formulation.

Using the same notation as for previous formulations, the linearized and time discrete version of problem (61) consists of finding $\delta u \in X_0$ such that

$$\langle v, \mathscr{D}_{\delta t}(\delta u)\rangle + B_{\text{ustl,lin}}(\bar{u}; \delta u, v) = L(v) - B_{\text{ustl}}(\bar{u}, v) - \langle v, \mathscr{D}_{\delta t}(\bar{u})\rangle \quad \text{for all } v \in X_0,$$

where the linear form $B_{\text{ustl,lin}}(\bar{u}; \delta u, v)$ is given by

$$\begin{aligned} B_{\text{ustl,lin}}(\bar{u}; \delta u, v) = &\big\langle \partial_A v_a, \partial_B \delta u_a \bar{S}_{BA}\big\rangle + \big\langle \partial_A v_a, \bar{F}_{aB} \delta S_{BA}\big\rangle \\ &+ \langle T_{AB}, \delta S_{AB}\rangle - \langle T_{AB}, C^{\tan}_{ABCD} F_{aC} \partial_D \delta u_a\rangle. \end{aligned} \tag{63}$$

In this expression, $C^{\tan}_{ABCD}$ is the $ABCD$ component of the tangent constitutive tensor $\boldsymbol{C}^{\tan}$ associated to the potential Ψ and $F_{aC}\partial_D \delta u_a$ arises from the linearization of the right Cauchy-Green tensor. Note that $B_{\text{ustl,lin}}(\bar{u}; \delta u, v)$ is not symmetric, but could be easily symmetrized if, as said above, Eq. (60) is multiplied by the inverse of $\boldsymbol{C}^{\tan}$.

The discrete stabilized FE approximation for the $\boldsymbol{u}$-$\boldsymbol{S}$ formulation using a TL description is obtained from Eqs. (45)-(44) (extended to the transient problem). Using quasi-static and linear SGS (i.e., neglecting their time derivatives and their nonlinear effects), the resulting problem reads: find $\delta u_h = [\delta \boldsymbol{u}_h, \delta p_h] \in X_{0,h}$ such that

$$\begin{aligned} &\langle v_h, \mathscr{D}_{\delta t}(\delta u_h)\rangle + B_{\text{ustl,lin}}(\bar{u}_h; \delta u_h, v_h) \\ &\quad + \sum_K \tau_K \langle \mathscr{P}_h^{\perp}[f - \mathscr{A}_{\text{ustl}}(\bar{u}_h) - \mathscr{L}_{\text{ustl}}(\bar{u}_h; \delta u_h)], \hat{\mathscr{L}}^*_{\text{ustl}}(\bar{u}_h; v_h)\rangle_K \\ &\quad = L(v_h) - B_{\text{ustl}}(\bar{u}_h, v_h) - \langle v_h, \mathscr{D}_{\delta t}(\bar{u}_h)\rangle \quad \text{for all } v_h \in X_{0,h}. \end{aligned}$$

As for previous problems, the key is to identify all the terms in this expression. From Eqs. (62) and (63) we can obtain $B_{\text{ustl}}(\bar{u}_h, v_h)$ and $B_{\text{ustl,lin}}(\bar{u}_h; \delta u_h, v_h)$, respectively, and the rest of terms required are:

$$\mathscr{A}_{\text{ustl}}(u) = \Big[-\partial_A(F_{aB}S_{BA}), S_{AB} - 2\frac{\partial \Psi}{\partial C_{AB}}\Big], \tag{64}$$

$$f = [\rho_0 f_a, 0],$$

$$\mathscr{L}_{\text{ustl}}(\bar{u}; \delta u) = [-\partial_A(\partial_B \delta u_a \bar{S}_{BA}) - \partial_A(\bar{F}_{aB}\delta S_{BA}), \delta S_{AB} - C^{\tan}_{ABCD}\bar{F}_{aC}\partial_D \delta u_a], \tag{65}$$

$$\hat{\mathscr{L}}^*_{\text{ustl}}(\bar{u}; v) = \mathscr{L}^*_{\text{ustl}}(\bar{u}; v) = [-\partial_A(\partial_B v_a \bar{S}_{AB}) + \partial_D(T_{AB}C^{\tan}_{ABCD}\bar{F}_{aC}), T_{AB} + \bar{F}_{aB}\partial_A v_a]. \tag{66}$$

Note that now $\hat{\mathscr{L}}^*_{\text{ustl}}(\bar{u}; v)$ is the adjoint of $\mathscr{L}_{\text{ustl}}(\bar{u}; u)$.

The matrix of stabilization parameters can be computed as for the displacement-pressure formulation. If $\bar{\mu}$ is a characteristic value of the tangent deviatoric part of the constitutive tensor in the linearized constitutive law, we may use the same expression for τ_K as for the linear case (given in Table III) using $\bar{\mu}$ instead of μ.

