MIXED STABILIZED FINITE ELEMENT METHODS IN LINEAR ELASTICITY FOR THE VELOCITY-STRESS EQUATIONS IN THE TIME AND THE FREQUENCY DOMAINS

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Abstract. *In this work we present stabilized finite element methods for the mixed velocity-stress elasticity equations and for its irreducible velocity form. This is done both for the time and frequency domains, the latter being obtained by assuming a harmonic behavior in time. Stabilization methods that belong to the computational framework of the Variational Multi-Scale formulation are used. It is shown that the adequate selection of the algorithmic parameters on which the formulation depends allows one to switch from the primal to the dual functional framework. The performance of the method is tested through several numerical examples, one of which includes a convergence study.*

Keywords: Elastodynamics, stabilized finite element methods, vector-tensor mixed formulation, frequency domain

1. Introduction

The most common way to study linear elasticity (under the infinitesimal strain assumption) has always been through the equation that uses displacements as unknowns and in the time domain, either in its irreducible form or in mixed form. In the former, just the displacements are computed, whereas the latter involves the computation of two or more different unknowns (see, e.g. [1, 2, 3]). However, with some simple manipulations, it is easy to obtain a new equation where velocity is the main unknown. This new equation can also be presented either in its irreducible form, where just the velocity is calculated, or in its mixed form, where it is computed alongside the stress tensor [4, 5, 6, 7]. Moreover, these two forms of the elasticity problem can also be studied in the frequency domain, which is obtained by considering a harmonic behavior in time of the solution and analyzing one mode or, equivalently, taking the Fourier transform in time. Therefore, we can enumerate (at least) four possible different problems to consider, namely, irreducible and mixed forms in either the time domain or the frequency domain; these are schematically shown in Figure 1. It is important to note that the number of variables is doubled for the equations in the frequency domain, because they have to be solved for the real and the imaginary parts of the unknowns. These four ways of writing the elasticity problem have in common that they have the structure of a *wave problem*, of second order for the irreducible velocity form and of first order for the mixed velocity-stress form. The mixed displacement-stress form also models wave phenomena, obviously, but it is of second order for displacements and zeroth order for stresses, without the structure of a first order wave equation; we will not consider this way to write the problem in this paper.

While the two irreducible forms, in displacements and velocities, are classical and the mixed form in the time domain is also well known, the mixed form in the frequency domain has attracted much less attention. The numerical approximation of this problem is one of

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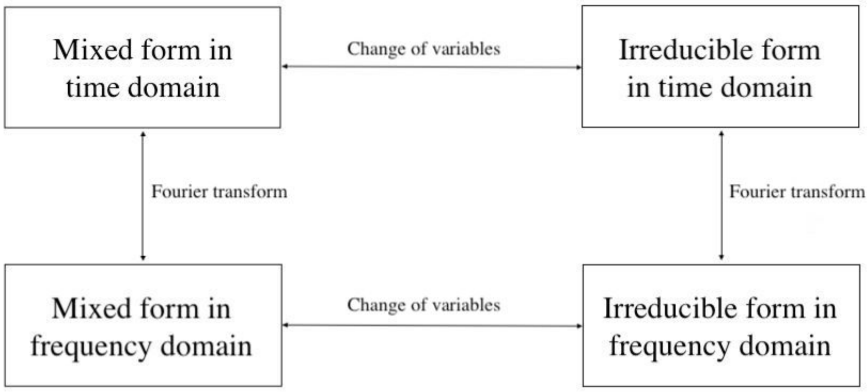


FIGURE 1. Scheme of the four possible forms of the elasticity problem.

the novelties of this paper. The aim of the present article is to study this velocity-stress mixed formulations in both the time and the frequency domains; the two corresponding irreducible velocity formulations will be considered and studied as well.

The two first problems to consider are the irreducible form for both time and frequency domains. These equations are completely analogous to the ones for the displacement; this fact makes them the most popular ones, as there is a vast literature about the displacement equation. On the contrary, the mixed equations we will consider are quite different from those for the displacement. They are in fact the mixed vector-tensor version of the well known mixed scalar-vector wave equations (see, e.g., [8, 9]). Therefore, they have the structure of mixed wave equations. Another critical issue is that these mixed formulations have two possible functional settings, each leading to well posed problems with different approximation properties for the velocity and the stress. The first of these functional settings is directly inherited from the irreducible formulation, with the same regularity for the velocities and the stresses. However, the fact that the stresses are independent unknowns in the mixed formulation allows one to transfer regularity from the velocities to the stresses. This yields what can be called a dual approach, which can provide more accuracy for the stress than the classical one. In the case of the equations in the frequency domain, we also have to take into account the computational saves associated to solving for a single frequency [10, 11].

The Finite Element (FE) method is used in this work for the spatial approximation of all problems (see e.g. [12]). For the irreducible problems, the standard Galerkin method yields stable and optimally convergent results. However, for mixed formulations there is an inf-sup condition to be satisfied in order to obtain stable solutions. This can be avoided by switching to a stabilized FE formulation, that is the approach we follow in this work. More precisely, the formulation we present in this paper belongs to the computational framework of Variational Multi-Scale (VMS) methods [13, 14], a family of techniques based on splitting the unknowns of the problem into two different scales: the one that will be approximated by the FE mesh and the sub-grid scale (SGS), which cannot be captured or represented by the FE space. The main idea of this formulation consists of approximating this SGS so as to arrive at a problem with enhanced stability but involving only the FE unknowns. In our approach we shall account for an approximation of the SGSs both in the element interiors and on the element boundaries [15, 16].

One of the main characteristics of methods motivated within the VMS framework is that a projection of the residual of the FE solution onto the space of the SGS is involved. The resulting stabilized FE method depends on how this projection is chosen. When the space of the SGS is considered the space of FE residuals, the projection can be taken as the identity and we call the resulting formulation the Algebraic Sub-Grid Scale method [17]. An alternative is to take the space of the SGS orthogonal to the FE space; the projection involved is then the L^2 orthogonal projection to the FE space; in this case, we call the resulting method the Orthogonal Sub-Grid-Scale stabilization [18]. With this choice, SGSs are only active in regions where the unknown cannot be resolved by the FE mesh.

The stabilization method employed implies the introduction of stabilization parameters that depend on the length scales and the constants of the problem. For the case of mixed formulations, a proper design of these parameters will allow us to switch from one functional setting of those mentioned above to the other, that is to say, whether the appropriate functional framework is the primal or the dual one will be determined by the stabilization parameters. The precise description of these functional settings is detailed below.

The use of a stabilized formulation has a beneficial side effect regarding the symmetry of the stress tensor. While in the standard Galerkin method it is difficult to construct FE spaces for the stress and the velocity satisfying the inf-sup condition in which stresses are symmetric [19], and thus sometimes the weak imposition of symmetry is preferred [20], this offers no difficulty in stabilized formulations. In our implementation we have made use of the symmetry of the stress tensor in a strong sense, interpolating only the upper triangular part of this tensor.

In summary, in this paper we will propose stabilized FE methods for the mixed forms of the linear elasticity problem in the time and frequency domains, allowing for independent velocity-stress interpolations. The formulation includes stabilization terms evaluated in the element interiors and on the interelement boundaries, and the stabilization parameters will allow us changing the functional framework of the problem.

The irreducible form of the elastodynamics problem is obviously not new. Regarding the mixed velocity-stress formulation in the time domain using stabilized formulations, it was studied in [6] for what can be called the most regular possible functional setting. In fact, the approach adopted in that paper was the extension of the mixed scalar-vector wave equation analyzed in [21] to the mixed vector-tensor problem resulting in the elasticity equations. This regular functional setting is permitted by the smoothing effect of the time derivative, which is justified in Hille-Yosida's theorem, as explained in [21]. Here we do not restrict ourselves to this framework, but consider instead the problems with either regular velocity or with regular stresses, i.e., the two settings that can be termed as the primal and the dual one. In this sense, the contribution of this paper is the extension to the vector-tensor mixed problem of the scalar-vector wave equation proposed and analyzed in [8, 22]. In fact, even though we do not undertake any numerical analysis in the present paper, we expect the theoretical results proved in these references to carry over to the present problem without significant changes. The Galerkin approximation of the velocity-stress formulation, satisfying and inf-sup condition for the interpolation spaces, is analyzed in [4, 5]. Finally, the translation to the frequency domain of the mixed velocity-stress approach has not been attempted before, as far as we are aware. It has to be remarked that the mathematical structure of this problem differs significantly from that of the counterpart in the time domain. Summarizing, the main novelties of this paper are the new functional frameworks (primal and dual) for the velocity-stress formulation in the time domain and the frequency domain counterpart of this problem, again considering two possible functional settings and using stabilized FE methods in all cases.

This paper is organized as follows. In section 2, the different forms of the equations of the problem and their variational versions are described. In section 3, the stabilized discrete problems using the ASGS and OSGS methods are presented, both for the time and for the frequency domains. In section 4, the algebraic matrix form of the problems are presented. In section 5, three different numerical examples are shown. Finally, in section 6 conclusions are drawn.

2. Problem statement

2.1. **Initial and boundary value problem.** The problems we consider are boundary value problems posed in a spatial domain $\Omega \subset \mathbb{R}^d$, where $d = 2, 3$. The ones in the time domain are also initial value problems posed in a time interval $(0, T)$.

Let $\partial\Omega$ be the boundary of the domain Ω. We split this boundary into two disjoint sets denoted as Γ_v and Γ_σ, where the boundary conditions corresponding to the velocity $\boldsymbol{v}$ and the normal component of the Cauchy stress tensor $\boldsymbol{\sigma}$ will be enforced, respectively.

For the following equations, we define $\boldsymbol{f}$ as the given forcing term, ∇^S as the symmetric gradient of a vector field, ρ as the density, $\boldsymbol{C}$ as the elastic fourth order constitutive tensor and ω as the angular frequency. The unit normal to the boundary of the domain will be denoted as $\boldsymbol{n}$.

The mixed problem in the time domain consists of finding the velocity $\boldsymbol{v} : \Omega\times(0, T) \to \mathbb{R}^d$ and the stress tensor $\boldsymbol{\sigma} : \Omega \times (0, T) \to \mathbb{R}^d \otimes \mathbb{R}^d$ such that

$$\rho\, \partial_t \boldsymbol{v} - \nabla \cdot \boldsymbol{\sigma} = \boldsymbol{f}, \tag{1}$$

$$\boldsymbol{C}^{-1} : \partial_t \boldsymbol{\sigma} - \nabla^S \boldsymbol{v} = \boldsymbol{0}, \tag{2}$$

with the initial conditions

$$\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x}), \;\; \boldsymbol{\sigma}(\boldsymbol{x}, 0) = \boldsymbol{\sigma}_0(\boldsymbol{x}), \;\; \boldsymbol{x} \in \Omega,$$

and with the boundary conditions

$$\boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_v, \qquad \boldsymbol{n} \cdot \boldsymbol{\sigma} = \boldsymbol{0} \text{ on } \Gamma_\sigma. \tag{3}$$

We have considered homogeneous boundary conditions to simplify the exposition. It can be observed that Eq. (1) is the standard Cauchy equation, whereas Eq. (2) is obtained taking the time derivative of the constitutive equation. The initial stress tensor $\boldsymbol{\sigma}_0(\boldsymbol{x})$ is assumed to be symmetric (see below), and therefore this equation yields that the stress tensor $\boldsymbol{\sigma}(\boldsymbol{x}, t)$ is symmetric for all time t.

The mixed form of the problem in the frequency domain consists of assuming a harmonic behavior of the unknowns in time, each mode having the form $\exp(-\mathrm{i}\omega t)\hat{\boldsymbol{v}}$ for the velocity and $\exp(-\mathrm{i}\omega t)\hat{\boldsymbol{\sigma}}$ for the stress. The same result is found considering the Fourier transform in time of the equations of the problem. Assuming the modes independent, the amplitudes of these modes, omitting the ˆ symbol for them and for the amplitude of the forcing term, are solution of: find $\boldsymbol{v} : \Omega \to \mathbb{C}^d$ and $\boldsymbol{\sigma} : \Omega \to \mathbb{C}^d \otimes \mathbb{C}^d$ such that

$$-\mathrm{i}\, \rho\, \omega\, \boldsymbol{v} - \nabla \cdot \boldsymbol{\sigma} = \boldsymbol{f}, \tag{4}$$

$$-\mathrm{i}\, \omega\, \boldsymbol{C}^{-1} : \boldsymbol{\sigma} - \nabla^S \boldsymbol{v} = \boldsymbol{0}, \tag{5}$$

with the same boundary conditions given by Eq. (3).

