Archives of Computational Methods in Engineering manuscript No.
(will be inserted by the editor)

Stabilised Variational Multi-Scale Finite Element Formulations for Viscoelastic Fluids

Ernesto Castillo · Laura Moreno · Joan Baiges · Ramon Codina

Received: date / Accepted: date

Abstract The objective of this article is to summarise the work that we have been doing as a group in the context of stabilised finite element formulations for viscoelastic fluid flows. Viscoelastic fluids are complex non-Newtonian fluids, characterised by having an irreducible constitutive equation that needs to be solved coupled with the momentum and continuity equations. The finite element approximation of this kind of fluids presents several numerical difficulties. It inherits obviously the problems associated with the approximation of the incompressible Navier-Stokes equations. But, on top of that, now the constitutive equation is highly non-linear, with an advective term that may lead to both global and local oscillations in the numerical approximation. Moreover, even in the case of smooth solutions, it is necessary to meet some additional compatibility conditions between the velocity and the stress interpolation in order to ensure control over velocity gradients. The stabilised methods detailed in this work allow one to use equal order or even arbitrary interpolation for the problem unknowns ($\boldsymbol{\sigma}$-$\boldsymbol{u}$-p) (elastic deviatoric stress-velocity-pressure) and to stabilise dominant convective terms, and all of them can be framed in the context of Variational Multi-scale Methods (VMS). Some additional numerical ingredients that are introduced in this article are the treatment of the non-linearities associated with the problem and the possibility to introduce a discontinuity-capturing technique to prevent local oscillations. Concerning the constitutive equation, both the standard as the logarithmic conformation

Ernesto Castillo
Departamento de Ingeniería Mecánica, Universidad de Santiago de Chile
Av. Libertador Bernardo O'Higgins 3363, Santiago, Chile
E-mail: ernesto.castillode@usach.cl

Laura Moreno · Joan Baiges · Ramon Codina
Universitat Politècnica de Catalunya
Jordi Girona 1-3, Edifici C1, 08034, Barcelona, Spain
E-mail: lmoreno@cimne.upc.edu
E-mail: joan.baiges@upc.edu
E-mail: ramon.codina@upc.edu

reformulation are discussed for stationary and time-dependent problems, and different versions of stabilised finite element formulations are presented in both cases.

Keywords Viscoelastic fluids · stabilised finite element methods · Variational Multi-Scale methods · Log-conformation reformulation (LCR)

1 Introduction

Viscoelastic fluids are a specific kind of non-Newtonian fluids that come from the blend of a solvent and a polymer. The latter gives to the fluid the ability to store and recover shear-energy, which is a key aspect of viscoelastic fluids [1]. This fact introduces the necessity to include an irreducible tensorial constitutive equation that allows one to describe the elastic nature of the fluid, yielding a coupled three-field problem, the unknowns of the problem being the elastic deviatoric stress, the velocity, and the pressure.

Viscoelastic fluids are present in several industrial processes involving paints, melts, plastics, food, adhesives, or 3D printing of polymeric materials, among others [2,3,4,5,6,7,8,9]. Due to the broad spectrum of applications, the study of this type of materials is carried out from different disciplines, ranging from chemistry to mathematics. In the last forty years, computational rheology has been shown as a powerful tool that enables to understand both fundamental aspects of rheology and applied aspects involved in specific industrial problems. The challenge today for numerical methods is to be able to approximate highly elastic flows in an accurate manner using efficient algorithms. In this respect, this paper summarises the work developed by our group in the context of stabilised finite element (FE) methods, including several numerical ingredients related to the non-linear nature of the problem and the treatment of local instabilities [10,11,12,13,14,15,16,17]. All these aspects will be studied from a FE approach and will be framed in the context of stabilised methods of Variational Multi-Scale (VMS) type.

The first part of the work (Section 2) is devoted to a state of the art review of physical and mathematical aspects needed for the numerical modelling of viscoelastic fluids. In Section 3, the continuous viscoelastic problem is presented both in standard and logarithmic formulation. In Section 4, the Galerkin problem is presented, emphasising the need for using a stabilised formulation. In Section 5, the design process of a stabilised formulation of VMS type is summarised, taking into account spatial and temporal instabilities. After that, both the standard viscoelastic approach and the logarithmic reformulation case are particularised in Section 6. In Section 7, some computational aspects are presented, including solution algorithms and some implementation aspects. Section 8 describes the main results available for the numerical analysis of the formulations proposed. The work closes with some general conclusions and an outlook in Section 9.

2 A numerically-oriented state of the art

2.1 From physical to numerical instabilities

From a chemical point of view, viscoelastic fluids are characterised by having complex and high-molecular-weight molecules with many internal degrees of freedom [18]. The essential feature of polymeric or viscoelastic fluids is the presence of long-chain molecules. In a flow, these chain molecules are stretched out by the drag forces exerted on them by the surrounding fluid [19]. The natural tendency of the molecule to retract from this stretched configuration generates an elastic force (spring element) which contributes to the macroscopic stress tensor and coexists and interacts with the viscous component of the fluid (dashpot element); this is the reason why these fluids are called viscoelastic.

From the more general point of view, one can build a very sophisticated nonlinear invariant theory from springs and dashpots [20]. Regarding this, the Maxwell constitutive model is the simplest one that takes into account viscous and elastic elements. Under the hypothesis that the relation between the stress tensor and the kinematic tensors of a fluid particle should be independent of the instantaneous orientation of that particle in space, the complexity of constitutive models increases [18]. Using a co-rotating coordinate frame for each fluid particle, the well known Oldroyd-B and Giesekus models can be obtained. The design of constitutive models is from any point of view, a very complicated task. In this respect, the molecular theories for polymeric liquids are an essential tool to define more realistic models that try to mimic the molecular structure and their interaction in a flow. For more details on these important topics that are beyond the scope of this article, see specific and classical texts such as [21,22,23].