5.3. Updated Lagrangian formulation: $\boldsymbol{u}$-$\boldsymbol{\sigma}$ approach

Let us consider the FE approximation of compressible Neo-Hookean hyperelastic materials using displacements and stresses as unknowns and the UL description of the geometry. The derivation of this approximation follows the same lines as for the rest of the models collected in this paper, but, to our knowledge, this particular one has not been proposed before.

The IVBP to be solved is the following: find $\boldsymbol{u} : \mathfrak{D} \longrightarrow \mathbb{R}^{\mathrm{d}}$ and $\boldsymbol{\sigma} : \mathfrak{D} \longrightarrow \mathbb{R}^{\mathrm{d}} \otimes \mathbb{R}^{\mathrm{d}}$ such that

$$\rho \partial^2_{tt} u_a - \partial_b \sigma_{ab} = \rho f_a \quad \text{in } \mathfrak{D}, \tag{67}$$

$$\begin{aligned} \sigma_{ab} - J^{-1}[(\lambda \log(J) - \mu)\delta_{ab} + \mu b_{ab}] = 0 \qquad & \text{in } \mathfrak{D}, \\ u_a = u_a^0,\ \partial_t u_a = \dot{u}_a^0 \qquad & \text{in } \Omega(0),\ t = 0 \\ u_a = u_{D,a} \qquad & \text{in } \Gamma_D,\ t \in (0, t_{\text{fin}}), \\ n_b \sigma_{ab} = t_{N,a} \qquad & \text{in } \Gamma_N,\ t \in (0, t_{\text{fin}}), \end{aligned} \tag{68}$$

the unknowns being u_a and σ_{ab}.

Let V and T be, respectively, the proper functional spaces where displacement and stress solutions are well-defined for each fixed time $t \in (0, t_{\text{fin}})$, and let $V_0 \subset V$ the subspace of V with vector fields vanishing on Γ_D. Testing Eqs. (67)-(68) against $[\boldsymbol{v}, \boldsymbol{\tau}] \in V_0 \times T$ we obtain the weak form of the problem: find $u = [\boldsymbol{u}, \boldsymbol{\sigma}] : (0, t_{\text{fin}}) \longrightarrow X$ such that the initial conditions are satisfied and

$$\langle v, \mathscr{D}_t u \rangle + B_{\text{usul}}(u, v) = L(v) \quad \text{for all } v = [\boldsymbol{v}, q, \boldsymbol{\tau}] \in X_0, \tag{69}$$

where $B_{\text{usul}}(u, v)$ is a semilinear form defined on $X \times X_0$ as

$$\begin{aligned} B_{\text{usul}}(u, v) := & \langle \partial_b v_a, \sigma_{ab} \rangle \\ & + \langle \sigma_{ab}, \tau_{ab} \rangle - \langle J^{-1}[(\lambda \log(J) - \mu)\delta_{ab} + \mu b_{ab}], \tau_{ab} \rangle. \end{aligned} \tag{70}$$

$L(v)$ and $\mathscr{D}_t(u)$ are the same as for the displacement-pressure UL formulation. As for the TL approach, the problem cannot derive from the stationary conditions of a functional and the test function $\boldsymbol{\tau}$ is in fact not a stress, but a strain; the possible fixing is also the same.

After time discretization and linearization using Newton-Raphson's scheme, the linear variational problem to solve consists of finding $\delta u \in X_0$ such that

$$\langle v, \mathscr{D}_{\delta t}(\delta u) \rangle + B_{\text{usul,lin}}(\bar{u}; \delta u, v) = L(v) - B_{\text{usul}}(\bar{u}, v) - \langle v, \mathscr{D}_{\delta t}(\bar{u}) \rangle \quad \text{for all } v \in X_0,$$

where the linear form $B_{\text{usul,lin}}(\bar{u}; \delta u, v)$ is given by

$$\begin{aligned} B_{\text{usul,lin}}(\bar{u}; \delta u, v) = & \langle \partial_b v_a, \delta\sigma_{ab} \rangle + \langle \delta\sigma_{ab}, \tau_{ab} \rangle \\ & + \langle \bar{J}^{-1}[\lambda \log(\bar{J}) - \mu - \lambda] \bar{F}_{Aa}^{-1} \partial_A \delta u_a, \tau_{cc} \rangle \\ & - \langle \bar{J}^{-1} \mu (\partial_A \delta u_a \bar{F}_{bA} + \bar{F}_{aA} \partial_A \delta u_b, \tau_{ab} \rangle. \end{aligned} \tag{71}$$