The irreducible elastodynamics equation for the velocity in the time domain is obtained by taking the time derivative of the first equation in Eq. (1) and making use of the second one to eliminate the stress tensor as unknown. Still calling $\boldsymbol{f}$ the resulting right-hand-side (RHS), the equation found consists of finding $\boldsymbol{v} : \Omega \times (0, T) \to \mathbb{R}^d$ such that

$$\rho \partial_{tt} \boldsymbol{v} - \nabla \cdot \boldsymbol{C} : \nabla^S \boldsymbol{v} = \boldsymbol{f}, \tag{6}$$

with the initial conditions

$$\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x}), \qquad \partial_t \boldsymbol{v}|_{t=0} = \dot{\boldsymbol{v}}_0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega, \tag{7}$$

where $\boldsymbol{v}_0(\boldsymbol{x})$ and $\dot{\boldsymbol{v}}_0(\boldsymbol{x})$ are given functions, and boundary conditions

$$\boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_v, \qquad \boldsymbol{n} \cdot \boldsymbol{C} : \nabla^S \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_\sigma. \tag{8}$$

Let us comment on the relationship between the initial conditions for the mixed and the irreducible forms of the problem in the time domain. In the case of the standard displacement approach, the initial conditions need to be given in terms of the initial displacements $\boldsymbol{u}_0(\boldsymbol{x})$ and the initial velocities $\boldsymbol{v}_0(\boldsymbol{x})$. From $\boldsymbol{u}_0(\boldsymbol{x})$ we can compute the initial stress as $\boldsymbol{\sigma}_0(\boldsymbol{x}) = \boldsymbol{C} : \nabla^S \boldsymbol{u}_0(\boldsymbol{x})$, and thus given the pair $\boldsymbol{u}_0(\boldsymbol{x})$, $\boldsymbol{v}_0(\boldsymbol{x})$ we know the pair of initial conditions $\boldsymbol{v}_0(\boldsymbol{x})$, $\boldsymbol{\sigma}_0(\boldsymbol{x})$ needed for the mixed form (1)-(2). However, the irreducible equation requires the initial velocity $\boldsymbol{v}_0(\boldsymbol{x})$ and the initial acceleration $\dot{\boldsymbol{v}}_0(\boldsymbol{x})$. To have the same solution as in the previous problems, this acceleration can be computed from Eq. (1) evaluated at $t = 0$, i.e.,

$$\dot{\boldsymbol{v}}_0 = \rho^{-1}[\boldsymbol{f} + \nabla \cdot \boldsymbol{\sigma}_0] = \rho^{-1}[\boldsymbol{f} + \nabla \cdot \boldsymbol{C} : \nabla^S \boldsymbol{u}_0].$$

The irreducible form of the problem in the frequency domain consists of finding $\boldsymbol{v} : \Omega \to \mathbb{C}^d$ such that

$$-\rho\,\omega^2 \boldsymbol{v} - \nabla \cdot \boldsymbol{C} : \nabla^S \boldsymbol{v} = \boldsymbol{f}, \tag{9}$$

with the boundary condition given again by Eq. (8). The same comments regarding the notation as in the mixed form in the frequency domain apply now. This equation can be found considering that $\boldsymbol{v}$ is the amplitude associated to the frequency ω of the solution of Eq. (6) or directly multiplying by $\mathrm{i}\,\omega$ Eq. (4) and making use of Eq. (5) to eliminate the stress tensor as an unknown of the problem.

The constitutive tensor $\boldsymbol{C}$ can be different depending on the kind of material we deal with. In this paper we will consider linear elastic and isotropic materials, and $\boldsymbol{C}$ can be written as

$$\boldsymbol{C} = 2\mu \boldsymbol{I}_4 + \lambda \boldsymbol{I}_2 \otimes \boldsymbol{I}_2, \qquad \text{or} \qquad \boldsymbol{C}^{-1} := \frac{1}{2\mu}\boldsymbol{I}_4 - \frac{\lambda}{2\mu(2\mu + 3\lambda)}\boldsymbol{I}_2 \otimes \boldsymbol{I}_2, \tag{10}$$

where $\boldsymbol{I}_4$ is the fourth order identity tensor, $\boldsymbol{I}_2$ is the second order identity tensor, μ is the shear modulus and λ is the Lamé first parameter. However, instead of using μ and λ, in the numerical examples we shall employ the Young modulus (E) and the Poisson ratio (ν), satisfying

$$\mu = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}. \tag{11}$$

2.2. **Variational formulation.** To write the weak form of the problem, we need to introduce some notation and specifications. Being $\mathfrak{D}$ a subdomain of Ω, $L^2(\mathfrak{D})$ is the space of square integrable functions in $\mathfrak{D}$ (scalars, vectors or tensors), $H^1(\mathfrak{D})$ is the space of functions in $L^2(\mathfrak{D})$ with derivatives in $L^2(\mathfrak{D})$ and $H(\mathrm{div}, \mathfrak{D})$ is the space of vector or tensor functions with components and divergence in $L^2(\mathfrak{D})$. The inner product in $L^2(\mathfrak{D})$ of two functions a, b will be denoted as $(a, b)_{\mathfrak{D}}$, with $(a, b)_\Omega \equiv (a, b)$, whereas the generic integral of the product of two functions will be denoted as $\langle a, b\rangle_{\mathfrak{D}}$, again with the simplification $\langle a, b\rangle_\Omega \equiv \langle a, b\rangle$.

2.2.1. *Mixed form: variational forms I and II.* Let V_v and V_σ be the appropriate spaces for the velocity and the stresses, respectively, to be specified below. For time dependent problems, $\boldsymbol{v} : (0, T) \to V_v$ and $\boldsymbol{\sigma} : (0, T) \to V_\sigma$. A test function in V_v will be denoted by $\boldsymbol{w}$ and a test function in V_σ by $\boldsymbol{\eta}$.

Let us now define $B([\boldsymbol{v}, \boldsymbol{\sigma}], [\boldsymbol{w}, \boldsymbol{\eta}]) = -\langle \nabla \cdot \boldsymbol{\sigma}, \boldsymbol{w}\rangle - \langle \nabla^S \boldsymbol{v}, \boldsymbol{\eta}\rangle$ and $L([\boldsymbol{w}, \boldsymbol{\eta}]) = \langle \boldsymbol{f}, \boldsymbol{w}\rangle$. The weak form of the mixed problems depends on the chosen regularity of the variables, which is determined by deciding which term of $B([\boldsymbol{v}, \boldsymbol{\sigma}], [\boldsymbol{w}, \boldsymbol{\eta}])$ is integrated by parts. This applies for both time and frequency domains.

On the one hand, what we will call **Variational form I** (VFI) corresponds to integrate by parts the first term of $B([\boldsymbol{v}, \boldsymbol{\sigma}], [\boldsymbol{w}, \boldsymbol{\eta}])$. Defining the spaces $V_v = \{\boldsymbol{w} \in H^1(\Omega)^d \mid \boldsymbol{w} = \boldsymbol{0} \text{ on } \Gamma_v\}$ and $V_\sigma = L^2(\Omega)^{d\times d}_{\mathrm{sym}}$ (symmetric second order tensors with components in $L^2(\Omega)$)

and taking $\boldsymbol{v}(\cdot,t), \boldsymbol{w} \in V_v$ and $\boldsymbol{\sigma}(\cdot,t), \boldsymbol{\eta} \in V_\sigma$, we may write

$$\begin{aligned} B([\boldsymbol{v},\boldsymbol{\sigma}],[\boldsymbol{w},\boldsymbol{\eta}]) &= (\boldsymbol{\sigma}, \nabla^S \boldsymbol{w}) - (\nabla^S \boldsymbol{v}, \boldsymbol{\eta}) - \langle \boldsymbol{n}\cdot\boldsymbol{\sigma}, \boldsymbol{w}\rangle_{\Gamma_\sigma} \\ &=: B_I([\boldsymbol{v},\boldsymbol{\sigma}],[\boldsymbol{w},\boldsymbol{\eta}]) - \langle \boldsymbol{n}\cdot\boldsymbol{\sigma}, \boldsymbol{w}\rangle_{\Gamma_\sigma}. \end{aligned}$$

Imposing weakly the boundary condition $\boldsymbol{n}\cdot\boldsymbol{\sigma} = \boldsymbol{0}$ on Γ_σ, VFI of the mixed form of the elasticity equations in the time domain consists of finding $\boldsymbol{v} : (0,T) \to V_v$ and $\boldsymbol{\sigma} : (0,T) \to V_\sigma$ such that

$$\rho\,(\partial_t \boldsymbol{v}, \boldsymbol{w}) + (\boldsymbol{C}^{-1} : \partial_t \boldsymbol{\sigma}, \boldsymbol{\eta}) + B_I([\boldsymbol{v},\boldsymbol{\sigma}],[\boldsymbol{w},\boldsymbol{\eta}]) = L([\boldsymbol{w},\boldsymbol{\eta}]), \tag{12}$$

for all $\boldsymbol{w} \in V_v$ and for all $\boldsymbol{\eta} \in V_\sigma$.

The initial conditions for Eq. (12) hold in $L^2(\Omega)$, that is to say, if $\boldsymbol{v}_0(\boldsymbol{x})$ and $\boldsymbol{\sigma}_0(\boldsymbol{x})$ are not $L^2(\Omega)$ functions, they have to be prescribed as

$$(\boldsymbol{v}|_{t=0}, \boldsymbol{w}) = (\boldsymbol{v}_0, \boldsymbol{w})\ \forall \boldsymbol{w} \in L^2(\Omega)^d, \quad (\boldsymbol{\sigma}|_{t=0}, \boldsymbol{\eta}) = (\boldsymbol{\sigma}_0, \boldsymbol{\eta})\ \forall \boldsymbol{\eta} \in L^2(\Omega)^{d\times d}_{\text{sym}}. \tag{13}$$

Similarly, VFI of the mixed form of the elasticity equations in the frequency domain consists of finding $\boldsymbol{v} \in V_v$ and $\boldsymbol{\sigma} \in V_\sigma$ such that

$$-\mathrm{i}\,\omega\,\rho\,(\boldsymbol{v}, \boldsymbol{w}) - \mathrm{i}\,\omega(\boldsymbol{C}^{-1} : \boldsymbol{\sigma}, \boldsymbol{\eta}) + B_I([\boldsymbol{v},\boldsymbol{\sigma}],[\boldsymbol{w},\boldsymbol{\eta}]) = L([\boldsymbol{w},\boldsymbol{\eta}]), \tag{14}$$

for all $\boldsymbol{w} \in V_v$ and for all $\boldsymbol{\eta} \in V_\sigma$, taking into account that spaces V_v and V_σ are now made of complex-valued functions. Note also that the $L^2(\Omega)$ inner product is not symmetric, but Hermitian, being defined as

$$(a, b) = \int_\Omega a\bar{b},$$

for any functions a, b, $\bar{b}$ being the complex conjugate of b.

Let us remark that the boundary conditions we have considered in both cases are

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{0}, && \text{strongly imposed on } \Gamma_v, \\ \boldsymbol{n}\cdot\boldsymbol{\sigma} &= \boldsymbol{0}, && \text{weakly imposed on } \Gamma_\sigma. \end{aligned}$$

On the other hand, **Variational form II** (VFII) corresponds to integrate by parts the second term of $B([\boldsymbol{v},\boldsymbol{\sigma}],[\boldsymbol{w},\boldsymbol{\eta}])$. Defining the spaces $V_v = L^2(\Omega)^d$ and $V_\sigma = \{\boldsymbol{\eta} \in H(\text{div},\Omega) \mid \boldsymbol{n}\cdot\boldsymbol{\eta} = \boldsymbol{0} \text{ on } \Gamma_\sigma \text{ and } \boldsymbol{\eta}^{\text{t}} = \boldsymbol{\eta}\}$, we may write

$$\begin{aligned} B([\boldsymbol{v},\boldsymbol{\sigma}],[\boldsymbol{w},\boldsymbol{\eta}]) &= -(\nabla\cdot\boldsymbol{\sigma}, \boldsymbol{w}) + (\boldsymbol{v}, \nabla\cdot\boldsymbol{\eta}) - \langle \boldsymbol{v}, \boldsymbol{n}\cdot\boldsymbol{\eta}\rangle_{\Gamma_v} \\ &=: B_{II}([p,\boldsymbol{u}],[q,\boldsymbol{v}]) - \langle \boldsymbol{v}, \boldsymbol{n}\cdot\boldsymbol{\eta}\rangle_{\Gamma_v}. \end{aligned}$$

Imposing weakly the boundary condition $\boldsymbol{v} = \boldsymbol{0}$ on Γ_v, VFII of the mixed form of the elasticity equations in the time domain consists of finding $\boldsymbol{v} : (0,T) \to V_v$ and $\boldsymbol{\sigma} : (0,T) \to V_\sigma$ such that

$$\rho\,(\partial_t \boldsymbol{v}, \boldsymbol{w}) + (\boldsymbol{C}^{-1} : \partial_t \boldsymbol{\sigma}, \boldsymbol{\eta}) + B_{II}([\boldsymbol{v},\boldsymbol{\sigma}],[\boldsymbol{w},\boldsymbol{\eta}]) = L([\boldsymbol{w},\boldsymbol{\eta}]), \tag{15}$$

for all $\boldsymbol{w} \in V_v$ and for all $\boldsymbol{\eta} \in V_\sigma$.

The initial conditions for this problem are the same as for VFI, i.e., those given by Eq. (13).

Similarly, VFII in the frequency domain reads: find $\boldsymbol{v} \in V_v$ and $\boldsymbol{\sigma} \in V_\sigma$ such that

$$-\mathrm{i}\,\omega\,\rho\,(\boldsymbol{v}, \boldsymbol{w}) - \mathrm{i}\,\omega(\boldsymbol{C}^{-1} : \boldsymbol{\sigma}, \boldsymbol{\eta}) + B_{II}([\boldsymbol{v},\boldsymbol{\sigma}],[\boldsymbol{w},\boldsymbol{\eta}]) = L([\boldsymbol{w},\boldsymbol{\eta}]), \tag{16}$$

for all $\boldsymbol{w} \in V_v$ and for all $\boldsymbol{\eta} \in V_\sigma$.