The flow patterns in viscoelastic fluids can be highly dynamic and in some cases chaotic, due to the elastic component of the fluid and the convective nature of the constitutive equation, even in quasi non-inertial flows [24], where non-linear rheological effects can manifest through the generation of large normal stresses which result in complex flow phenomena causing elastic turbulence [25,26]. Typical situations where this physical phenomenon can be seen are the flow in micro-channels or cross-flows, where the increase in normal stresses promote the generation of boundary layers with patterns similar to those of viscous turbulence [27,28]. In [29], the effect of the contraction ratio in the dynamic response of the flow in square-square three-dimensional contractions is analysed using experimental and numerical results, where asymmetric flows in a symmetric problem are created increasing the elasticity of the flow. In [30], the instabilities and the asymmetry of flow in a symmetric domain are analysed for flows with high Deborah number using the Leonov constitutive model. In this line, the turbulent flow of a viscoelastic solution is a new challenge from the numerical approximation perspective, on the one hand for the complex non-linear system that needs to be solved, and on the

other, for the computational resources needed to approximate the unavoidable time-dependent three-field problem.

The numerical approximation of high Weissenberg viscoelastic fluid flows is one of the biggest challenges in computation rheology since the 1970s, and it is called the High Weissenberg Number Problem (HWNP) [31]. It is defined as a numerical phenomenon that causes the iterative computations to breakdown for relatively low Weissemberg numbers. A distinctive feature of this instability is that the breakdown occurs suddenly for a critical value of the Weissemberg number, which is caused by a lack of convergence in the iterative method due to the hyperbolic nature of the differential constitutive equations. The numerical instability is brought about by the failure of the proper balance of the deformation rate and the convection, and it was identified and discussed by Fattal and Kupferman [32]. It is a fundamental instability, present in all constitutive models and standard numerical methods. Nevertheless, it is demonstrated that constitutive models can predict other instabilities of mathematical character [33,34], referred to as constitutive instabilities, which can be classified in two: the Hadamard instability, associated with the non-linear fast response of constitutive equations, and the dissipative instability, related to the formulation of the dissipative behaviour of viscoelastic models [35]. The consequences of Hadamard's instability are devastating. As the solution cannot be continued or it is not continuous along the timeline, very quick blow-up instabilities with very short-wave disturbances occur with devastating results for numerical computations, which fail to converge. Every constitutive equation includes a dissipative term or terms. The formulation of the non-equilibrium terms may give rise to another type of instability known as the dissipative instability. This is a relatively new issue in the study of instabilities which plague non-linear viscoelastic constitutive equations and was initiated by Kwon and Leonov in [36]. A constitutive equation may be both Hadamard and dissipative unstable, and a constitutive equation which is Hadamard stable may turn out to be dissipative unstable. Dissipative stability requires that firstly in any flow the free energy and dissipation functionals remain bounded, and secondly, steady flow curves in simple shear and in simple elongation must be monotonically and unboundedly increasing with respect to the strain rate (see [37] for more details about the stability of viscoelastic constitutive models).

Nowadays, the causes of the HWNP have been identified: on the one hand, the loss of positive-definiteness of the conformation tensor, which is an internal variable that should be symmetric positive-definite to be physically admissible [38,39] and, on the other hand, the large stress gradients, regions with particular high deformation rate, or near stagnation points, favour the breakdown of the numerical method, as explained in Fattal and Kupferman in [38,32]. They describe this phenomenon to be caused by the use of inappropriate approximations to represent the stress tensor, emphasising the importance of preserving its positivity, which is the key of the log-conformation representation (denoted by LCR). The reformulation of the traditional equations of viscoelastic fluids using a logarithmic transformation eliminates some instabilities and linearises the exponential stress profiles near the stress sin-

gularities. Therefore, the formulation seeks to treat the exponential growth of the elastic stresses, allowing to approximate more elastic flows. After the original works of Fattal and Kupferman [38], alternative schemes have been proposed. For example, Vaithianathan and Collins [40] presented two matrix decomposition schemes in order to construct the positive definiteness conformational tensor, employing the FENE-P model. Balci et al. [41] proposed a square root conformation representation. Afonso et al. [42], developed several matrix kernel-transformation families which can be applied to the conformation tensor equation. It is important to note that, although there is a variety of proposals to deal with the lack of positive-definiteness in the conformation tensor, the logarithm representation is the only one capable of linearising the exponential stress profile.

In summary, what has been previously described allows us to realise the complex nature of viscoelastic fluids, which strongly affects the design of numerical methods that must face the instabilities of the continuous problem. On the discrete problem side, other instabilities and difficulties appear, which are summarised in the next subsection

2.2 Purely numerical challenges

From the mathematical point of view, a viscoelastic constitutive equation is a convective-reactive non-linear time-dependent equation that must be solved coupled with the momentum and continuity equations, giving place to a mixed irreducible three-field problem which contains a tensorial, a vectorial and a scalar unknown, in the isothermal case. This equation, coupled with the conservation of momentum and mass, yields a problem whose mathematical analysis is not yet complete (see Subsection 2.3). Assuming the problem to be well-posed, let us discuss some delicate numerical issues.

The treatment of the nonlinearity is an aspect that deserves to be studied in detail. Apart from the nonlinearity in the convective term of the momentum equation, the constitutive equation has two additional nonlinear terms, the convective one and the rotational term. Fixed point type schemes are robust, but with a very low convergence rate when the elastic component increases [43]. Newton-Raphson schemes are the most extensively used in the literature [44, 45], although they often need to be complemented with additional numerical tools, such as continuation methods or relaxation schemes [10]. The other option is to solve the non-linear problem, either stationary or time-dependent, using a decoupling in time algorithm, which is the basis of some finite volume solution algorithms [46,47,48]. In the context of FE methods, fractional step schemes also allow one to uncouple unknowns [49,11,50].