Making use of Eqs. (45)-(44) in the present case, the resulting problem is: find $\delta u_h = [\delta \boldsymbol{u}_h, \delta \boldsymbol{\sigma}_h] \in X_{0,h}$ such that

$$\begin{aligned} & \langle v_h, \mathscr{D}_{\delta t}(\delta u_h) \rangle + B_{\text{usul,lin}}(\bar{u}_h; \delta u_h, v_h) \\ & + \sum_K \tau_K \langle \mathscr{P}_h^{\perp}[f - \mathscr{A}_{\text{usul}}(\bar{u}_h) - \mathscr{L}_{\text{usul}}(\bar{u}_h; \delta u_h)], \hat{\mathscr{L}}_{\text{usul}}^*(\bar{u}_h; v_h) \rangle_K \\ & = L(v_h) - B_{\text{usul}}(\bar{u}_h, v_h) - \langle v_h, \mathscr{D}_{\delta t}(\bar{u}_h) \rangle \quad \text{for all } v_h \in X_{0,h}. \end{aligned}$$

with $B_{\text{usul}}(\bar{u}_h, v_h)$ and $B_{\text{usul,lin}}(\bar{u}_h; \delta u_h, v_h)$ obtained from Eqs. (70) and (71), respectively, and:

$$\begin{aligned} \mathscr{A}_{\text{usul}}(u) &= \left[-\partial_b \sigma_{ab}, \sigma_{ab} - J^{-1}[(\lambda \log(J) - \mu)\delta_{ab} + \mu b_{ab}] \right], \\ f &= [\rho f_a, 0], \\ \mathscr{L}_{\text{usul}}(\bar{u}; \delta u) &= \left[-\partial_b \delta\sigma_{ab}, \right. \\ & \qquad \left. \delta\sigma_{ab} + \bar{J}^{-1}[\lambda \log(\bar{J}) - \mu - \lambda] \bar{F}_{Ac}^{-1} \partial_A \delta u_c \delta_{ab} - \bar{J}^{-1} \mu (\partial_A \delta u_a \bar{F}_{bA} + \bar{F}_{aA} \partial_A \delta u_b) \right], \\ \hat{\mathscr{L}}_{\text{usul}}^*(\bar{u}; v) = \mathscr{L}_{\text{usul}}^*(\bar{u}; v) &= \left[-\partial_A \{ \bar{J}^{-1}[\lambda \log(\bar{J}) - \mu - \lambda] \bar{F}_{Aa}^{-1} \tau_{cc} \} + 2\partial_A (\bar{J}^{-1} \mu \bar{F}_{bA} \tau_{ab}), \right. \\ & \qquad \left. (\partial_a v_b)^{\text{s}} + \tau_{ab} \right]. \end{aligned}$$

Finally, we may again use the same expression for τ_K as in linear Elasticity (given in Table III), μ being the first Lamé's parameter of the Neo-Hookean model.

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6. MIXED DISPLACEMENT-PRESSURE-STRESS FORMULATION

The last approach we wish to consider is the mixed displacement-pressure-stress one. The introduction of the pressure is of interest when the material is incompressible, and thus this is the situation considered in the following.

6.1. Linear Elasticity: $\boldsymbol{u}$-p-$\boldsymbol{\sigma}^{\mathrm{dev}}$ approach

As for the previous problems, the bilinear form of the continuous problem is given in Table II, and the formulation we propose is given by Eqs. (36)-(35) extended to the transient case, with the stabilization matrix given in Table III. Even though one could think of a primal and a dual formulation in this case, as for the displacement-stress approach, we are not aware of any stability result for what could be the dual formulation, and thus we restrict the discussion to the classical (primal) formulation. Once again, for this linear problem we consider $\boldsymbol{u}_D = \boldsymbol{0}$.

Thus, the discrete problem we propose to solve is the following: for each time step, find $[\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h^{\mathrm{dev}}] \in V_h \times Q_h \times T_h$ such that