For VFII the boundary conditions we have considered in both cases are

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{0}, && \text{weakly imposed on } \Gamma_v, \\ \boldsymbol{n}\cdot\boldsymbol{\sigma} &= \boldsymbol{0}, && \text{strongly imposed on } \Gamma_\sigma. \end{aligned}$$

What we have called VFI could also be called primal form of the problem, by analogy with what is done for Darcy's equations. Similarly, what we have called VFII could be called dual form of the problem.

	VFI	VFII
Time domain	$\rho\,(\partial_t \boldsymbol{v},\boldsymbol{w}) + (\boldsymbol{C}^{-1} : \partial_t\boldsymbol{\sigma},\boldsymbol{\eta})$ $+B_I([\boldsymbol{v},\boldsymbol{\sigma}],[\boldsymbol{w},\boldsymbol{\eta}]) = L([\boldsymbol{w},\boldsymbol{\eta}])$	$\rho\,(\partial_t \boldsymbol{v},\boldsymbol{w}) + (\boldsymbol{C}^{-1} : \partial_t\boldsymbol{\sigma},\boldsymbol{\eta})$ $+B_{II}([\boldsymbol{v},\boldsymbol{\sigma}],[\boldsymbol{w},\boldsymbol{\eta}]) = L([\boldsymbol{w},\boldsymbol{\eta}])$
Frequency domain	$-\mathrm{i}\,\omega\,\rho\,(\boldsymbol{v},\boldsymbol{w}) - \mathrm{i}\,\omega(\boldsymbol{C}^{-1} : \boldsymbol{\sigma},\boldsymbol{\eta})$ $+B_I([\boldsymbol{v},\boldsymbol{\sigma}],[\boldsymbol{w},\boldsymbol{\eta}]) = L([\boldsymbol{w},\boldsymbol{\eta}])$	$-\mathrm{i}\,\omega\,\rho\,(\boldsymbol{v},\boldsymbol{w}) - \mathrm{i}\,\omega(\boldsymbol{C}^{-1} : \boldsymbol{\sigma},\boldsymbol{\eta})$ $+B_{II}([\boldsymbol{v},\boldsymbol{\sigma}],[\boldsymbol{w},\boldsymbol{\eta}]) = L([\boldsymbol{w},\boldsymbol{\eta}])$
Bilinear form	$B_I([\boldsymbol{v},\boldsymbol{\sigma}],[\boldsymbol{w},\boldsymbol{\eta}])$ $= (\boldsymbol{\sigma},\nabla^S\boldsymbol{w}) - (\nabla^S\boldsymbol{v},\boldsymbol{\eta})$	$B_{II}([\boldsymbol{v},\boldsymbol{\sigma}],[\boldsymbol{w},\boldsymbol{\eta}])$ $= -(\nabla\cdot\boldsymbol{\sigma},\boldsymbol{w}) + (\boldsymbol{v},\nabla\cdot\boldsymbol{\eta})$
V_v	$\{\boldsymbol{w}\in H^1(\Omega)^d \mid \boldsymbol{w}=\boldsymbol{0} \text{ on } \Gamma_v\}$	$L^2(\Omega)^d$
V_σ	$L^2(\Omega)^{d\times d}_{\mathrm{sym}}$	$\{\boldsymbol{\eta}\in H(\mathrm{div},\Omega) \mid \boldsymbol{n}\cdot\boldsymbol{\eta}=\boldsymbol{0} \text{ on } \Gamma_\sigma \text{ and } \boldsymbol{\eta}^{\mathrm{t}}=\boldsymbol{\eta}\}$
Boundary conditions	$\boldsymbol{v}=\boldsymbol{0}$ on Γ_v strongly $\boldsymbol{n}\cdot\boldsymbol{\sigma}=\boldsymbol{0}$ on Γ_σ weakly	$\boldsymbol{v}=\boldsymbol{0}$ on Γ_v weakly $\boldsymbol{n}\cdot\boldsymbol{\sigma}=\boldsymbol{0}$ on Γ_σ strongly

TABLE 1. Mixed form of linear elasticity problems

Table 1 summarizes the ingredients of VFI and VFII for the mixed form of the elasticity equations both in the time domain and in the frequency domain.

2.2.2. *Irreducible form.* The functional setting of the irreducible form of the problem is the same as for VFI, but now the stresses are eliminated from the equations. Thus, if $V_v = \{\boldsymbol{w}\in H^1(\Omega)^d \mid \boldsymbol{w}=\boldsymbol{0} \text{ on } \Gamma_v\}$, the weak form of the irreducible problem in the time domain consists of finding $\boldsymbol{v} : (0,T)\to V_v$ such that

$$\rho\,(\partial_{tt}\boldsymbol{v},\boldsymbol{w}) + (\boldsymbol{C} : \nabla^S\boldsymbol{v},\nabla^S\boldsymbol{w}) = \langle\boldsymbol{f},\boldsymbol{w}\rangle, \tag{17}$$

for all $\boldsymbol{w}\in V_v$, whereas in the frequency domain the problem consists of finding a complex valued $\boldsymbol{v}\in V_v$ such that

$$-\rho\,\omega^2(\boldsymbol{v},\boldsymbol{w}) + (\boldsymbol{C} : \nabla^S\boldsymbol{v},\nabla^S\boldsymbol{w}) = \langle\boldsymbol{f},\boldsymbol{w}\rangle, \tag{18}$$

for all $\boldsymbol{w}\in V_v$. The boundary conditions are now

$$\begin{aligned}\boldsymbol{v}=\boldsymbol{0}, &\quad \text{strongly imposed on } \Gamma_v,\\ \boldsymbol{n}\cdot\boldsymbol{C} : \nabla^S\boldsymbol{v}=\boldsymbol{0}, &\quad \text{weakly imposed on } \Gamma_\sigma,\end{aligned}$$

in both cases.

Regarding the initial conditions for the problem in the time domain, since this is a second problem in time with an elliptic second order differential operator in space, the initial condition for the velocity has to hold in $H^1(\Omega)$ and for the acceleration it has to hold in $L^2(\Omega)$, i.e.:

$$(\boldsymbol{v}|_{t=0},\boldsymbol{w})_{H^1} = (\boldsymbol{v}_0,\boldsymbol{w})_{H^1}\ \forall\boldsymbol{w}\in H^1(\Omega)^d,\quad (\partial_t\boldsymbol{v}|_{t=0},\boldsymbol{w}) = (\dot{\boldsymbol{v}}_0,\boldsymbol{w})\ \forall\boldsymbol{w}\in L^2(\Omega)^d, \tag{19}$$

where $(\cdot,\cdot)_{H^1}$ is the inner product in $H^1(\Omega)^d$.

2.2.3. *Well posedness.* The problems presented at the continuous level are all well posed, but proving their stability requires ingredients that are not necessarily inherited at the discrete level. Referring first to the problem posed in the time domain, the irreducible formulation is a standard linear wave equation with well understood properties. However, the mixed formulation requires a compatibility condition between velocities and stresses that can be expressed as an inf-sup condition, the same as for Darcy's problem [23]. This condition is fulfilled at the continuous level, both for VFI and for VFII, but not necessarily in the FE approximation discussed below. Circumventing this condition justifies the need of stabilized FE formulations.

As for the problem in the time domain, there is an inf-sup condition to be met between the velocity and stress spaces for the mixed form of the problem in the frequency domain, that is satisfied at the continuous level. However, this problem has a particular feature that needs to be highlighted. First, the problem is solvable if the frequency chosen is not an eigenvalue of the spatial wave operator, either in mixed form (i.e., the divergence in the first component and the symmetric gradient in the second) or in irreducible form. But in this case, stability can only be proved through an inf-sup condition, as even the irreducible formulation is not coercive [24, 25]. Furthermore, at the discrete level instabilities may appear due to high frequencies, leading to the so called pollution effect. Even if those can also be treated using stabilized FE formulations, we shall not attempt to treat them in this paper.

3. Finite element approximation

Let us assume for simplicity that Ω is a polyhedral domain and let $\mathcal{T}_h = \{K\}$ be a FE partition of size h. The collection of interior edges will be denoted by $\mathcal{E}_h = \{E\}$. For conciseness, we will assume that the family of FE partitions $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform. The L^2 inner product in K will be represented as $(\cdot,\cdot)_K$, on its boundary as $(\cdot,\cdot)_{\partial K}$, and on E as $(\cdot,\cdot)_E$. The FE spaces to approximate $\boldsymbol{v}$ and $\boldsymbol{\sigma}$ will be written as $V_{v,h} \subset V_v$ and $V_{\sigma,h} \subset V_\sigma$, respectively, i.e., only conforming approximations will be considered. The crucial aspect of our approximation is that spaces $V_{v,h}$ and $V_{\sigma,h}$ can be *of any polynomial order*. In the case of mixed formulations, special care is needed to construct the stress tensor FE space to yield a stable numerical formulation; the approach we favor is to use arbitrary interpolations for velocity and stress and modify the standard Galerkin method using a stabilized FE formulation, in our case case based on the Variational Multi-Scale (VMS) approach [13, 14, 26]. This stabilized formulation is obtained by introducing the Sub-Grid Scale (SGS) spaces for $\boldsymbol{v}$ and $\boldsymbol{\sigma}$, V_v' and V_σ', respectively, such that

$$V_v = V_{v,h} \oplus V_v', \quad V_\sigma = V_{\sigma,h} \oplus V_\sigma'.$$

This implies that the unknowns of the problem and their respective test functions can also be decomposed into the FE part and the SGS part, as $\boldsymbol{v} = \boldsymbol{v}_h + \boldsymbol{v}'$, $\boldsymbol{w} = \boldsymbol{w}_h + \boldsymbol{w}'$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}_h + \boldsymbol{\sigma}'$ and $\boldsymbol{\eta} = \boldsymbol{\eta}_h + \boldsymbol{\eta}'$. Here and below, the prime will be used to denote SGS functions and SGS spaces.

3.1. Finite element approximation in the time domain.

Let us start with the mixed formulation of the problem in the time domain. In this case, let us introduce the differential operator $\mathcal{L} = [\mathcal{L}_v, \mathcal{L}_\sigma]$ and its transpose $\mathcal{L}^{\mathrm{t}} = [\mathcal{L}_v^{\mathrm{t}}, \mathcal{L}_\sigma^{\mathrm{t}}]$, defined as

$$\mathcal{L}_v([\boldsymbol{v}_h, \boldsymbol{\sigma}_h]) = -\nabla\cdot\boldsymbol{\sigma}_h, \qquad \mathcal{L}_\sigma([\boldsymbol{v}_h, \boldsymbol{\sigma}_h]) = -\nabla^S \boldsymbol{v}_h,$$
$$\mathcal{L}_v^{\mathrm{t}}([\boldsymbol{w}_h, \boldsymbol{\eta}_h]) = \nabla\cdot\boldsymbol{\eta}_h, \qquad \mathcal{L}_\sigma^{\mathrm{t}}([\boldsymbol{w}_h, \boldsymbol{\eta}_h]) = \nabla^S \boldsymbol{w}_h,$$

as well as the temporal operator $\mathcal{D}_t = [\mathcal{D}_{t,v}, \mathcal{D}_{t,\sigma}]$ defined as

$$\mathcal{D}_{t,v}([\boldsymbol{v}_h, \boldsymbol{\sigma}_h]) = \rho\partial_t \boldsymbol{v}_h, \qquad \mathcal{D}_{t,\sigma}([\boldsymbol{v}_h, \boldsymbol{\sigma}_h]) = \boldsymbol{C}^{-1} : \partial_t\boldsymbol{\sigma}.$$

This notation allows us to write the differential equations of the problem in mixed form as

$$\mathcal{D}_t([\boldsymbol{v}_h,\boldsymbol{\sigma}_h]) + \mathcal{L}([\boldsymbol{v}_h,\boldsymbol{\sigma}_h]) = [\boldsymbol{f},\boldsymbol{0}].$$

We will not describe in detail all the steps to arrive to the VMS formulation we propose (see [26] for an overview), but we will only point out those that are relevant to the problem at hand and, in particular, to the different approximation required for VFI and VFII.

For the sake of conciseness, and since our objective is not to analyze what happens when the time steps of the discretization in time introduced later are very small, we will consider throughout quasi-static SGSs, i.e., the time derivative of both the velocity and the stress SGS will be considered negligible (see again [26] for a general discussion on this point).