The FE approximation of the flow of viscoelastic fluids presents several numerical difficulties. It inherits obviously the problems associated with the approximation of the incompressible Navier-Stokes equations, mainly the compatibility between the velocity-pressure approximation and the treatment of the nonlinear advective term [51]. But, on top of that, now the constitutive

equation is highly nonlinear, with an advective term that may lead to both global and local oscillations in the numerical approximation.

Even in the case of smooth solutions, it is necessary to meet some additional compatibility conditions between the velocity and the stress interpolation in order to control velocity gradients [52]. Elements that satisfy the compatibility requirements velocity-pressure and stress-velocity are rare and expensive to use from the computational point of view [53]. These compatibility conditions of inf-sup type are inherited from the Newtonian three-field Stokes or Navier-Stokes problem [54, 55], and they consist of two restrictions on the interpolation spaces, one between pressure and velocity, and the other between velocity and elastic stress (see e.g [56, 57] and [58] for background). These two restrictions reduce drastically the choices of stable FE spaces that allow one to discretise the unknowns. For example, in the work of Marchal and Crochet [59] one can find different inf-sup stable elements capable of solving the viscoelastic problem. In this classical reference, the authors propose a family of bi-quadratic velocity and bilinear pressure elements with a multi-bilinear (2×2 or 3×3 or 4×4) stress element for the 2D case. The mathematical analysis of these elements can be found in [60]. It is a clear example of the difficulties to satisfy the two inf-sup conditions associated with the three-field formulation needed in the viscoelastic flow problem. For the three-dimensional case, Bogaerds et al. [53] propose a DEVSS-DG stable spatial discretisation using tri-quadratic interpolation for velocity, tri-linear interpolation for both pressure and discrete rate of deformation, while discontinuous tri-linear polynomials approximate the viscoelastic stresses. In [61] one can find a useful review of mixed methods that satisfy the two compatibility conditions required.

Once the equations have been properly linearised, the advective nature of the constitutive equation, which becomes dominant when the Weissenberg number increases or when the normal stresses are high, makes it necessary to use a stabilised FE formulation to avoid global oscillations. The most widespread method to account for the convective term in the constitutive equation is the so-called SUPG method of Brooks and Hughes [62], first applied to viscoelastic flows by Marchal and Crochet [59]. In a more recent work, Masud et al. [44] use a VMS stabilised method for the momentum-continuity equations and the same SUPG method for the constitutive equation. Other stabilised methods for the viscoelastic fluid problem are the GLS-type methods used for example by Fan et al. [63] and Coronado et al. [64]. Different families of stabilised formulations can also be found in the literature, which are in general extensions of those used for the standard Navier-Stokes equations. For example, Li et al. [65] proposed the so-called I-PS-DEVSS-CNBS scheme to stabilise the viscoelastic problem, based on the finite incremental calculus (FIC, [66]) pressure stabilisation process, the discrete elastic-viscous stress-splitting method (DEVSS), the use of the Crank-Nicolson-based-splitting (CNBS) scheme, and the use of the non-consistent SU method to stabilise the viscoelastic equation. Damanik et al. [67], proposed a consistent edge-oriented FE stabilisation technique, including special geometrical multigrid solvers. Other two options to circumvent the dominant convective nature of the problem are the fully ex-

plicit characteristic based split (CBS) scheme (see [68]) proposed by Nithiarasu [69], with a good performance for a wide range of Weissenberg numbers, and the nonlinear weighted least-squares FE method proposed by Lee [43]. More recently, Venkatesan and Ganesan [70] proposed a stabilised FE method based on the Local Projection Stabilisation technique showing accuracy and robustness in classical benchmarks. In the same line, Varchanis et al. [71], implemented equal order interpolation for $\boldsymbol{\sigma}$-$\boldsymbol{u}$-p using a combination of classical FE stabilisation techniques (PSPG/DEVSS-TG/SUPG) with the log-conformation representation of the constitutive equation to obtain numerically stable solutions at high Weissenberg numbers.

The starting point of a VMS approach is to split the unknowns of the problem into two components, namely, the component that can be approximated by the FE mesh and the unresolvable one, called sub-grid scale or sub-scale in what follows. The latter needs to be approximated in a simple manner in terms of the former, so as to capture its main effect and yield a stable formulation for the FE unknown. The number of degrees of freedom is, therefore, the same as for the Galerkin method. There are different ways to approximate the sub-scale and, in particular, to choose the (finite dimensional) space where it is taken. On this regard, two possible choices will be discussed in this work. Another discussed aspect in this paper will be the structure of the sub-scale method, which can be residual one or non-residual [14]. The last option has been found crucial in the approximation of highly elastic flows. Finally, the time-dependent nature of the sub-scales will be discussed, a key point to ensure stability in anisotropic space-time discretisations [72,73,74,17,75], which we believe is a crucial ingredient in the approximation of elastic turbulent flows.

In general, a stabilised FE method ensures globally stable solutions. However, viscoelastic flows are characterised by having strong gradients when the elastic component of the fluid is important or when geometrical singularities, like a rigid body or a non-convex corner, get in the way with the flow. The use of discontinuity-capturing (DC) techniques is not a popular topic in the analysis of viscoelastic flows, but the high elastic stress gradients that appear when the Weissenberg number is increased make it a typical situation where the application of a DC scheme can help [76]. In this last article, Bonito et al. proposed a weakly consistent artificial viscosity term that vanishes at an optimal rate under mesh refinement. Carew et al. in [77] have shown that the inclusion of such a DC technique in a stabilised formulation can improve the stability properties and permits to analyse fluids with a higher elasticity. In that work, the numerical diffusion of the discontinuity-capturing term is based on the FE residual of the constitutive equation, in a similar way to that used by Codina [78] (see also [79]). In [10], we showed that a DC technique based on the orthogonal projection of the elastic stress gradient, which represents the *non-captured* part in the FE approximation, can improve the numerical approximation of the stresses, eliminating spurious peaks without physical sense. This numerical technique can be crucial if standard viscoelastic formulations are used.