$$\begin{aligned}
(\boldsymbol{v}_h, \rho\delta_{tt}^2\boldsymbol{u}_h) &+ (\nabla^{\mathrm{s}}\boldsymbol{v}_h, \boldsymbol{\sigma}_h^{\mathrm{dev}}) - (p_h, \nabla\cdot\boldsymbol{v}_h) + (q_h, \nabla\cdot\boldsymbol{u}_h) \\
&+ (\boldsymbol{C}^{\mathrm{dev}^{-1}} : \boldsymbol{\sigma}_h^{\mathrm{dev}}, \boldsymbol{\tau}_h) - (\nabla^{\mathrm{s}}\boldsymbol{u}_h, \boldsymbol{\tau}_h) \\
&+ \sum_K c_1\frac{h^2}{\mu}\langle -\nabla\cdot\boldsymbol{\tau}_h + \nabla q_h, \mathscr{P}_h^{\perp}[-\nabla\cdot\boldsymbol{\sigma}^{\mathrm{dev}} + \nabla p_h - \boldsymbol{f}]\rangle_K \\
&+ \sum_K c_2\mu\langle\nabla\cdot\boldsymbol{v}_h, \mathscr{P}_h^{\perp}[\nabla\cdot\boldsymbol{u}_h]\rangle_K \\
&+ \sum_K c_2\mu\langle\nabla^{\mathrm{s}}\boldsymbol{v}_h, \mathscr{P}_h^{\perp}[\nabla^{\mathrm{s}}\boldsymbol{u}_h]\rangle_K \\
&= \langle\boldsymbol{v}_h, \boldsymbol{f}\rangle + \langle\boldsymbol{v}_h, \boldsymbol{t}_N\rangle_{\Gamma_N} \qquad \text{for all } [\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h] \in V_h \times Q_h \times T_h.
\end{aligned}$$

In the stationary problem the formulation is stable and optimally convergent in the norm [16]:

$$|\!|\!|[\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h]|\!|\!|^2 = \mu\|\nabla\boldsymbol{v}_h\|^2 + \frac{1}{\mu}\|q_h\|^2 + \frac{1}{\mu}\|\boldsymbol{\tau}_h\|^2$$

If k_u, k_p and k_σ are the interpolation orders in V_h, Q_h and T_h, respectively, and r_u, r_p and r_σ the Sobolev regularities of the continuous solution $\boldsymbol{u}$, p, and $\boldsymbol{\sigma}^{\mathrm{dev}}$, respectively, one can prove that

$$|\!|\!|[\boldsymbol{u} - \boldsymbol{u}_h, p - p_h, \boldsymbol{\sigma}^{\mathrm{dev}} - \boldsymbol{\sigma}_h^{\mathrm{dev}}]|\!|\!|^2 \lesssim h^{s_u - 1}\mu\|\boldsymbol{u}\|^2_{H^{s_u}(\Omega)} + h^{s_p}\frac{1}{\mu}\|p\|^2_{H^{s_p}(\Omega)} + h^{s_\sigma}\frac{1}{\mu}\|\boldsymbol{\sigma}^{\mathrm{dev}}\|^2_{H^{s_p}(\Omega)},$$

where $s_u = \min\{k_u + 1, r_u\}$, $s_p = \min\{k_p + 1, r_p\}$ and $s_\sigma = \min\{k_\sigma + 1, r_\sigma\}$. In particular, for smooth solutions and equal order interpolation $k_u = k_p = k_\sigma = k$, we have that

$$|\!|\!|[\boldsymbol{u} - \boldsymbol{u}_h, p - p_h, \boldsymbol{\sigma}^{\mathrm{dev}} - \boldsymbol{\sigma}_h^{\mathrm{dev}}]|\!|\!|^2 \lesssim h^k\Big(\mu\|\boldsymbol{u}\|^2_{H^{k+1}(\Omega)} + \frac{1}{\mu}\|p\|^2_{H^k(\Omega)} + \frac{1}{\mu}\|\boldsymbol{\sigma}^{\mathrm{dev}}\|^2_{H^k(\Omega)}\Big).$$

6.2. Total Lagrangian formulation: $\boldsymbol{u}$-p-$\boldsymbol{S}'$ approach

The problem to be solved is formally identical to Eqs. (46)-(47), with the same initial and boundary conditions. The difference is that now we consider $\boldsymbol{S}' : \mathfrak{D}_0 \longrightarrow \mathbb{R}^{\mathrm{d}} \otimes \mathbb{R}^{\mathrm{d}}$ as an independent unknown of the problem, and the constitutive equation for it needs to be added as an additional equation of the problem, i.e.,

$$S'_{AB} - 2\frac{\partial W}{\partial C_{AB}} = 0. \tag{72}$$

Tensor $\boldsymbol{F}$, its determinant J and tensor $\boldsymbol{C}$ are understood to be written in terms of the displacement field $\boldsymbol{u}$.