Splitting the unknowns into their FE components and their SGS components and integrating by parts terms containing derivatives of the SGSs, the stabilized form of the discrete version of VFI projected onto the FE space can be written as: find $\boldsymbol{v}_h : (0,T) \to V_{v,h}$ and $\boldsymbol{\sigma}_h : (0,T) \to V_{\sigma,h}$ such that

$$\begin{aligned}\rho\,(\partial_t \boldsymbol{v}_h, \boldsymbol{w}_h) + (\boldsymbol{C}^{-1} : \partial_t \boldsymbol{\sigma}_h, \boldsymbol{\eta}_h) + B_I([\boldsymbol{v}_h,\boldsymbol{\sigma}_h],[\boldsymbol{w}_h,\boldsymbol{\eta}_h]) + \sum_K (\boldsymbol{v}', \mathcal{L}_v^{\mathrm{t}}([\boldsymbol{w}_h,\boldsymbol{\eta}_h]))_K \\ + \sum_K (\boldsymbol{\sigma}', \mathcal{L}_\sigma^{\mathrm{t}}([\boldsymbol{w}_h,\boldsymbol{\eta}_h]))_K - \sum_E (\boldsymbol{v}', [\![\boldsymbol{n}\cdot\boldsymbol{\eta}_h]\!])_E = L([\boldsymbol{w}_h,\boldsymbol{\eta}_h]),\end{aligned} \tag{20}$$

which must hold for all $[\boldsymbol{w}_h,\boldsymbol{\eta}_h] \in V_{v,h} \times V_{\sigma,h}$. The last term in the left-hand-side (LHS) of this equation stems from assuming $\boldsymbol{v}'$ to be single valued on the edges and noting that

$$\sum_K (\boldsymbol{v}', \boldsymbol{n}\cdot\boldsymbol{\eta}_h)_{\partial K} = \sum_E (\boldsymbol{v}', [\![\boldsymbol{n}\cdot\boldsymbol{\eta}_h]\!])_E,$$

where we have introduced the jump

$$[\![\boldsymbol{n}\cdot\boldsymbol{T}]\!]_E := \begin{cases} \boldsymbol{n}\cdot\boldsymbol{T}|_{\partial K_1} + \boldsymbol{n}\cdot\boldsymbol{T}|_{\partial K_2} & \text{if } E \text{ is shared by elements } K_1 \text{ and } K_2, \\ \boldsymbol{n}\cdot\boldsymbol{T}|_{\partial K\cap\partial\Omega} & \text{if } E\subset\partial K \text{ belongs to } \partial\Omega. \end{cases}$$

In this definition, $\boldsymbol{T}$ can be either a tensor or a vector.

Likewise, the stabilized FE approximation for VFII can be written as: find $\boldsymbol{v}_h : (0,T) \to V_{v,h}$ and $\boldsymbol{\sigma}_h : (0,T) \to V_{\sigma,h}$ such that

$$\begin{aligned}\rho\,(\partial_t \boldsymbol{v}_h, \boldsymbol{w}_h) + (\boldsymbol{C}^{-1} : \partial_t \boldsymbol{\sigma}_h, \boldsymbol{\eta}_h) + B_{II}([\boldsymbol{v}_h,\boldsymbol{\sigma}_h],[\boldsymbol{w}_h,\boldsymbol{\eta}_h]) + \sum_K (\boldsymbol{v}', \mathcal{L}_v^{\mathrm{t}}([\boldsymbol{w}_h,\boldsymbol{\eta}_h]))_K \\ + \sum_K (\boldsymbol{\sigma}', \mathcal{L}_\sigma^{\mathrm{t}}([\boldsymbol{w}_h,\boldsymbol{\eta}_h]))_K - \sum_E (\boldsymbol{\sigma}', [\![\boldsymbol{n}\otimes\boldsymbol{w}_h]\!])_E = L([\boldsymbol{w}_h,\boldsymbol{\eta}_h]),\end{aligned} \tag{21}$$

where now we have considered $\boldsymbol{\sigma}'$ single valued on the edges.

The key point is now to approximate $\boldsymbol{v}'$ and $\boldsymbol{\sigma}'$ in the element interiors and on the edges. We do this as follows (see [15, 8, 26] for background):

$$\boldsymbol{v}'|_K = \tau_v \tilde{P}_h[\boldsymbol{f} - \mathcal{D}_{t,v}([\boldsymbol{v}_h,\boldsymbol{\sigma}_h]) - \mathcal{L}_v([\boldsymbol{v}_h,\boldsymbol{\sigma}_h])]_K, \tag{22}$$

$$\boldsymbol{\sigma}'|_K = \tau_\sigma \tilde{P}_h[-\mathcal{D}_{t,\sigma}([\boldsymbol{v}_h,\boldsymbol{\sigma}_h]) - \mathcal{L}_\sigma([\boldsymbol{v}_h,\boldsymbol{\sigma}_h])]_K, \tag{23}$$

$$\boldsymbol{v}'|_E = -\delta_\sigma [\![\boldsymbol{n}\cdot\boldsymbol{\sigma}_h]\!]|_E \qquad \text{for VF I}, \tag{24}$$

$$\boldsymbol{\sigma}'|_E = -\delta_v [\![\boldsymbol{n}\otimes\boldsymbol{v}_h]\!]|_E \qquad \text{for VF II}, \tag{25}$$

where $\tilde{P}_h$ is the projection onto the space of SGSs, still to be chosen, and $\tau_v, \tau_\sigma, \delta_v, \delta_\sigma$ are stabilization parameters whose expression is introduced later on. Observe that $\tilde{P}_h$ projects onto the space of velocity SGSs in Eq. (22) and onto the space of stress SGSs in Eq. (23). We will not distinguish both, as the argument determines which is the projection considered.

When these expressions are inserted into Eq. (20) and Eq. (21) we obtain the final discrete version of VFI and VFII, respectively. It only remains to state the initial conditions for these problems. In both cases, the initial velocity and the initial stress can be taken as the $L^2(\Omega)$ projection of $\boldsymbol{v}_0$ and $\boldsymbol{\sigma}_0$, respectively, onto the corresponding FE space, that is to say:

$$(\boldsymbol{v}_h|_{t=0}, \boldsymbol{w}_h) = (\boldsymbol{v}_0, \boldsymbol{w}_h) \qquad \forall \boldsymbol{w}_h \in \bar{V}_{v,h},$$
$$(\boldsymbol{\sigma}_h|_{t=0}, \boldsymbol{\eta}_h) = (\boldsymbol{\sigma}_0, \boldsymbol{\eta}_h) \qquad \forall \boldsymbol{w}_h \in \bar{V}_{\sigma,h},$$

where $\bar{V}_{v,h}$ and $\bar{V}_{\sigma,h}$ are respectively constructed as $V_{v,h}$ and $V_{\sigma,h}$ without imposing the boundary conditions.

Even though it is not our purpose to study in detail the stability of the formulation, let us briefly indicate from where does this stability come. The Galerkin terms neither contribute to stability nor worsen it, as $B_I([\boldsymbol{v}_h, \boldsymbol{\sigma}_h], [\boldsymbol{v}_h, \boldsymbol{\sigma}_h]) = B_{II}([\boldsymbol{v}_h, \boldsymbol{\sigma}_h], [\boldsymbol{v}_h, \boldsymbol{\sigma}_h]) = 0$. It can be noted that the SGS $\boldsymbol{v}'$ will lead to a contribution of the form $\sum_K \tau_v(\tilde{P}_h[\nabla \cdot \boldsymbol{\sigma}_h], \nabla \cdot \boldsymbol{\eta}_h)_K$, whereas $\boldsymbol{\sigma}'$ will lead to the term $\sum_K \tau_\sigma(\tilde{P}_h[\nabla^S \boldsymbol{v}_h], \nabla^S \boldsymbol{w}_h)_K$. Thus, $\boldsymbol{v}'$ will provide some stability on the divergence of the stress and $\boldsymbol{\sigma}'$ on the gradient of the velocity. The amount of stabilization required depends on the variational form of the problem, and is driven by the stabilization parameters, as discussed below. Furthermore, VFI does not require continuity of the normal stresses, as well as VFII does not require continuity of the velocity. If we choose FE spaces without this continuity, $\boldsymbol{v}'$ on the edges contributes to enhance the stability of the stresses when their normal component is discontinuous in VFI, and $\boldsymbol{\sigma}'$ on the edges contributes to enhance the stability of the velocities when these are chosen discontinuous in VFII.

Regarding the choice of the projection $\tilde{P}_h$, if the space of SGSs is taken as the space of FE residuals, $\tilde{P}_h$ is the identity, and we call Algebraic Sub-Grid Scale (ASGS) the resulting method [17]. Alternatively, if the space of SGSs is taken as the orthogonal to the FE space, the projection onto this space is $\tilde{P}_h = I - P_h$, I being the identity and P_h the projection onto the FE space; this is what we call the Orthogonal Sub-Grid Scale (OSGS) method [18].

In the case of the irreducible equation in the time domain, the standard Galerkin method is stable. The semi-discrete problem (discretized in space, continuous in time) reads: find $\boldsymbol{v} : (0, T) \to V_{v,h}$ such that

$$\rho(\partial_{tt}\boldsymbol{v}_h, \boldsymbol{w}_h) + (\boldsymbol{C} : \nabla^S \boldsymbol{v}_h, \nabla^S \boldsymbol{w}_h) = \langle \boldsymbol{f}, \boldsymbol{w}_h \rangle, \qquad \forall \boldsymbol{w}_h \in V_{v,h}, \tag{26}$$
$$(\boldsymbol{v}_h|_{t=0}, \boldsymbol{w}_h)_{H^1} = (\boldsymbol{v}_0, \boldsymbol{w}_h)_{H^1}, \qquad \forall \boldsymbol{w}_h \in \bar{V}_{v,h}, \tag{27}$$
$$(\partial_t \boldsymbol{v}_h|_{t=0}, \boldsymbol{w}_h) = (\dot{\boldsymbol{v}}_0, \boldsymbol{w}_h) \qquad \forall \boldsymbol{w}_h \in \bar{V}_{v,h}. \tag{28}$$

3.2. **Time discretization.** The two equations studied in the previous section were in the time domain, and therefore we also need to discretize them in time. Being our discussion independent of the time integration scheme employed, let us just mention that in the numerical examples we will use backward differences (BDF) of second order (BDF2).

Let $0 = t^0 < \cdots < t^n < \cdots < t^N = T$ be a partition of the time interval $(0, T)$, for simplicity uniform and of size δt. Being $f(t)$ a generic time dependent function and $f^n = f(t^n)$, the BDF2 scheme for the first time derivative consists of approximating

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=t^n} \approx \frac{3f^n - 4f^{n-1} + f^{n-2}}{2\delta t},$$

and the BDF2 scheme for the second time derivative consists of approximating

$$\left.\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}\right|_{t=t^n} \approx \frac{2f^n - 5f^{n-1} + 4f^{n-2} - f^{n-3}}{\delta t^2}.$$

In both cases, the first step is computed with the corresponding first order approximation to initialize the integration in time.

Despite BDF schemes are not common to integrate wave problems, at least in their irreducible version, we have found them quite effective in the mixed form of the elasticity problem. Nevertheless, any other classical finite difference scheme can be employed.

3.3. **Finite element approximation in the frequency domain.** In the case of the mixed form of the problem in the frequency domain, we will proceed to present the FE approximation following the same steps as in the time domain version. The main difference is that the term coming from the temporal derivatives assuming a harmonic time dependence will be considered part of the spatial operator.

Let us introduce the differential operator $\mathcal{L} = [\mathcal{L}_v, \mathcal{L}_\sigma]$ and its transpose as

$$\mathcal{L}_v([\boldsymbol{v}_h, \boldsymbol{\sigma}_h]) = -\mathrm{i}\,\rho\,\omega\,\boldsymbol{v}_h - \nabla\cdot\boldsymbol{\sigma}_h, \qquad \mathcal{L}_\sigma([\boldsymbol{v}_h, \boldsymbol{\sigma}_h]) = -\mathrm{i}\,\omega\,\boldsymbol{C}^{-1} : \boldsymbol{\sigma}_h - \nabla^S \boldsymbol{v}_h,$$
$$\mathcal{L}_v^{\mathrm{t}}([\boldsymbol{w}_h, \boldsymbol{\eta}_h]) = -\mathrm{i}\,\rho\,\omega\,\boldsymbol{w}_h + \nabla\cdot\boldsymbol{\eta}_h, \qquad \mathcal{L}_\sigma^{\mathrm{t}}([\boldsymbol{w}_h, \boldsymbol{\eta}_h]) = -\mathrm{i}\,\omega\,\boldsymbol{C}^{-1} : \boldsymbol{\eta}_h + \nabla^S \boldsymbol{w}_h.$$

which allow us to write the differential equations of the problem in mixed form as

$$\mathcal{L}([\boldsymbol{v}_h, \boldsymbol{\sigma}_h]) = [\boldsymbol{f}, \boldsymbol{0}].$$

Note that all functions involved are now complex valued. In particular, the adjoint of $\mathcal{L}$ is the complex conjugate of $\mathcal{L}^{\mathrm{t}}$, although we prefer to write the FE formulation proposed in terms of $\mathcal{L}^{\mathrm{t}}$.

The stabilized FE formulation for VFI in the frequency domain can be written as: find $\boldsymbol{v}_h \in V_{v,h}$ and $\boldsymbol{\sigma}_h \in V_{\sigma,h}$ such that

$$\begin{aligned} &-\mathrm{i}\,\rho\,\omega\,(\boldsymbol{v}_h, \boldsymbol{w}_h) - \mathrm{i}\,\omega(\boldsymbol{C}^{-1} : \boldsymbol{\sigma}_h, \boldsymbol{\eta}_h) + B_I([\boldsymbol{v}_h, \boldsymbol{\sigma}_h], [\boldsymbol{w}_h, \boldsymbol{\eta}_h]) + \sum_K (\boldsymbol{v}', \mathcal{L}_v^{\mathrm{t}}([\boldsymbol{w}_h, \boldsymbol{\eta}_h]))_K \\ &\quad + \sum_K (\boldsymbol{\sigma}', \mathcal{L}_\sigma^{\mathrm{t}}([\boldsymbol{w}_h, \boldsymbol{\eta}_h]))_K - \sum_E (\boldsymbol{v}', [\![\boldsymbol{n}\cdot\boldsymbol{\eta}_h]\!])_E = L([\boldsymbol{w}_h, \boldsymbol{\eta}_h]), \end{aligned} \tag{29}$$

which must hold for all $[\boldsymbol{w}_h, \boldsymbol{\eta}_h] \in V_{v,h} \times V_{\sigma,h}$. Note that having chosen the transpose rather that the adjoint operator in the stabilization terms, this equation is the same as (20) except for the terms involving temporal derivatives.