Since the logarithmic formulation was presented, a great number of works have been written following this novel strategy, applying different methodologies and schemes, in finite volume and FE codes. Referring to FEs, the first work applying the log-conformation reformulation is due to Hulsen et al. [80] using the DEVSS/DG formulation for the discretisation and a first-order upwind scheme to treat convective terms. Later, Coronado et al. [81] proposed a simple alternative form of the log-conformation formulation implemented in the DEVSS-TG/SUPG FE method, which in comparison with the previous work required fewer code modifications with respect to the standard formulation. An analysis between the two previous publications and two new implementations was presented by Kane et al. [82], emphasising particularly in the treatment of the advective term of the constitutive equation. The final conclusion is that all four formulations are very similar, except the one described by Coronado et al. [81], that is a little less robust due to the linear interpolation of the convective term. Damanik et al. [67] proposed a fully coupled monolithic FE approach using the edge-oriented FE stabilisation for the convective term. Saramito [83] and Knechtges [84] derived fully implicit versions of the log-conformation formulation that do not involve an algebraic decomposition of the velocity gradient tensor, and which can be linearised and solved by the Newton-Raphson method.

We summarise in this work a slightly different formulation of the log-conformation formulation, that has been recently proposed in [16] in the context of a stabilised VMS method. A key point of this formulation is the capability to use it even when the Weissemberg number is close to zero. The same idea was followed by Saramito [83]; both formulations can be reduced to the standard Navier-Stokes equations when the Weissemberg number is set to zero. Due to this, continuation methods can be successfully employed to get convergence in highly nonlinear problems. Also, we have to remark that the steady problem can be solved directly, while in most of the references indicated the logarithmic formulation shows a strong time-dependency.

2.3 Some numerical analysis results

For viscoelastic fluid flows, in contrast to the Navier-Stokes equations, well-posedness for general models is not well understood. Global existence in time of solutions has been proved only if the initial conditions are small perturbations of the rest state, and for the steady-state case existence of solutions can be proved only for small perturbations of the Newtonian case (see [19,85] for comprehensive reviews).

The existence of slow steady flows of viscoelastic fluids using differential constitutive equations was proved in [86] for Hilbert spaces. For the time-dependent case, the existence of solutions locally in time, and for small data globally in time, has been proved for Hilbert spaces in [87]. The extension to Banach spaces and a complete review of uniqueness, regularity, well-posedness and stability results can be found in [85]. The existence of global weak solu-

tions for general initial conditions using a co-rotational Oldroyd-B model has been proved in [88] using a simplification (without physical justification) which consists in replacing the velocity gradient in the stress equation by its skew-symmetric part. In [89] the authors proved global existence of weak solutions in two dimensions to the Oldroyd-B model regularised with the introduction of a diffusion term in the constitutive equation and assuming homogeneous natural boundary conditions associated to this term. An analysis of the effects it has on the numerical approximation can be found in [90].

In the context of the FE approximation, for the steady-state case, one of the first works where the existence of approximate solutions and error analysis were presented is that of Baranger and Sandri in [52]. The authors used a discontinuous interpolation (Lesaint-Raviart method) to treat the viscoelastic stresses. Later, Sandri in [91] showed by using a fixed point method that the discrete approximate problem using a P_1(continuous)-P_2 (continuous)-P_1 (continuous) interpolation for stress, velocity, and pressure, respectively, and the SUPG method to treat the convective term in the constitutive equation, has a unique solution for which error bounds can be found. Picasso and Rappaz [92] analysed a stationary non-linear Stokes problem, and they proved a priori and a posteriori error estimates for the FE approximation for small Weissenberg numbers using a GLS method and an Elastic Viscous Split Stress (EVSS) scheme, aimed to circumvent the inf-sup condition between velocities and stresses. More recently, in [93] the authors present an error analysis of a particular Oldroyd-B model with the limiting Weissenberg number going to infinity, assuming a suitable regularity of the exact solution for FE and finite volume methods.

The extension to the time-dependent case was treated in [76] for the same non-linear Stokes problem, proving global existence in time in Banach spaces under the small data assumption. For a Stokes/Oldroyd-B linearised problem, Bonito and Burman presented in [94] optimal a priori error estimates using the Interior-Penalty method. In this work, the authors showed that adding some type of artificial viscosity in the constitutive equation, the problem can be solved for a large range of Weissenberg numbers. A similar problem was studied by Ervin et al. in [95] for the steady state case, but using the Johnson-Segalman linearised constitutive model, proving existence and uniqueness of the continuous problem and of a FE approximation under the small data assumption. Ervin and Miles in [96] analysed the Oldroyd-B time-dependent case both in the semi-discrete and in the fully discrete cases using the SUPG method, proving existence and deriving a priori error estimates for the numerical approximation, assuming a Taylor-Hood pair approximation for the velocity and pressure and a continuous approximation for the viscoelastic stresses. In [97] the authors analysed the time behaviour of the viscoelastic Oldroyd model in two dimensions using a Galerkin formulation in space; in this work, the stress is eliminated through a proper projection operator, resulting in an integro-differential equation in terms of velocity and pressure.