Let V, Q and T be, respectively, the proper functional spaces where the displacement, the pressure and the component $\boldsymbol{S}'$ of tensor $\boldsymbol{S}$, solution to the problem, are defined for each fixed time $t \in (0, t_{\text{fin}})$. In general, $V \subset H^1(\Omega)^d$, $Q \subset L^2(\Omega)$ and $T \subset L^2(\Omega)^{\text{d}\times\text{d}}$, although the precise definition of these spaces depends on the constitutive law. Let $V_0 \subset V$ be the space of functions in V that vanish on $\Gamma_{0,D}$. In this case, $X = V \times Q \times T$, and we shall also need $X_0 = V_0 \times Q \times T$. The variational statement of the problem is derived by testing system (46)-(47) against arbitrary test functions, $\boldsymbol{v} \in V_0$ and $q \in Q$, and testing Eq. (72) against arbitrary test functions $\boldsymbol{T} \in T$. The weak form of the problem reads: find $u = [\boldsymbol{u}, p, \boldsymbol{S}'] : (0, t_{\text{fin}}) \longrightarrow X$ such that the initial conditions are satisfied and

$$\langle v, \mathscr{D}_t u \rangle + B_{\text{upstl}}(u, v) = L(v) \quad \text{for all } v = [\boldsymbol{v}, q] \in X_0, \tag{73}$$

where $B_{\text{upstl}}(u, v)$ is a semilinear form defined on $X \times X_0$ as

$$\begin{aligned} B_{\text{upstl}}(u, v) := {} & \langle \partial_A v_a, F_{aB} S'_{BA} \rangle - \left\langle \partial_A v_a, pJ F^{-1}_{Aa} \right\rangle + \left\langle q, \frac{\mathrm{d}G}{\mathrm{d}J} \right\rangle \\ & + \langle T_{AB}, S'_{AB} \rangle - \left\langle T_{AB}, 2\frac{\partial W}{\partial C_{AB}} \right\rangle. \end{aligned} \tag{74}$$

The form $L(v)$ and the operator $\mathscr{D}_t(u)$ are the same as for the displacement-stress approach. The same comments regarding the symmetrization of the problem apply in the present case, as $B_{\text{upstl}}(u, v)$ defined in Eq. (74) is not symmetric.

Let us proceed now to the linearization and time discretization of the problem. Using the same notation as in the previous sections, the resulting problem consists of finding $\delta u \in X_0$ such that

$$\langle v, \mathscr{D}_{\delta t}(\delta u) \rangle + B_{\text{upstl,lin}}(\bar{u}; \delta u, v) = L(v) - B_{\text{upstl}}(\bar{u}, v) - \langle v, \mathscr{D}_{\delta t}(\bar{u}) \rangle \quad \text{for all } v \in X_0,$$

where the linear form $B_{\text{upstl,lin}}(\bar{u}; \delta u, v)$ is given by

$$\begin{aligned} B_{\text{upstl,lin}}(\bar{u}; \delta u, v) = {} & \left\langle \partial_A v_a, \partial_B \delta u_a \bar{S}'_{BA} \right\rangle + \left\langle \partial_A v_a, \bar{F}_{aB} \delta S'_{BA} \right\rangle - \left\langle \partial_A v_a, \bar{J}\bar{p}\bar{F}^{-1}_{Bb} \partial_B \delta u_b \bar{F}^{-1}_{Aa} \right\rangle \\ & + \left\langle \partial_A v_a, \bar{J}\bar{p}\bar{F}^{-1}_{Ab} \partial_B \delta u_b \bar{F}^{-1}_{Ba} \right\rangle - \left\langle \partial_A v_a, \bar{J}\delta p \bar{F}^{-1}_{Aa} \right\rangle + \left\langle q, \gamma(\bar{J}) \bar{F}^{-1}_{Aa} \partial_A \delta u_a \right\rangle \\ & + \langle T_{AB}, S'_{AB} \rangle - \langle T_{AB}, C'^{\tan}_{ABCD} F_{aC} \partial_D \delta u_a \rangle. \end{aligned} \tag{75}$$

The notation involved in this expression is the same as in Eqs. (50) and (63). In particular, $C'^{\tan}_{ABCD}$ is the $ABCD$ component of the tangent constitutive tensor of the constitutive law for $\boldsymbol{S}'$.

Let us move now to the discrete stabilized FE approximation to the problem for the $\boldsymbol{u}$-p-$\boldsymbol{S}'$ formulation using a TL description, which reads: find $\delta u_h = [\delta \boldsymbol{u}_h, \delta p_h, \delta \boldsymbol{S}'_h] \in X_{0,h}$ such that

$$\begin{aligned} & \langle v_h, \mathscr{D}_{\delta t}(\delta u_h) \rangle + B_{\text{upstl,lin}}(\bar{u}_h; \delta u_h, v_h) \\ & \quad + \sum_K \tau_K \langle \mathscr{P}_h^{\perp} [f - \mathscr{A}_{\text{upstl}}(\bar{u}_h) - \mathscr{L}_{\text{upstl}}(\bar{u}_h; \delta u_h)], \hat{\mathscr{L}}^*_{\text{upstl}}(\bar{u}_h; v_h) \rangle_K \\ & \quad = L(v_h) - B_{\text{upstl}}(\bar{u}_h, v_h) - \langle v_h, \mathscr{D}_{\delta t}(\bar{u}_h) \rangle \quad \text{for all } v_h \in X_{0,h}. \end{aligned}$$