Likewise, the stabilized FE formulation for VFII in the frequency domain can be written as: find $\boldsymbol{v}_h \in V_{v,h}$ and $\boldsymbol{\sigma}_h \in V_{\sigma,h}$ such that

$$\begin{aligned} &-\mathrm{i}\,\rho\,\omega\,(\boldsymbol{v}_h, \boldsymbol{w}_h) - \mathrm{i}\,\omega(\boldsymbol{C}^{-1} : \boldsymbol{\sigma}_h, \boldsymbol{\eta}_h) + B_{II}([\boldsymbol{v}_h, \boldsymbol{\sigma}_h], [\boldsymbol{w}_h, \boldsymbol{\eta}_h]) + \sum_K (\boldsymbol{v}', \mathcal{L}_v^{\mathrm{t}}([\boldsymbol{w}_h, \boldsymbol{\eta}_h]))_K \\ &\quad + \sum_K (\boldsymbol{\sigma}', \mathcal{L}_\sigma^{\mathrm{t}}([\boldsymbol{w}_h, \boldsymbol{\eta}_h]))_K - \sum_{\partial K} (\boldsymbol{\sigma}', [\![\boldsymbol{n}\otimes\boldsymbol{w}_h]\!])_E = L([\boldsymbol{w}_h, \boldsymbol{\eta}_h]), \end{aligned} \tag{30}$$

which must hold for all $[\boldsymbol{w}_h, \boldsymbol{\eta}_h] \in V_{v,h} \times V_{\sigma,h}$. Even though this equations looks very similar to Eq. (21), let us emphasize that now the operator $\mathcal{L}^{\mathrm{t}}$ contains additional terms.

Finally, the approximation we propose for the SGSs in the element interiors and on the edges is:

$$\begin{aligned} \boldsymbol{v}'|_K &= \tau_v \tilde{P}_h[\boldsymbol{f} - \mathcal{L}_v([\boldsymbol{v}_h, \boldsymbol{\sigma}_h])]_K, \\ \boldsymbol{\sigma}'|_K &= \tau_\sigma \tilde{P}_h[-\mathcal{L}_\sigma([\boldsymbol{v}_h, \boldsymbol{\sigma}_h])]_K, \\ \boldsymbol{v}'|_E &= -\delta_\sigma [\![\boldsymbol{n}\cdot\boldsymbol{\sigma}_h]\!]|_E \qquad \text{for VF I}, \\ \boldsymbol{\sigma}'|_E &= -\delta_v [\![\boldsymbol{n}\otimes\boldsymbol{v}_h]\!]|_E \qquad \text{for VF II}. \end{aligned}$$

The same comments concerning these expressions as in the problem in the time domain apply now.

In the case of the irreducible equation in the frequency domain, the standard Galerkin method is stable in most cases. It is known that for high frequencies ω the pollution error can appear in the same way as it does for the Helmholtz equation [24, 27]. The literature about this topic is vast; let us only mention that this instability can be solved by adding stabilization on the edges of the elements and a proper design of the stabilization parameters (see e.g. [16]). Taking into account that solving this specific instability issue is not the goal of this paper, we consider that the problem reads: find $\boldsymbol{v}_h \in V_{v,h}$ such that

$$-\rho\,\omega^2(\boldsymbol{v}_h, \boldsymbol{w}_h) + (\boldsymbol{C} : \nabla^S \boldsymbol{v}_h, \nabla^S \boldsymbol{w}_h) = \langle \boldsymbol{f}, \boldsymbol{w}_h \rangle, \qquad \forall \boldsymbol{w}_h \in V_{v,h}. \tag{31}$$

3.4. **Stabilization parameters.** To complete the description of the formulation, we need to choose the stabilization parameters. The values of τ_v and τ_σ will be the same for the two mixed formulations and are motivated in the same way as it was done in [8], with the arguments and approximations exposed in [28, 21]. The final form of these parameters is

$$\tau_v = C_{\tau_v}\sqrt{\frac{1}{E\rho}}\,h\sqrt{\frac{l_v}{l_\sigma}}, \qquad \tau_\sigma = C_{\tau_\sigma}\sqrt{E\rho}\,h\sqrt{\frac{l_\sigma}{l_v}}, \tag{32}$$

where C_{τ_v} and C_{τ_σ} are dimensionless algorithmic constants, h is the length of the element in which the stabilization parameter is computed and l_v, l_σ, are the length scales of the problem. The values of these length scales are taken as shown in Table 2, where L_0 is a characteristic length scale of the problem domain (Ω). Note that *the length scales l_v and l_σ depend on whether VFI or VFII is used.* See also [8] for further discussion.

Variational form	I	II
τ_v	$\mathcal{O}(h^2)$	$\mathcal{O}(1)$
τ_σ	$\mathcal{O}(1)$	$\mathcal{O}(h^2)$
l_v	h	L_0^2/h
l_σ	L_0^2/h	h

TABLE 2. Stabilization parameters order and length scale definitions.

The stabilization parameters on the element edges that appear in the mixed formulations are calculated as

$$\delta_v = C_\delta\sqrt{E\rho}\frac{h}{L_0}, \qquad \delta_\sigma = C_\delta\sqrt{\frac{1}{E\rho}}\frac{h}{L_0}, \tag{33}$$

where C_δ is again a dimensionless algorithmic constant. Note that δ_v is only needed for VFI and δ_σ only for VFII.

4. Matrix version of the discrete problem

In this section we introduce the matrix version of the formulations presented in the previous one and discuss some issues regarding its numerical implementation. The time discretization has been discussed in section 3.2; however, for clearness, time is not discretized in the following equations.

Let us start with the irreducible form of the problem, which offers no difficulty. In the time domain, the matrix version of the differential equation (26) is

$$\mathsf{M}_v\ddot{\mathsf{v}} + \mathsf{K}_v\mathsf{v} = \mathsf{f},$$

whereas in the frequency domain the matrix version of Eq. (31) takes the form

$$-\omega^2\mathsf{M}_v\mathsf{v} + \mathsf{K}_v\mathsf{v} = \mathsf{f}.$$

In these expressions, v is the array of degrees of freedom of $\boldsymbol{v}_h$, the dot denotes time differentiation and the identification of array f coming from the forcing term, the mass matrix M_v and the stiffness matrix K_v is straightforward. Note that M_v includes the density and K_v the material properties.

Let us move our attention to the mixed form of the problem, starting with the time domain version given by Eqs. (20) and (21). The stabilization terms in these equations have been written in terms of the differential operators of the problem to emphasize their origin, but they can be explicitly written as

$$\sum_K \tau_v(\tilde{P}_h[\boldsymbol{f} - \rho\,\partial_t \boldsymbol{v}_h + \nabla\cdot\boldsymbol{\sigma}_h], \nabla\cdot\boldsymbol{\eta}_h)_K + \sum_K \tau_\sigma(\tilde{P}_h[-\boldsymbol{C}^{-1} : \partial_t\boldsymbol{\sigma}_h + \nabla^S\boldsymbol{v}_h], \nabla^S\boldsymbol{w}_h)_K$$
$$+ \sum_E \delta_\sigma([\![\boldsymbol{n}\cdot\boldsymbol{\sigma}_h]\!], [\![\boldsymbol{n}\cdot\boldsymbol{\eta}_h]\!])_E + \sum_E \delta_v([\![\boldsymbol{n}\otimes\boldsymbol{v}_h]\!], [\![\boldsymbol{n}\otimes\boldsymbol{w}_h]\!])_E.$$

The term multiplied by δ_σ is only needed for VFI, as in this case $V_{v,h}$ must be made of continuous functions and the last term vanishes, whereas the term multiplied by δ_v is only needed for VFII, since in this case the normal component of the stresses in $V_{\sigma,h}$ must be continuous and the term multiplied by δ_σ vanishes.

Let us comment on the treatment of the orthogonal projection when $\tilde{P}_h = P_h^\perp = I - P_h$. That could be treated implicitly, particularly using iterative solvers, but in order to avoid increasing the stencil of the matrices of the problem we will treat it explicitly. Furthermore, $P_h^\perp[\rho\,\partial_t\boldsymbol{v}_h] = \boldsymbol{0}$ and $P_h^\perp[\boldsymbol{C}^{-1} : \partial_t\boldsymbol{\sigma}_h] = \boldsymbol{0}$, and we can also neglect $P_h^\perp[\boldsymbol{f}]$, as this does not alter the accuracy of the method. Thus, we can compute:

$$\sum_K \tau_v(P_h^\perp[\nabla\cdot\boldsymbol{\sigma}_h], \nabla\cdot\boldsymbol{\eta}_h)_K = \sum_K \tau_v(\nabla\cdot\boldsymbol{\sigma}_h, \nabla\cdot\boldsymbol{\eta}_h)_K - \sum_K \tau_v(P_h[\nabla\cdot\boldsymbol{\sigma}_h], \nabla\cdot\boldsymbol{\eta}_h)_K,$$
$$\sum_K \tau_\sigma(P_h^\perp[\nabla^S\boldsymbol{v}_h], \nabla^S\boldsymbol{w}_h)_K = \sum_K \tau_\sigma(\nabla^S\boldsymbol{v}_h, \nabla^S\boldsymbol{w}_h)_K - \sum_K \tau_\sigma(P_h[\nabla^S\boldsymbol{v}_h], \nabla^S\boldsymbol{w}_h)_K.$$

The last term in these expressions will be evaluated from known values of the unknowns, and thus it will contribute to the RHS of the discrete system.

As before, let v be the array of degrees of freedom of $\boldsymbol{v}_h$, and let now s be the array of degrees of freedom of $\boldsymbol{\sigma}_h$. To write the matrix version of the problem, we will use the following notation. The contribution from the stabilization terms in the element interiors will be labelled with subscript τ, whereas the contribution from the edges will be labelled with subscript δ. A double subscript will be used to indicate the position of a certain block matrix, v referring to the momentum equation and blocks that multiply v and σ referring to the constitutive equation and blocks that multiply s. Terms that only appear using the ASGS method are indicated with the superscript 'asgs' and those that only appear in the OSGS formulation with the superscript 'osgs'.

Let G_S be the matrix that arises from the symmetric gradient, coming from the contribution of the Galerkin terms. Problem (20) for VFI yields the following system of ordinary differential equations:

$$\begin{bmatrix} \mathsf{M}_{vv} & \mathsf{M}^{\rm asgs}_{\tau,v\sigma} \\ \mathsf{M}^{\rm asgs}_{\tau,\sigma v} & \mathsf{M}_{\sigma\sigma} \end{bmatrix} \begin{bmatrix} \dot{\mathsf{v}} \\ \dot{\mathsf{s}} \end{bmatrix} + \begin{bmatrix} \mathsf{S}_{\tau,vv} & \mathsf{G}_S^{\rm t} \\ -\mathsf{G}_S & \mathsf{J}_{\delta,\sigma\sigma} + \mathsf{S}_{\tau,\sigma\sigma} \end{bmatrix} \begin{bmatrix} \mathsf{v} \\ \mathsf{s} \end{bmatrix} = \begin{bmatrix} \mathsf{f} + \mathsf{f}^{\rm osgs}_{\tau,v}(\mathsf{v}) \\ \mathsf{f}^{\rm asgs}_{\tau} + \mathsf{f}^{\rm osgs}_{\tau,\sigma}(\mathsf{s}) \end{bmatrix}.$$

The dependence of $\mathsf{f}^{\rm osgs}_{\tau,v}(\mathsf{v})$ and $\mathsf{f}^{\rm osgs}_{\tau,\sigma}(\mathsf{s})$ on the unknowns has been explicitly displayed. When the equations are discretized in time, they can be evaluated using values of v and s of previous time steps, or an iterative strategy can be employed and then values of previous iterations can be used. Note that a delicate issue of the ASGS method is the modification of the mass matrix of the system. Now it can be shown to be symmetric, but there is no guarantee of being positive definite if the stabilization parameters are not small.

Moving to VFII, let D be the matrix arising from the divergence of a tensor tested with a vector function. Problem (21) for VFII yields now the following system of ordinary differential equations:

$$\begin{bmatrix} \mathsf{M}_{vv} & \mathsf{M}^{\rm asgs}_{\tau,v\sigma} \\ \mathsf{M}^{\rm asgs}_{\tau,\sigma v} & \mathsf{M}_{\sigma\sigma} \end{bmatrix} \begin{bmatrix} \dot{\mathsf{v}} \\ \dot{\mathsf{s}} \end{bmatrix} + \begin{bmatrix} \mathsf{J}_{\delta,vv} + \mathsf{S}_{\tau,vv} & -\mathsf{D} \\ \mathsf{D}^{\rm t} & \mathsf{S}_{\tau,\sigma\sigma} \end{bmatrix} \begin{bmatrix} \mathsf{v} \\ \mathsf{s} \end{bmatrix} = \begin{bmatrix} \mathsf{f} + \mathsf{f}^{\rm osgs}_{\tau,v}(\mathsf{v}) \\ \mathsf{f}^{\rm asgs}_{\tau} + \mathsf{f}^{\rm osgs}_{\tau,\sigma}(\mathsf{s}) \end{bmatrix}.$$

Only the stiffness matrix changes with respect to the problem associated to VFI.