In particular, for VMS-type stabilisation methods, the three-field Stokes problem was presented and analysed in [55], and the same approach was fol-

lowed in [13]. In this case, it was applied to a linearised version of the stationary *standard* formulation of the viscoelastic flow problem, using the Oldroyd-B model. The linearisation is based on considering known the advection velocity and the velocity gradient in the rotational terms of the constitutive equation. The analysis follows a classical approach to prove stability and convergence, first using a mesh-dependent working norm, and then extending the results to natural norms. The analysis of the *logarithmic* formulation [98] allows explaining its improved performance with respect to the standard formulation when the Weissenberg number is high. This would be difficult to observe in the full nonlinear problem, since conditions to ensure the existence of solutions and convergence of their FE approximation pose stringent requirements on the Reynolds and Weissenberg numbers of the problem, even in the time-dependent case; for the standard formulation, this analysis can be found in [15]. Moreover, apart from analysing a time-dependent linearised problem, the novelty of this work is also the treatment of some of the terms that appear in the analysis. The main results of these works are summarised in Section 8.

3 The viscoelastic flow problem

Having stated the physical and numerical difficulties involved in the approximation of viscoelastic fluid flows, we start with the definition of the problem at the continuous level and then proceed to explain the discretisation tools employed to approximate its solution.

3.1 Standard initial and boundary value problem

Let us start presenting the standard equations associated to a viscoelastic fluid flow moving in a domain Ω of $\mathbb{R}^d$ ($d = 2$ or 3) during the time interval $[0, t_\mathrm{f}]$; the boundary of Ω is denoted by $\partial\Omega$. Assuming the flow to be incompressible and isothermal, the governing equations are the conservation of momentum and mass, which can be expressed as follows:

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla\cdot\boldsymbol{T} + \nabla p = \boldsymbol{f} \text{ in } \Omega, t\in(0,t_\mathrm{f}), \tag{1}$$

$$\nabla\cdot\boldsymbol{u} = 0 \text{ in } \Omega, t\in(0,t_\mathrm{f}), \tag{2}$$

where ρ denotes the density of the fluid, $p : \Omega\times(0,t_\mathrm{f}) \to \mathbb{R}$ represents the pressure, $\boldsymbol{u} : \Omega\times(0,t_\mathrm{f}) \to \mathbb{R}^d$ is the velocity field, $\boldsymbol{f} : \Omega\times(0,t_\mathrm{f}) \to \mathbb{R}^d$ is the right hand side force vector and $\boldsymbol{T} : \Omega\times(0,t_\mathrm{f}) \to \mathbb{R}^d\otimes\mathbb{R}^d$ is the deviatoric extra stress tensor. For viscoelastic fluids, $\boldsymbol{T}$ is defined in terms of a viscous and a viscoelastic contribution as

$$\boldsymbol{T} = 2\eta_s\nabla^\mathrm{s}\boldsymbol{u} + \boldsymbol{\sigma}, \tag{3}$$

where $\nabla^\mathrm{s}\boldsymbol{u}$ is the symmetric part of the velocity gradient, $\beta\in[0,1]$ is a real parameter used to define the amount of viscous or solvent viscosity, $\eta_s = \beta\eta_0$,

and elastic or polymeric viscosity, $\eta_p = (1-\beta)\,\eta_0$, in the fluid. The key point in viscoelastic fluids is that the problem is incomplete without the definition of a constitutive equation for the elastic part of the extra stress tensor ($\boldsymbol{\sigma}$). A large variety of approaches exist to define it; see [21,22] for a complete description, where both differential and integral expressions can be found.

The most popular differential constitutive model used to check new numerical formulations is the Oldroyd-B model, that reads as

$$\frac{1}{2\eta_p}\boldsymbol{\sigma} - \nabla^{\mathrm{s}}\boldsymbol{u} + \frac{\lambda}{2\eta_p}\left(\frac{\partial\boldsymbol{\sigma}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^T\cdot\boldsymbol{\sigma}\right) = \mathbf{0} \quad \text{in } \Omega,\ t\in(0,t_{\mathrm{f}}). \tag{4}$$

This model can be regarded as an extension of the Upper Convected Maxwell model and is equivalent to a fluid filled with elastic bead and spring dumbbells. From the rheological point of view, the Oldroyd-B model is a linear viscoelastic constitutive model, and therefore its use is restricted to motions with very small displacement gradients. However, the critical point of its use lies in the fact that it is the basis of non-linear models such as the Giesekus or Phan-Thien-Tanner (PTT) models, which contain only one additional term that is quadratic with respect to the stress field component. For this article, the Oldroyd-B model allows us to expose the key aspects of a stabilised formulation. The extension to more complex constitutive models is straightforward.

Let us introduce some notation to write the viscoelastic problem in compact form, which will simplify the presentation of the problem in the following sections. Calling $\boldsymbol{U} = [\boldsymbol{u}, p, \boldsymbol{\sigma}]$, $\boldsymbol{F} = [\boldsymbol{f}, 0, \mathbf{0}]$ and defining

$$\mathcal{D}_t(\boldsymbol{U}) := \begin{pmatrix} \rho\dfrac{\partial\boldsymbol{u}}{\partial t} \\ 0 \\ \dfrac{\lambda}{2\eta_p}\dfrac{\partial\boldsymbol{\sigma}}{\partial t} \end{pmatrix}, \tag{5}$$

and

$$\mathcal{L}(\hat{\boldsymbol{u}};\boldsymbol{U}) := \begin{pmatrix} \rho\hat{\boldsymbol{u}}\cdot\nabla\boldsymbol{u} - 2\eta_s\nabla\cdot(\nabla^{\mathrm{s}}\boldsymbol{u}) - \nabla\cdot\boldsymbol{\sigma} + \nabla p \\ \nabla\cdot\boldsymbol{u} \\ \dfrac{1}{2\eta_p}\boldsymbol{\sigma} - \nabla^{\mathrm{s}}\boldsymbol{u} + \dfrac{\lambda}{2\eta_p}\left(\hat{\boldsymbol{u}}\cdot\nabla\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot\nabla\hat{\boldsymbol{u}} - (\nabla\hat{\boldsymbol{u}})^T\cdot\boldsymbol{\sigma}\right) \end{pmatrix}, \tag{6}$$