We need to identify all the terms in this expression. From Eqs. (74) and (75) we can obtain $B_{\text{upstl}}(\bar{u}_h, v_h)$ and $B_{\text{upstl,lin}}(\bar{u}_h; \delta u_h, v_h)$, respectively, and the rest of terms are:

$$\begin{aligned} \mathscr{A}_{\text{upstl}}(u) &= \Big[-\partial_A (F_{aB} S'_{BA}) + \partial_A (pJ F^{-1}_{Aa}), \frac{\mathrm{d}G}{\mathrm{d}J}, S'_{AB} - 2\frac{\partial W}{\partial C_{AB}} \Big], \\ f &= [\rho_0 f_a, 0, 0], \end{aligned} \tag{76}$$

$$\begin{aligned}
\mathscr{L}_{\mathrm{upstl}}(\bar{u};\delta u) = [&-\partial_A(\partial_B\delta u_a \bar{S}'_{BA}) - \partial_A(\bar{F}_{aB}\delta S'_{BA}) + \partial_A(\bar{J}\bar{p}\bar{F}^{-1}_{Bb}\partial_B\delta u_b \bar{F}^{-1}_{Aa})\\
&+ \partial_A(\bar{J}\delta p \bar{F}^{-1}_{Aa}) - \partial_A(\bar{J}\bar{p}\bar{F}^{-1}_{Ab}\partial_B\delta u_b \bar{F}^{-1}_{Ba}),\\
&\gamma(\bar{J})\bar{F}^{-1}_{Aa}\partial_A\delta u_a,\\
&\delta S'_{AB} - C'^{\tan}_{ABCD}\bar{F}_{aC}\partial_D\delta u_a], \qquad (77)\\
\hat{\mathscr{L}}^{*}_{\mathrm{upstl}}(\bar{u};v) = \mathscr{L}^{*}_{\mathrm{upstl}}(\bar{u};v) = [&-\partial_A(\partial_B(v_a\bar{S}'_{AB})) + \partial_A(\bar{J}\bar{p}\bar{F}^{-1}_{Aa}\partial_B v_b \bar{F}^{-1}_{Bb})\\
&- \partial_A(\gamma(\bar{J})q\bar{F}^{-1}_{Aa}) - \partial_A(\bar{J}\bar{p}\bar{F}^{-1}_{Ba}\partial_B v_b \bar{F}^{-1}_{Ab}) + \partial_D(T_{AB}C'^{\tan}_{ABCD}\bar{F}_{aC}),\\
&- \bar{J}\bar{F}^{-1}_{Aa}\partial_A v_a,\\
&\bar{F}_{aB}\partial_A v_a + T_{AB}]. \qquad (78)
\end{aligned}$$

The expression of the stabilization parameters τ_K requires a characteristic value of the tangent constitutive tensor whose components are $C'^{\tan}_{ABCD}$, so that the design condition for the stabilization parameters is satisfied. If $\bar{\mu}$ is this characteristic value, we may use the same expression for τ_K as for the linear case (given in Table III) using $\bar{\mu}$ instead of μ.

6.3. Updated Lagrangian formulation: $\boldsymbol{u}$-p-$\boldsymbol{\sigma}^{\mathrm{dev}}$ *approach*

To close the list of formulations proposed, let us consider the FE approximation of Neo-Hookean hyperelastic materials in the incompressible limit using displacements, pressure and deviatoric stresses as unknowns and the UL description of the geometry. The IVBP to be solved is the following: find $\boldsymbol{u} : \mathfrak{D} \longrightarrow \mathbb{R}^{\mathrm{d}}$, $p : \mathfrak{D} \longrightarrow \mathbb{R}$ and $\boldsymbol{\sigma}^{\mathrm{dev}} : \mathfrak{D} \longrightarrow \mathbb{R}^{\mathrm{d}} \otimes \mathbb{R}^{\mathrm{d}}$ such that

$$\begin{aligned}
\rho\partial^2_{tt}u_a - \partial_b\sigma^{\mathrm{dev}}_{ab} + \partial_a p &= \rho f_a \quad &&\text{in } \mathfrak{D}, &(79)\\
\log(J) &= 0 &&\text{in } \mathfrak{D}, &(80)\\
\sigma^{\mathrm{dev}}_{ab} - J^{-1}\mu\Big(b_{ab} - \frac{b_{cc}}{\mathrm{d}}\delta_{ab}\Big) &= 0 &&\text{in } \mathfrak{D}, &(81)\\
u_a = u^0_a,\ \partial_t u_a &= \dot{u}^0_a &&\text{in } \Omega(0),\ t=0\\
u_a &= u_{D,a} &&\text{in } \Gamma_D,\ t\in(0,t_{\mathrm{fin}}),\\
n_b\sigma^{\mathrm{dev}}_{ab} - n_a p &= t_{N,a} &&\text{in } \Gamma_N,\ t\in(0,t_{\mathrm{fin}}),
\end{aligned}$$

the unknowns being u_a, p and $\sigma^{\mathrm{dev}}_{ab}$.