Let us consider now the equations in the frequency domain. Now the stabilization terms in the interior of the elements can be explicitly written as

$$\sum_K \tau_v(\tilde{P}_h[\boldsymbol{f} + \mathrm{i}\,\omega\,\rho\,\boldsymbol{v}_h + \nabla\cdot\boldsymbol{\sigma}_h], \mathrm{i}\,\omega\,\rho\,\boldsymbol{w}_h + \nabla\cdot\boldsymbol{\eta}_h)_K$$
$$+ \sum_K \tau_\sigma(\tilde{P}_h[\mathrm{i}\,\omega\,\boldsymbol{C}^{-1}:\boldsymbol{\sigma}_h + \nabla^S\boldsymbol{v}_h], \mathrm{i}\,\omega\,\boldsymbol{C}^{-1}:\boldsymbol{\eta}_h + \nabla^S\boldsymbol{w}_h)_K.$$

When the OSGS method is used, all zero order terms disappear, as their projection orthogonal to the FE space is zero. For the ASGS method, apart from changing $\dot{\mathsf{v}}$ by $-\mathrm{i}\,\omega\,\mathsf{v}$ and $\dot{\mathsf{s}}$ by $-\mathrm{i}\,\omega\,\mathsf{s}$, there are new contributions form the stabilization terms due to the presence of $\mathrm{i}\,\omega\,\rho\,\boldsymbol{w}_h$ and $\mathrm{i}\,\omega\,\boldsymbol{C}^{-1}:\boldsymbol{\eta}_h$. The final algebraic system to be solved has the following matrix structure for VFI:

$$\begin{bmatrix} -\mathrm{i}\,\omega\,\mathsf{M}_{vv} + \omega^2\mathsf{M}^{\mathrm{asgs}}_{\tau,vv} + \mathsf{S}_{\tau,vv} & -\mathrm{i}\,\omega\,\mathsf{M}^{\mathrm{asgs}}_{\tau,v\sigma} + \mathrm{i}\,\omega\,\mathsf{S}^{\mathrm{asgs}}_{\tau,v\sigma} + \mathsf{G}^{\mathrm{t}}_S \\ -\mathrm{i}\,\omega\,\mathsf{M}^{\mathrm{asgs}}_{\tau,\sigma v} + \mathrm{i}\,\omega\,\mathsf{S}^{\mathrm{asgs}}_{\tau,\sigma v} - \mathsf{G}_S & -\mathrm{i}\,\omega\,\mathsf{M}_{\sigma\sigma} + \omega^2\mathsf{M}^{\mathrm{asgs}}_{\tau,\sigma\sigma} + \mathsf{J}_{\delta,\sigma\sigma} + \mathsf{S}_{\tau,\sigma\sigma} \end{bmatrix} \begin{bmatrix} \mathsf{v} \\ \mathsf{s} \end{bmatrix}$$
$$= \begin{bmatrix} \mathsf{f} + \mathsf{f}^{\mathrm{osgs}}_{\tau,v}(\mathsf{v}) \\ \mathrm{i}\,\omega\,\mathsf{f}^{\mathrm{asgs}}_{\tau} + \mathsf{f}^{\mathrm{osgs}}_{\tau,\sigma}(\mathsf{s}) \end{bmatrix}.$$

The modifications for VFII are the same as for the problem in the time domain.

It is observed that there are many terms in the ASGS formulation that do not appear in the OSGS method, although in this case one needs to evaluate the projection of the unknown to compute an additional RHS term. In the problem in the frequency domain, this needs to be done in an iterative scheme if one wishes to maintain the compactness of the stencil of the Galerkin method.

5. Numerical Examples

5.1. **The swinging plate.** This example is an usual 2D accuracy test that allows closed-form solutions for all state variables (see [29, 6]). The domain of the problem is a square $\Omega = (0,2)$ m $\times$ $(0,2)$ m with coordinates $\boldsymbol{x} = [x_1, x_2]$. We basically want to use this example to prove that our stabilization methods work for the mixed equations in both the time and the frequency domains and in both variational forms, VFI and VFII. In order to do that, it is necessary to have the analytical solutions of this problem, which will be different for the time and frequency domains.

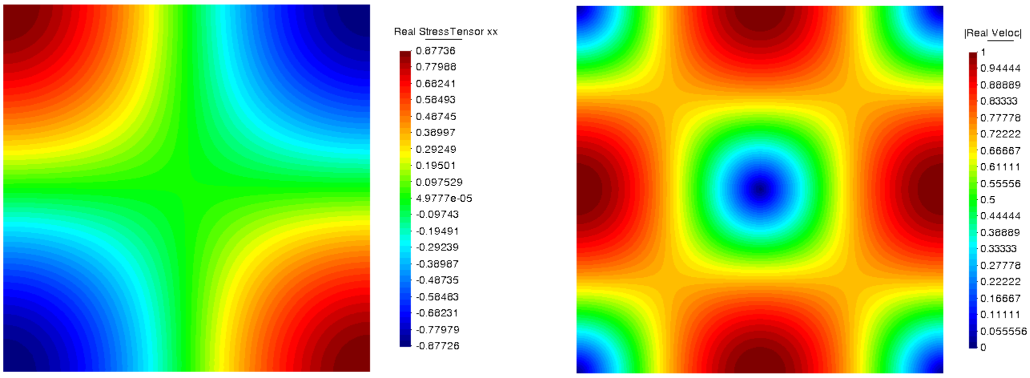
On the one hand, the analytical velocity and stress in the time domain are

$$\boldsymbol{v}(\boldsymbol{x},t) = \omega V_0\cos(\omega t)\begin{bmatrix} -\sin(\frac{\pi}{2}x_1)\cos(\frac{\pi}{2}x_2) \\ \cos(\frac{\pi}{2}x_1)\sin(\frac{\pi}{2}x_2) \end{bmatrix}\ \mathrm{m/s},$$
$$\boldsymbol{\sigma}(\boldsymbol{x},t) = \mu\pi\sin(\omega t)\cos\left(\frac{\pi}{2}x_1\right)\cos\left(\frac{\pi}{2}x_2\right)\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\ \mathrm{N/m^2},$$

where V_0 is a positive constant and we define $\omega = \pi\sqrt{\mu/2\rho}$. This angular frequency will also be used as the dominant frequency for the problem in the frequency domain, whose analytical velocity and stresses are

$$\boldsymbol{v}(\boldsymbol{x}) = V_0\begin{bmatrix} -\sin(\frac{\pi}{2}x_1)\cos(\frac{\pi}{2}x_2) \\ \cos(\frac{\pi}{2}x_1)\sin(\frac{\pi}{2}x_2) \end{bmatrix} + \mathrm{i}\,V_0\begin{bmatrix} \sin(\frac{\pi}{2}x_1)\cos(\frac{\pi}{2}x_2) \\ -\cos(\frac{\pi}{2}x_1)\sin(\frac{\pi}{2}x_2) \end{bmatrix}\ \mathrm{m/s},$$
$$\boldsymbol{\sigma}(\boldsymbol{x}) = \frac{\mu\pi}{\omega}\cos\left(\frac{\pi}{2}x_1\right)\cos\left(\frac{\pi}{2}x_2\right)\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + \mathrm{i}\,\frac{\mu\pi}{\omega}\cos\left(\frac{\pi}{2}x_1\right)\cos\left(\frac{\pi}{2}x_2\right)\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\ \mathrm{N/m^2}.$$

For the problem in the time domain, we need to apply the initial conditions ($\boldsymbol{v}(\boldsymbol{x},0)$ and $\boldsymbol{\sigma}(\boldsymbol{x},0)$) in the whole domain; moreover, we also need to prescribe the velocity or the normal stresses on the boundaries. The variable that will be prescribed depends on which variational form we use; the details are explained in section 2. For the problem in



(a) $\Re\{\sigma_{xx}\}$ (b) $|\Re\{\boldsymbol{v}\}|$

FIGURE 2. Solution of the swinging plate test (numerical example 5.1). Real part of the first component of the stress tensor (a) and module of the real part of the velocity (b) for the mixed equation in the frequency domain. The used mesh size is $h = 0.025$ m.

the frequency domain, we just have to prescribe the velocity or the normal stresses on the boundaries, also depending on the chosen variational form.

The chosen material parameters are $E = 1.7 \times 10^7$ N/m^2, $\rho = 1.1 \times 10^3$ kg/m^3 and $\nu = 0.3$. The characteristic length in the calculation of the stabilization parameters is $L_0 = 2$ m and the algorithmic constants are $C_{\tau_v} = C_{\tau_\sigma} = 10^{-5}$. We take the parameter $V_0 = 1$ m/s. For the time domain problem, we take the time step as $\delta t = 2 \times 10^{-6}$ s, the initial time as $t_0 = 0$ s and the final time as $T = 5 \times 10^{-4}$ s.

In Fig. 2, the solution for the real part of the first component of the stress tensor and the module of the the real part of the velocity for the mixed equation in the frequency domain are depicted. The velocity and stress for the imaginary parts of the solution of the problem and for the problem in the time domain are really similar to the ones that are shown.

We have performed a mesh convergence study using bilinear elements. Regarding the equation in the time domain, in Fig. 3 we plot the convergence errors in the $L^2(\Omega)$ norm of the first component of the velocity and the first component of the stress tensor at time $t = T$. Likewise, regarding the equation in the frequency domain, in Fig. 4 we plot the convergence errors in the $L^2(\Omega)$ norm of the real part of the first component of the velocity and the real part of the first component of the stress tensor. For both domains, the errors are presented for both VFI and VFII and for the two different stabilization methods (ASGS and OSGS).

Even though we have not shown the analysis in this norm, the convergence rates to be expected would be $m = 1$ for the stress using VFI and for the velocity using VFII and $m = 2$ for the velocity using VFI and for the stress using VFII. It is observed from Fig. 3 and Fig. 4 that this is indeed the case. In fact, the slope in some of the convergence curves for the velocity using VFI and for the stress using VFII are higher than $m = 1$. The results for the other components of the variables are similar to the ones shown.

5.2. **Cook's membrane.** For this second example we consider Cook's membrane problem (see, e.g., [30, 31, 32, 33]). The domain of the problem is the quadrilateral with the corner coordinates $(0, 0)$ m, $(0, 4.4)$ m, $(4.8, 6)$ m, and $(4.8, 4.4)$ m. The boundary between the previous two first coordinates is clamped, and we fix a periodic vertical velocity to the boundary between the third and the fourth coordinates. For the problem in the time

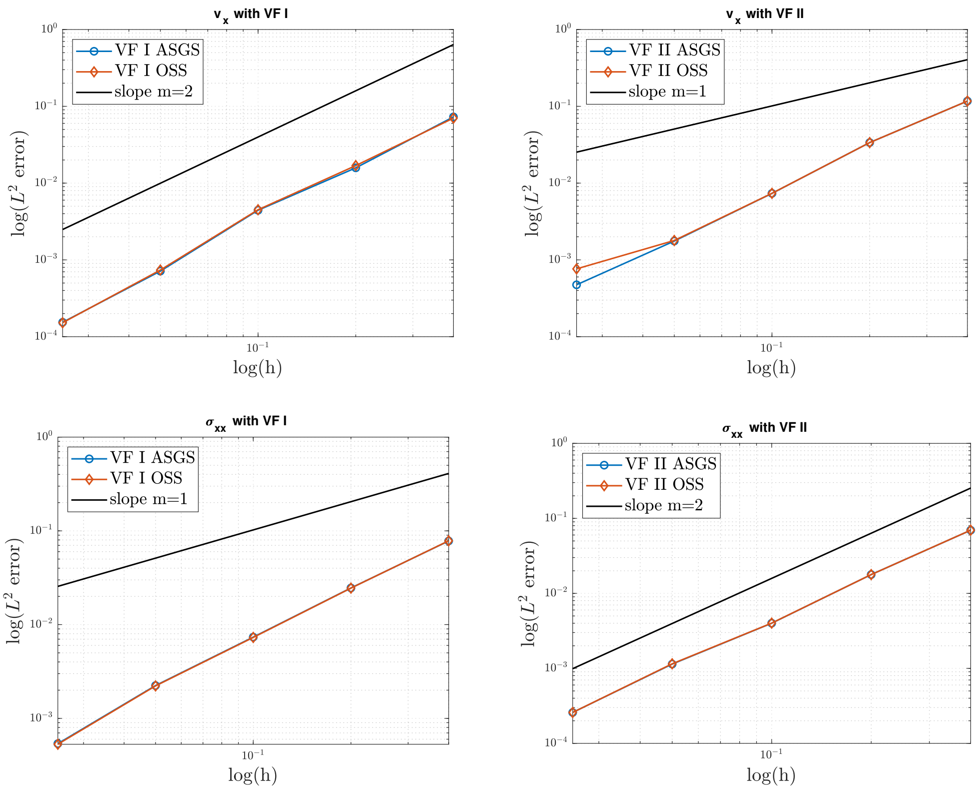


FIGURE 3. Convergence plots of the swinging plate test (numerical example 5.1) for VFI (left) and VFII (right) in the time domain at $t = T$, for both the first component of the velocity (top) and the first component of the stress tensor (bottom). Errors measured in the $L^2(\Omega)$ norm.

domain, this velocity has the form $v_y = V_0 \sin(\omega t)$, where V_0 is the amplitude and ω is the angular frequency. For the problem in the frequency domain this velocity will be simply set to $v_y = V_0(1 + \mathrm{i})$ and the frequency taken has been the same as that used for the velocity boundary condition of the problem in the time domain. The point now is which is the value of ω selected. From the modal analysis presented in [33], it turns out that the frequency associated to the fourth mode is $\omega = 10.6672$ s^{-1}, and that of the eighth mode is $\omega = 18.5002$ s^{-1}. Therefore, we have chosen as frequencies $\omega = 10.6$ s^{-1} and $\omega = 18.5$ s^{-1}, close to those of the fourth and eighth eigenmodes but that have allowed us to solve the problem in the frequency domain and, indeed, get solutions close to these eigenmodes.