where $\hat{\boldsymbol{u}}$ represents an auxiliary variable used to distinguish the velocity with the role of advection (note that this velocity introduces the convective nonlinearity of the problem), we may write (1), (2) and (4) using the definition (3) as:

$$\mathcal{D}_t(\boldsymbol{U}) + \mathcal{L}(\boldsymbol{u};\boldsymbol{U}) = \boldsymbol{F}, \tag{7}$$

which represents a mixed parabolic-hyperbolic system.

To close problem (7), initial and boundary conditions both in the velocity and the elastic stress fields are needed. In principle the elastic stresses can be fixed only on the inflow part of the boundary $\Gamma_{\mathrm{in}} = \{\boldsymbol{x}\in\partial\Omega \mid (\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{x}) < 0\}$,

where $\boldsymbol{n}$ is the outward unit normal vector to $\partial\Omega$. For simplicity in the exposition, we will consider homogeneous Dirichlet conditions for the velocity field and no boundary conditions for $\boldsymbol{\sigma}$; however, all the methods summarised in the following are entirely general, and any boundary condition can be included.

3.2 The log-conformation reformulation

As commented above, the HWNP is a fundamental instability that all numerical methods suffer and can be overcome using the logarithmic reformulation. The logarithmic reformulation of the constitutive equation will be exposed starting from the standard approach, and following the nomenclature and structure presented in [16].

First, the *conformation tensor*, an internal variable that represents the macromolecular configuration of the polymer chains, is defined, taking into account that it must be symmetric and positive-definite to be physically-admissible; this is a consequence of statistical mechanics arguments. It is defined as

$$\boldsymbol{\tau} = \frac{\lambda\boldsymbol{\sigma}}{\eta_p} + \boldsymbol{I}.$$

Consequently, and based on the work of [38], the stress tensor can be expressed as a function of the conformation tensor as $\boldsymbol{\sigma} = \frac{\eta_p}{\lambda}(\boldsymbol{\tau} - \boldsymbol{I})$. Then, replacing $\boldsymbol{\sigma}$ in the constitutive equation (4) by $\boldsymbol{\tau}$, we can rewrite the Oldroyd-B model in terms of this new variable as

$$\frac{1}{2\lambda}(\boldsymbol{\tau} - \boldsymbol{I}) - \nabla^{\mathrm{s}}\boldsymbol{u} + \frac{1}{2}\left(\frac{\partial\boldsymbol{\tau}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\tau} - \boldsymbol{\tau}\cdot\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^T\cdot\boldsymbol{\tau}\right) = \boldsymbol{0}. \tag{8}$$

Note that this expression is valid only for $\lambda > 0$. To allow the use of this approach even for $\lambda = 0$, a small but important modification will be done, following the idea presented in [16]. To this end, we introduce the relaxation-time parameter $\lambda_0(\lambda)$ linearly dependent with λ, which could be defined as $\lambda_0 = \max\{k\lambda, \lambda_{0,\min}\}$, k being a constant and $\lambda_{0,\min}$ a given threshold. So, if $k = 1$ and $\lambda_{0,\min} = 0$, the original change of variables proposed by Fattal and Kupferman [38] is recovered; however, if k is taken equal to zero, then the three-field Navier-Stokes problem for Newtonian fluids is obtained. From the convergence point of view, in [16] it is reported that it is useful to take k small. With this in mind, the conformation stress tensor is replaced by the new tensor, still denoted by $\boldsymbol{\tau}$,

$$\boldsymbol{\tau} = \frac{\lambda_0(\lambda)\boldsymbol{\sigma}}{\eta_p} + \boldsymbol{I}.$$

Note that if $\lambda_0 \leq \lambda$, $\boldsymbol{\tau}$ will be positive definite if so is the original conformation tensor. To avoid excessive nomenclature, from this point we will use λ_0 instead

of $\lambda_0(\lambda)$, and we will still call $\boldsymbol{\tau}$ the conformation tensor. The constitutive equation (4) can be rewritten by following the proposed modification as

$$\frac{1}{2\lambda_0}(\boldsymbol{\tau}-\boldsymbol{I})-\nabla^{\mathrm{s}}\boldsymbol{u}+\frac{\lambda}{2\lambda_0}\left(\frac{\partial\boldsymbol{\tau}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{\tau}-\boldsymbol{\tau}\cdot\nabla\boldsymbol{u}-(\nabla\boldsymbol{u})^T\cdot\boldsymbol{\tau}+2\nabla^{\mathrm{s}}\boldsymbol{u}\right)=\boldsymbol{0}. \tag{9}$$

The log-conformation reformulation consists in a change of variables in terms of the matrix-logarithm of the conformation tensor, that is to say, the conformation tensor is replaced by a new variable $\boldsymbol{\psi}=\log(\boldsymbol{\tau})$. This can be calculated through an eigenvalue computation, rotating the $\boldsymbol{\tau}$ tensor into its main principle axis, and can be expressed as $\boldsymbol{\psi}=\boldsymbol{R}\log(\boldsymbol{\Lambda})\boldsymbol{R}^T$; this operation is possible because $\boldsymbol{\tau}$ is a symmetric positive definite tensor and therefore it can always be diagonalised, the eigenvalues being non-negative. In the expression introduced, $\boldsymbol{\Lambda}$ is a diagonal matrix with these eigenvalues of $\boldsymbol{\tau}$, and $\boldsymbol{R}$ is the orthogonal matrix of the eigenvectors of $\boldsymbol{\tau}$.