Proceeding as in the previous models, let V, Q and T be, respectively, the proper functional spaces where displacement, pressure and deviatoric stress solutions are well-defined for each fixed time $t \in (0, t_{\mathrm{fin}})$, and let $V_0 \subset V$ the subspace of V with vector fields vanishing on Γ_D. Testing Eqs. (79)-(81) against $[\boldsymbol{v}, q, \boldsymbol{\tau}] \in V_0 \times Q \times T$ we obtain the weak form of the problem: find $u = \big[\boldsymbol{u}, p, \boldsymbol{\sigma}^{\mathrm{dev}}\big] : (0, t_{\mathrm{fin}}) \longrightarrow X$ such that the initial conditions are satisfied and

$$\langle v, \mathscr{D}_t u\rangle + B_{\mathrm{upsul}}(u, v) = L(v) \quad \text{for all } v = [\boldsymbol{v}, q, \boldsymbol{\tau}] \in X_0, \qquad (82)$$

where $B_{\mathrm{upsul}}(u, v)$ is a semilinear form defined on $X \times X_0$ as

$$\begin{aligned}
B_{\mathrm{upsul}}(u,v) := &\langle \partial_b v_a, \sigma^{\mathrm{dev}}_{ab}\rangle - \langle \partial_a v_a, p\rangle\\
&+ \langle q, \log(J)\rangle + \langle \sigma^{\mathrm{dev}}_{ab}, \tau_{ab}\rangle - \Big\langle J^{-1}\mu\Big(b_{ab} - \frac{b_{cc}}{\mathrm{d}}\delta_{ab}\Big), \tau_{ab}\Big\rangle. \qquad (83)
\end{aligned}$$

$L(v)$ and $\mathscr{D}_t(u)$ are the same as for the previous UL formulations. Also, the same comments regarding the non-symmetry of the formulation apply. However, in this particular case it is only necessary to symmetrize the contribution arising from the deviatoric part of the constitutive law, which is easily done by multiplying Eq (81) by μ^{-1}.

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After time discretization and linearization using Newton-Raphson's scheme, the linear variational problem to solve consists of finding $\delta u \in X_0$ such that

$$\langle v, \mathscr{D}_{\delta t}(\delta u)\rangle + B_{\text{upsul,lin}}(\bar{u};\delta u, v) = L(v) - B_{\text{upsul}}(\bar{u}, v) - \langle v, \mathscr{D}_{\delta t}(\bar{u})\rangle \quad \text{for all } v \in X_0,$$

where the linear form $B_{\text{upsul,lin}}(\bar{u};\delta u, v)$ is given by

$$\begin{aligned} B_{\text{upsul,lin}}(\bar{u};\delta u, v) &= \langle \partial_b v_a, \delta\sigma_{ab}^{\text{dev}}\rangle - \langle \partial_a v_a, \delta p\rangle \\ &+ \langle q, \bar{F}^{-1}_{Aa}\partial_A \delta u_a\rangle + \langle \delta\sigma_{ab}^{\text{dev}}, \tau_{ab}\rangle \\ &- \Big\langle \frac{\mu}{\bar{J}}(\partial_A \delta u_a \bar{F}_{bA} + \bar{F}_{aA}\partial_A \delta u_b), \tau_{ab}\Big\rangle + \Big\langle 2\frac{\mu}{\bar{J}\text{d}}\bar{F}_{cA}\partial_A \delta u_c, \tau_{aa}\Big\rangle \\ &- \Big\langle \frac{\mu}{\bar{J}}\bar{F}^{-1}_{Ac}\partial_A \delta u_c\Big(\bar{b}_{ab} - \frac{\bar{b}_{cc}}{\text{d}}\delta_{ab}\Big), \tau_{ab}\Big\rangle \end{aligned} \tag{84}$$