We have used VFI, and thus we have imposed strongly the velocity in two of the four boundaries, for both the time and the frequency domains. Should we have used VFII, the velocity would have been imposed weakly and the normal stresses strongly to zero on the upper and lower boundaries.

In this example the ASGS method has been used as stabilization technique. The discretization of the problem in space is given by an irregular mesh with linear triangular elements of size $h = 0.05$ m and 13135 elements, using equal continuous interpolation for all variables. For the time domain problem, we take the time step size as $\delta t = 1 \times 10^{-2}$ s, the initial time as $t_0 = 0$ s and the final time as $T = 5$ s. The chosen material parameters are $E = 250$ N/m^2, $\rho = 1$ kg/m^3 and $\nu = 0.3$. The characteristic length in the calculation of the stabilization parameters is $L_0 = 4$ m and the algorithmic constants are $C_{\tau_v} = C_{\tau_\sigma} = 10^{-5}$.

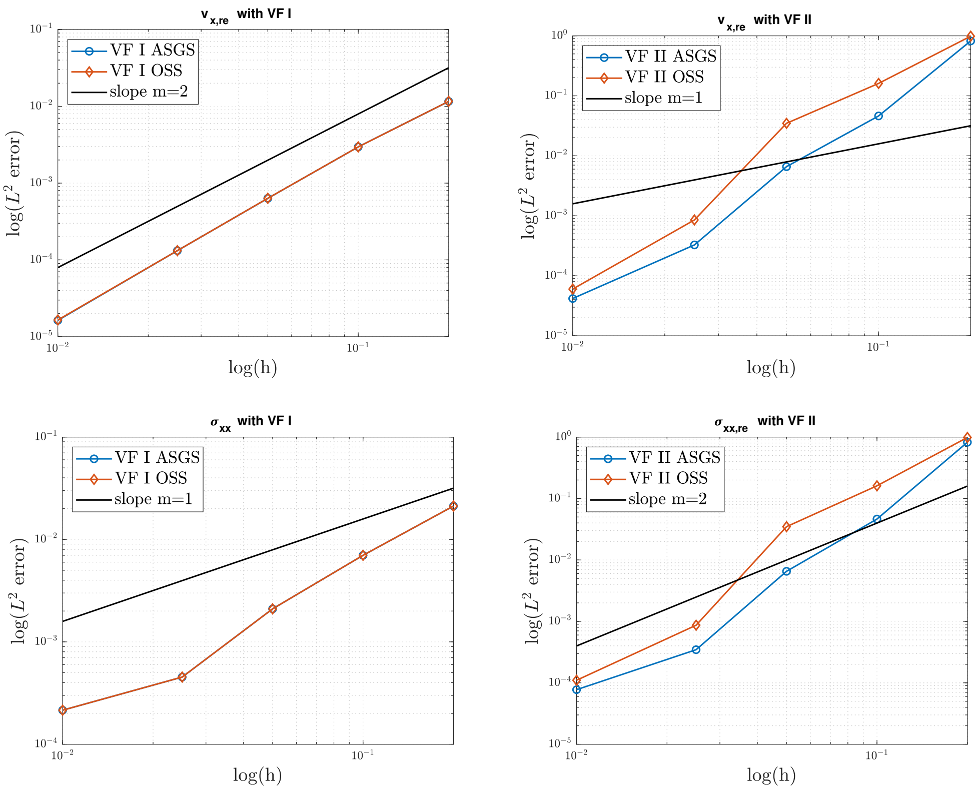
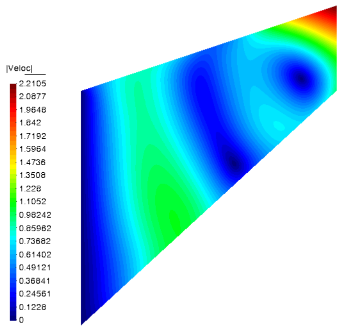


FIGURE 4. Convergence plots of the swinging plate test (numerical example 5.1) for VFI (left) and VFII (right) in the frequency domain, for both the real part of the first component of the velocity (top) and the real part of the first component of the stress tensor (bottom). Errors measured in the $L^2(\Omega)$ norm.

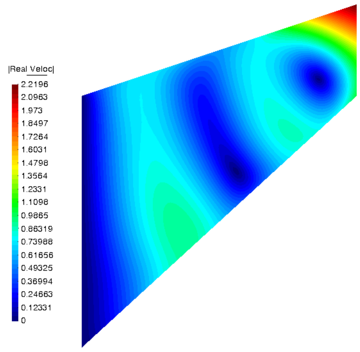
In Fig. 5 the solutions close to the fourth eigenmode for the unknowns of the problem are depicted. On the left part, results at time step $t = 1.67$ s are shown, and the results for the real part of the problem in the frequency domain are shown on the right part. As it can be seen, the time step has been chosen in order to get almost opposite results for the solutions in different domains; the velocity looks similar because what is shown is the module of its two components. Since we are dealing with harmonic oscillations, we can consider that the results are satisfactory, so the same pattern of oscillation can be identified in both domains. Note that for the solution in the time domain, all modes are in fact active, and we cannot expect full coincidence with the solution in the frequency domain. In Fig. 6 the solutions close to the eighth eigenmode for the unknowns of the problems are depicted. As before, on the left part results at time step $t = 1.40$ s are shown, and the results for the real part of the problem in the frequency domain are shown on the right part. Since the results are also for harmonic oscillations, the same comments as before can be applied.

5.3. **Rotational 3D clamped beam.** This third example is devoted to a 3D case. Contrary to the previous example, and to complement it, in this case we have used VFII. Again, the ASGS has been employed, being the results of the OSGS formulation very similar.

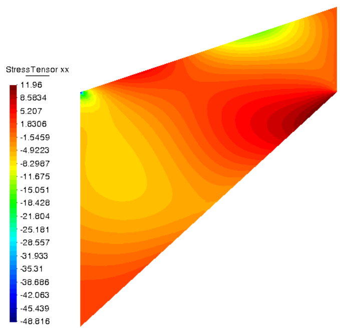
The domain of the problem is a cylinder of square cross section, the corner coordinates of the base being $(1, 1, 0)$ m, $(1, -1, 0)$ m, $(-1, -1, 0)$ m, and $(-1, 1, 0)$ m, and the corner coordinates of the top the same but with the vertical coordinate z equal to 10 m. The base is clamped and we have fixed the boundary conditions on the top in order to make it rotate. To do that, we have weekly imposed the velocities equal to zero in the base and



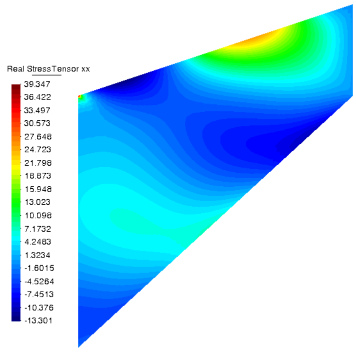
(a) $|\boldsymbol{v}|$



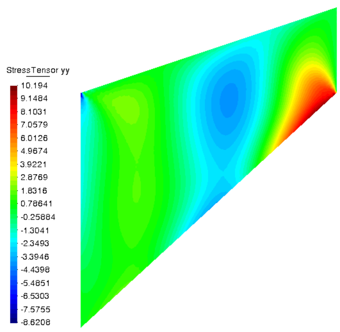
(b) $|\Re\{\boldsymbol{v}\}|$



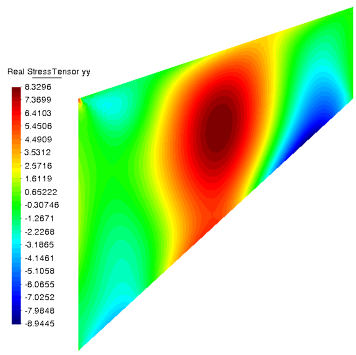
(c) σ_{xx}



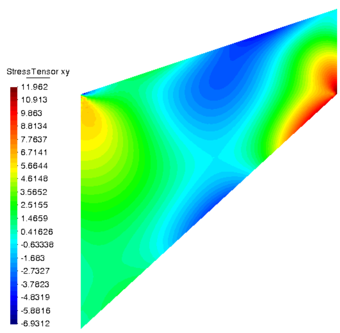
(d) $\Re\{\sigma_{xx}\}$



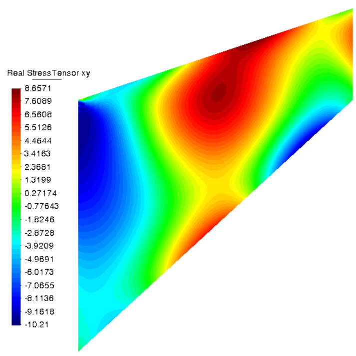
(e) σ_{yy}



(f) $\Re\{\sigma_{yy}\}$

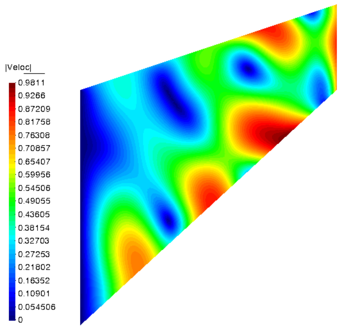


(g) σ_{xy}

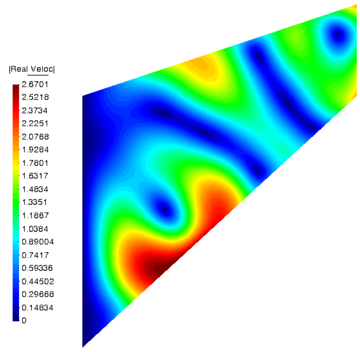


(h) $\Re\{\sigma_{xy}\}$

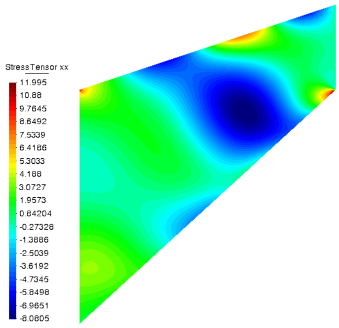
FIGURE 5. Solution close to the fourth eigenmode of Cook's membrane problem (numerical example 5.2). Unknowns of the problem in the time domain at time step $t = 1.67$ s (left), and real part of the unknowns of the problem in the frequency domain (right).



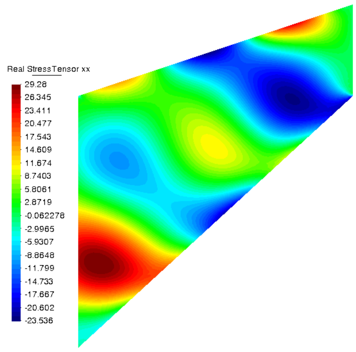
(a) $|\boldsymbol{v}|$



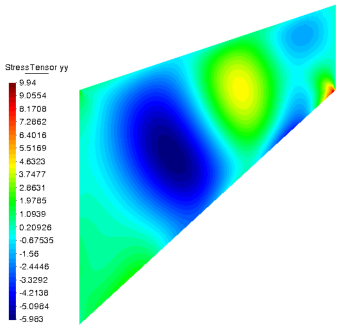
(b) $|\Re\{\boldsymbol{v}\}|$



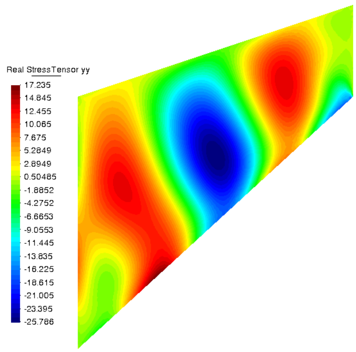
(c) σ_{xx}



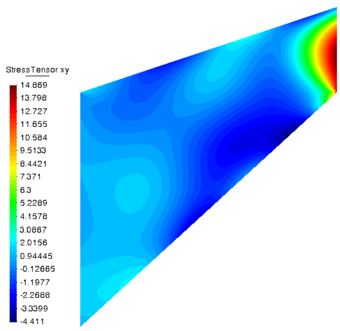
(d) $\Re\{\sigma_{xx}\}$



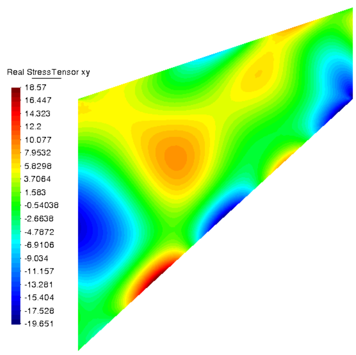
(e) σ_{yy}



(f) $\Re\{\sigma_{yy}\}$



(g) σ_{xy}



(h) $\Re\{\sigma_{xy}\}$

FIGURE 6. Solution close to the eighth eigenmode of Cook's membrane problem (numerical example 5.2). Unknowns of the problem in the time domain at time step $t = 1.67$ s (left), and real part of the unknowns of the problem in the frequency domain (right).

strongly imposed the normal stresses at the top, these stresses being $\boldsymbol{t}_z = (0,0,1)\cdot\boldsymbol{\sigma} = (\sigma_{xz}, \sigma_{yz}, \sigma_{zz})$. On the one hand, for the problem in the time domain, periodical boundary conditions have been applied, so that $\boldsymbol{t}_z = (-y, x, 0)\ \sin(\omega t)$ N/m^2. On the other hand, for the problem in the frequency domain we have just set $\boldsymbol{t}_z = (-y, x, 0)(1+\mathrm{i})$ N/m^2, and the same ω used in the boundary conditions of the problem in the time domain has been used now as a dominant frequency. Since in this problem we are not looking for harmonic oscillations, the definition of this angular frequency is less important, and we have set it to $\omega = 0.5$ s^{-1}.