To sum up, in order to obtain the new formulation the stress tensor must be replaced by $\boldsymbol{\sigma}=\frac{\eta_p}{\lambda_0}(\boldsymbol{\tau}-\boldsymbol{I})$, and in turn, the conformation tensor $\boldsymbol{\tau}$ must be written as $\boldsymbol{\tau}=\exp(\boldsymbol{\psi})$ in the standard viscoelastic formulation detailed above, (1), (2) and (4). The new equations of the log-conformation formulation are now expressed as follows:

$$\rho\frac{\partial\boldsymbol{u}}{\partial t}-\frac{\eta_p}{\lambda_0}\nabla\cdot\exp(\boldsymbol{\psi})-2\eta_s\nabla\cdot(\nabla^{\mathrm{s}}\boldsymbol{u})+\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla p=\boldsymbol{f}, \tag{10}$$

$$\nabla\cdot\boldsymbol{u}=0, \tag{11}$$

$$\begin{aligned}\frac{1}{2\lambda_0}(\exp(\boldsymbol{\psi})-\boldsymbol{I})-\nabla^{\mathrm{s}}\boldsymbol{u}+\frac{\lambda}{2\lambda_0}\Big(&\frac{\partial\exp(\boldsymbol{\psi})}{\partial t}+\boldsymbol{u}\cdot\nabla\exp(\boldsymbol{\psi})\\&-\exp(\boldsymbol{\psi})\cdot\nabla\boldsymbol{u}-(\nabla\boldsymbol{u})^T\cdot\exp(\boldsymbol{\psi})+2\nabla^{\mathrm{s}}\boldsymbol{u}\Big)=\boldsymbol{0},\end{aligned} \tag{12}$$

where the unknowns are the velocity, the pressure, and the tensor $\boldsymbol{\psi}$, which depends directly on the viscoelastic stress tensor $\boldsymbol{\sigma}$.

In the same way as for the standard case, we can write (10)-(12) in compact form, redefining $\boldsymbol{U}_{\log}=[\boldsymbol{u},p,\boldsymbol{\psi}]$ and $\boldsymbol{F}_{\log}=[\boldsymbol{f},0,\frac{1}{2\lambda_0}\boldsymbol{I}]$ and defining

$$\mathcal{D}_{t,\log}(\boldsymbol{U}):=\begin{pmatrix}\rho\dfrac{\partial\boldsymbol{u}}{\partial t}\\0\\\dfrac{\lambda}{2\lambda_0}\dfrac{\partial\exp(\boldsymbol{\psi})}{\partial t}\end{pmatrix}, \tag{13}$$

$$\mathcal{L}_{\log}(\hat{\boldsymbol{u}};\boldsymbol{U}):=\begin{pmatrix}-\dfrac{\eta_p}{\lambda_0}\nabla\cdot\exp(\boldsymbol{\psi})-2\eta_s\nabla\cdot(\nabla^{\mathrm{s}}\boldsymbol{u})+\rho\hat{\boldsymbol{u}}\cdot\nabla\boldsymbol{u}+\nabla p\\\nabla\cdot\boldsymbol{u}\\\dfrac{1}{2\lambda_0}\exp(\boldsymbol{\psi})-\nabla^{\mathrm{s}}\boldsymbol{u}+\dfrac{\lambda}{2\lambda_0}\left(\hat{\boldsymbol{u}}\cdot\nabla\exp(\boldsymbol{\psi})\right.\\\left.-\exp(\boldsymbol{\psi})\cdot\nabla\hat{\boldsymbol{u}}-(\nabla\hat{\boldsymbol{u}})^T\cdot\exp(\boldsymbol{\psi})+2\nabla^{\mathrm{s}}\boldsymbol{u}\right)\end{pmatrix}, \tag{14}$$

where again $\hat{\boldsymbol{u}}$ represents an auxiliary variable used to distinguish the velocity with the role of advection. The compact form is now:

$$\mathcal{D}_{t,\log}(\boldsymbol{U}) + \mathcal{L}_{\log}(\boldsymbol{u};\boldsymbol{U}) = \boldsymbol{F}_{\log}. \tag{15}$$

Again, problem (15) needs to be complemented with initial and boundary conditions, with the same remarks as for the standard case. The problem is completely defined by the initial conditions for the velocity and the new variable $\boldsymbol{\psi}$, which are denoted by $\boldsymbol{u} = \boldsymbol{u}^0$, and $\boldsymbol{\psi} = \boldsymbol{\psi}^0$ at time $t = 0$, with $\boldsymbol{u}^0$ and $\boldsymbol{\psi}^0$ functions defined on the whole domain Ω.

To avoid excessive nomenclature in what follows, we will omit the subindex log in (13) and (14), even if we refer to the logarithmic problem, but taking into account that the logarithmic problem (15) is different to the standard one (7).

3.3 Variational form

In order to write the weak form of the problem, let us introduce some standard notation. The space of square integrable functions in a domain ω is denoted by $L^2(\omega)$, and the space of functions whose distributional derivatives of order up to $m \geq 0$ (integer) belong to $L^2(\omega)$ is denoted by $H^m(\omega)$. The norm in $H^m(\Omega)$ will be denoted $\|\cdot\|_m$, with $m < \infty$, whereas $\|\cdot\|_\infty$ will stand for the norm in $L^\infty(\Omega)$. The norm in $L^2(\Omega)$ will be simply written as $\|\cdot\|$.