Applying now Eqs. (45)-(44) to the problem at hand, the resulting problem is: find $\delta u_h = [\delta \boldsymbol{u}_h, \delta p_h, \delta \boldsymbol{\sigma}^{\text{dev}}] \in X_{0,h}$ such that

$$\begin{aligned} &\langle v_h, \mathscr{D}_{\delta t}(\delta u_h)\rangle + B_{\text{upsul,lin}}(\bar{u}_h;\delta u_h, v_h) \\ &\quad + \sum_K \tau_K \langle \mathscr{P}_h^{\perp}[f - \mathscr{A}_{\text{upsul}}(\bar{u}_h) - \mathscr{L}_{\text{upsul}}(\bar{u}_h;\delta u_h)], \hat{\mathscr{L}}^*_{\text{upsul}}(\bar{u}_h; v_h)\rangle_K \\ &\quad = L(v_h) - B_{\text{upsul}}(\bar{u}_h, v_h) - \langle v_h, \mathscr{D}_{\delta t}(\bar{u}_h)\rangle \quad \text{for all } v_h \in X_{0,h}. \end{aligned}$$

with $B_{\text{upsul}}(\bar{u}_h, v_h)$ and $B_{\text{upsul,lin}}(\bar{u}_h;\delta u_h, v_h)$ obtained from Eqs. (83) and (84), respectively, and:

$$\begin{aligned} \mathscr{A}_{\text{upsul}}(u) &= \Big[-\partial_b \sigma_{ab}^{\text{dev}} + \partial_a p, \log(J), \sigma_{ab}^{\text{dev}} - J^{-1}\mu\Big(b_{ab} - \frac{b_{cc}}{\text{d}}\delta_{ab}\Big)\Big], \\ f &= [\rho f_a, 0, 0], \\ \mathscr{L}_{\text{upsul}}(\bar{u};\delta u) &= \Big[-\partial_b \delta\sigma_{ab}^{\text{dev}} + \partial_a p, \\ &\quad \bar{F}^{-1}_{Aa}\partial_A \delta u_a, \\ &\quad \delta\sigma_{ab}^{\text{dev}} - \frac{\mu}{\bar{J}}(\partial_A \delta u_a \bar{F}_{bA} + \bar{F}_{aA}\partial_A \delta u_b) \\ &\quad\quad + 2\frac{\mu}{\bar{J}\text{d}}\bar{F}_{cA}\partial_A \delta u_c \delta_{ab} - \frac{\mu}{\bar{J}}\bar{F}^{-1}_{Ac}\partial_A \delta u_c\Big(\bar{b}_{ab} - \frac{\bar{b}_{cc}}{\text{d}}\delta_{ab}\Big)\Big], \\ \hat{\mathscr{L}}^*_{\text{upsul}}(\bar{u}; v) &= \mathscr{L}^*_{\text{upsul}}(\bar{u}; v) = \Big[-\partial_A(\bar{F}^{-1}_{Aa} q) + 2\partial_A\Big(\frac{\mu}{\bar{J}}\tau_{ab}\bar{F}_{bA}\Big) \\ &\quad\quad - 2\partial_A\Big(\frac{\mu}{\bar{J}\text{d}}\tau_{cc}\bar{F}_{aA}\Big) + \partial_A\Big\{\frac{\mu}{\bar{J}}\bar{F}^{-1}_{Aa}\Big(\bar{b}_{bc} - \frac{\bar{b}_{dd}}{\text{d}}\delta_{bc}\Big)\tau_{bc}\Big\}, \\ &\quad - \partial_a v_a, \\ &\quad (\partial_a v_b)^{\text{s}} + \tau_{ab}\Big]. \end{aligned}$$

Finally, we may again use the same expression for τ_K as for the linear case (given in Table III), μ being the first Lamé's parameter of the Neo-Hookean model.

7. CONCLUSIONS

In this paper we have reviewed several formulations for linear elastic and finite strain hyperelastic problems. The objective has been to write stabilized FE methods for all formulations in a unified format, and discuss the main features of each. It has been shown that from a common structure of all formulations each particular case can be easily derived. This common structure arises from the VMS concept, choosing appropriately the space of SGSs and the treatment of the nonlinearity. We have concentrated on continuous interpolation for all variables, but the ideas presented can be generalized to discontinuous interpolations as soon as they are conforming.

Little attention has been paid to time, and in fact the inertial term has been included for the sole purpose of highlighting that any finite difference approximation can be accommodated. In the case of transient problems, we definitely favor the use of dynamic SGSs, which are crucial if the time step is small relative to the size of the FE partition. However, we have avoided to describe its implications in detail and we have focused our attention to the mixed structure of the equations to be solved.

The resulting formulations are easy to implement, allowing in particular equal interpolation for all variables. Furthermore, the choice of the stabilization parameters proposed guarantees stability and optimal order of convergence in natural norms, at least for the linear problems for which the analysis is available.

Other mixed formulations can be easily designed following the guidelines presented here. In particular, in some situations it can be convenient to introduce strains as new unknowns. In nonlinear problems, the possibility of using different measures of deformation (deformation tensor or right or left Cauchy-Green tensors, for example) is open.

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