Since we have employed VFII, we also have strongly imposed the normal stresses on the four lateral surfaces of the domain. For the two that are perpendicular to the x axis we have strongly imposed that $\boldsymbol{t}_x = (1,0,0)\cdot\boldsymbol{\sigma} = (0,0,0)$ N/m^2, and for the two surfaces that are perpendicular to the y axis we have strongly imposed that $\boldsymbol{t}_y = (0,1,0)\cdot\boldsymbol{\sigma} = (0,0,0)$ N/m^2. In the case that we would solve the problem with VFI, the velocities would be strongly imposed and the tractions would be weekly imposed, as we did in the previous example.

In order to overcome some modelling issues at the edges that are intersections of the surfaces of the domain, where the normal is not defined, some arrangements have been made. On the one hand, we have strongly imposed $\boldsymbol{t}_x$ and $\boldsymbol{t}_y$ equal to zero in the four lines that are vertical edges, except for the components that also belong to $\boldsymbol{t}_z$; in other words, we have just fixed σ_{xx}, σ_{yy} and σ_{xy} to zero. On the other hand, not only the mentioned $\boldsymbol{t}_z$ has been imposed on the edges of the top surface, but also $\boldsymbol{t}_x = (0,0,0)$ N/m^2 and $\boldsymbol{t}_y = (0,0,0)$ N/m^2 have been imposed.

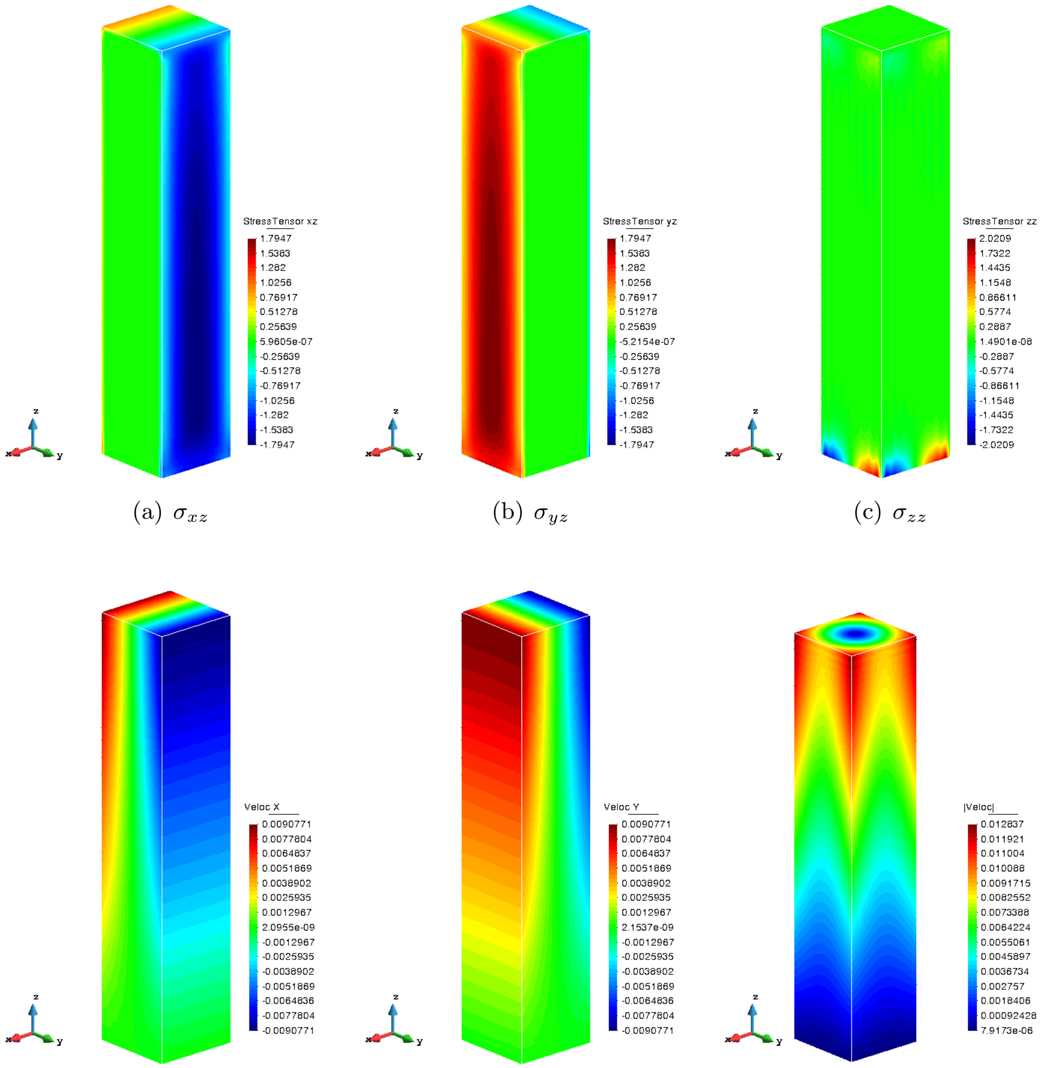
The chosen material parameters are $E = 2500$ N/m^2, $\rho = 10$ kg/m^3 and $\nu = 0.3$. The characteristic length in the calculation of the stabilization parameters is $L_0 = 10$ m. The algorithmic constants for the case in frequency domain are $C_{\tau_v} = 0.9$ and $C_{\tau_\sigma} = 60$, whereas the algorithmic constants for the case in time domain are $C_{\tau_v} = 0.001$ and $C_{\tau_\sigma} = 10$. Note that these algorithmic constants may depend on whether the problem is 2D or 3D and on whether VFI or VFII are used. The computational domain is discretized using a regular mesh of linear hexahedra of size $0.13 \times 0.13 \times 0.1$ m^3 and 22 500 elements, using equal continuous interpolation for all variables. For the time domain problem, we take the time step as $\delta t = 0.05$ s, the initial time as $t_0 = 0$ s and the final time as $T = 2.3$ s.

In Fig. 7 the solutions of the three components of $\boldsymbol{t}_z$, the two first components of $\boldsymbol{v}$ and the module of the velocity for the problem in the time domain at time $t = 2.3$ s are depicted. The other variables of the problem are not shown due their lack of relevance in this example.

In Fig. 8 the solutions of the three components of $\boldsymbol{t}_z$, the two first components of $\boldsymbol{v}$ and the module of the velocity for the problem in the frequency domain are depicted. If we compare these results with the previously commented ones of Fig. 7 we can confirm that the rotation of the beam is well approximated and that we have successfully demonstrated that this formulation in the frequency domain also works well for these kind of examples in 3D.

6. Conclusions

In this paper, FE approximations for the mixed velocity-stress elasticity equation in the time and the frequency domains have been presented. The irreducible elasticity equations for the velocity have been included as a reference. It the time domain, stabilized FE formulations for the mixed problem had been proposed before, but not allowing to consider the two functional frameworks presented here, namely, VFI and VFII. To our knowledge, mixed formulations of the elasticity equations in the frequency domain are original of this work.



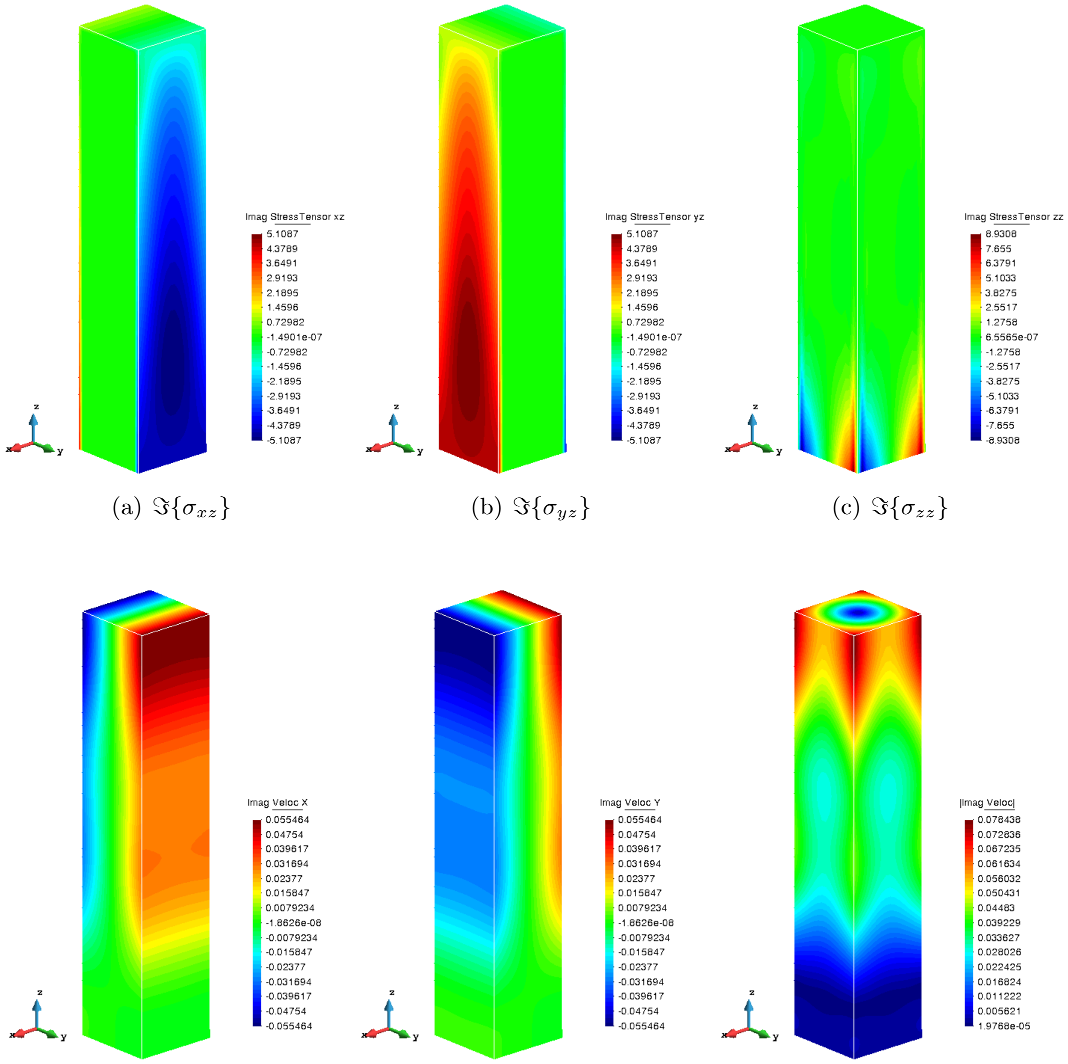
(a) σ_{xz} (b) σ_{yz} (c) σ_{zz}

(d) v_x (e) v_y (f) $|\boldsymbol{v}|$

FIGURE 7. Solutions of the components of $\boldsymbol{t}_z$ and $\boldsymbol{v}$ for the rotational clamped beam problem (numerical example 5.3) solved in the time domain at time $t = 2.3$ s.

Two different stabilized FE methods in the framework of the VMS formulation, ASGS and OSGS, have been applied to the two proposed variational forms of the problems, VFI and VFII. The assumed FE interpolation is conforming, but very general, in the sense that the regularity requirements of the continuous solution are assumed at the discrete level. Thus, continuous velocities and stresses with a discontinuous normal component can be used for VFI, whereas discontinuous velocities and stresses with a continuous normal component can be used for VFII. This is possible because of the introduction of stabilizing terms on the interelement boundaries. It has been shown that, apart from the continuity requirements in the FE interpolation, one can switch from VFI to VFII and vice-versa simply by properly designing the stabilization parameters.

The performance of the stabilization methods has been tested in three different numerical examples. The purpose of the first example has been to check the convergence rates that could be expected. The purpose of the second example has been to illustrate the use of the



(a) $\Im\{\sigma_{xz}\}$ (b) $\Im\{\sigma_{yz}\}$ (c) $\Im\{\sigma_{zz}\}$

(d) $\Im\{v_x\}$ (e) $\Im\{v_y\}$ (f) $|\Im\{\boldsymbol{v}\}|$

FIGURE 8. Imaginary parts of the solutions of the components of $\boldsymbol{t}_z$ and $\boldsymbol{v}$ for the rotational clamped beam problem (numerical example 5.3) solved in the frequency domain.

mixed forms of the elasticity equations in a more complex 2D simulation, in which VFI has been used in both the time and the frequency domains. The purpose of the last example has been to prove that the formulations also work in a 3D problem; to complement the results of the second example, VFII has been used to solve this problem.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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