The space $H^1_0(\omega)$ is made up of functions in $H^1(\omega)$ vanishing on $\partial\omega$. The topological dual of $H^1_0(\Omega)$ is denoted by $H^{-1}(\Omega)$, the duality pairing being $\langle\cdot,\cdot\rangle$. The L^2 inner product in ω (for scalars, vectors and tensors) is denoted by $(\cdot,\cdot)_\omega$ and the integral over ω of the product of two general functions is written as $\langle\cdot,\cdot\rangle_\omega$, the subscript being omitted when $\omega = \Omega$. The norm in a space X is denoted by $\|\cdot\|_X$, except in the case $X = L^2(\Omega)$, case in which the subscript is omitted. With this notation in mind, we will write the weak forms for the standard and for the logarithmic formulations.

In the standard case, the velocity and pressure spaces are $\boldsymbol{\mathcal{V}}_0 = H^1_0(\Omega)^d$ and $\mathcal{Q} = L^2(\Omega)/\mathbb{R}$, respectively, while stresses are assumed to belong to $\boldsymbol{\Upsilon} := \left\{\boldsymbol{\chi} \mid \boldsymbol{\chi} \in (L^2(\Omega))^{d\times d}_{\text{sym}},\ \boldsymbol{w}\cdot\nabla\boldsymbol{\chi} \in (L^2(\Omega))^{d\times d}_{\text{sym}} \quad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{V}}_0\right\}$, where the subscript sym stands for symmetric tensors. Under this regularity requirements, the weak form in the standard case consists in finding $\boldsymbol{U} = [\boldsymbol{u}, p, \boldsymbol{\sigma}] : (0, t_{\text{f}}) \longrightarrow \boldsymbol{\mathcal{X}} := \boldsymbol{\mathcal{V}}_0 \times \mathcal{Q} \times \boldsymbol{\Upsilon}$, such that the initial conditions are satisfied and:

$$\left(\rho\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v}\right) + 2\left(\eta_s \nabla^{\text{s}}\boldsymbol{u}, \nabla^{\text{s}}\boldsymbol{v}\right) + \langle \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v}\rangle + (\boldsymbol{\sigma}, \nabla^{\text{s}}\boldsymbol{v}) - (p, \nabla\cdot\boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle, \tag{16}$$

$$(q, \nabla\cdot\boldsymbol{u}) = 0, \tag{17}$$

$$\left(\frac{1}{2\eta_p}\boldsymbol{\sigma} - \nabla^{\text{s}}\boldsymbol{u}, \boldsymbol{\chi}\right) + \frac{\lambda}{2\eta_p}\left(\frac{\partial \boldsymbol{\sigma}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\sigma}, \boldsymbol{\chi}\right)$$

$$-\frac{\lambda}{2\eta_p}\left(\boldsymbol{\sigma}\cdot\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T\cdot\boldsymbol{\sigma},\boldsymbol{\chi}\right)=\mathbf{0}, \tag{18}$$

for all $\boldsymbol{V}=[\boldsymbol{v},q,\boldsymbol{\chi}]\in\boldsymbol{\mathcal{X}}$, where it is assumed that $\boldsymbol{f}$ is such that $\langle\boldsymbol{f},\boldsymbol{v}\rangle$ is well defined.

For the logarithmic reformulation, we shall consider the case in which the change of variables in the stress is made for the unknown, but not for the test function. The reason is that this option simplifies the FE implementation, since we shall see that after linearisation we can use standard FE spaces. From the physical point of view, the unknown will be the logarithm of the conformation tensor, a dimensionless variable, whereas the test function will be a stress, and therefore will belong to the same space $\boldsymbol{\Upsilon}$ introduced before. Up to scaling by η_p/λ_0, this is also the space for the conformation tensor $\boldsymbol{\tau}$, and we can formally write the space for the unknown $\boldsymbol{\psi}$ as $\boldsymbol{\Upsilon}_{\boldsymbol{\psi}}=\log(\boldsymbol{\Upsilon})$. Thus, the weak form consists of finding $\boldsymbol{U}=[\boldsymbol{u},p,\boldsymbol{\psi}]:(0,t_{\mathrm{f}})\longrightarrow\boldsymbol{\mathcal{X}}_{\boldsymbol{\psi}}:=\boldsymbol{\mathcal{V}}_0\times\mathcal{Q}\times\boldsymbol{\Upsilon}_{\boldsymbol{\psi}}$, such that the initial conditions are satisfied and:

$$\left(\rho\frac{\partial\boldsymbol{u}}{\partial t},\boldsymbol{v}\right)+\frac{\eta_p}{\lambda_0}(\exp(\boldsymbol{\psi}),\nabla^{\mathrm{s}}\boldsymbol{v})+2(\eta_s\nabla^{\mathrm{s}}\boldsymbol{u},\nabla^{\mathrm{s}}\boldsymbol{v})$$
$$+\langle\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v}\rangle-(p,\nabla\cdot\boldsymbol{v})=\langle\boldsymbol{f},\boldsymbol{v}\rangle, \tag{19}$$
$$(q,\nabla\cdot\boldsymbol{u})=0, \tag{20}$$
$$\left(\frac{1}{2\lambda_0}\exp(\boldsymbol{\psi})-\nabla^{\mathrm{s}}\boldsymbol{u},\boldsymbol{\chi}\right)+\frac{\lambda}{2\lambda_0}\left(\frac{\partial\exp(\boldsymbol{\psi})}{\partial t},\boldsymbol{\chi}\right)$$
$$+\frac{\lambda}{2\lambda_0}\left(\boldsymbol{u}\cdot\nabla\exp(\boldsymbol{\psi})-\exp(\boldsymbol{\psi})\cdot\nabla\boldsymbol{u},\boldsymbol{\chi}\right)$$
$$+\frac{\lambda}{2\lambda_0}\left(-(\nabla\boldsymbol{u})^T\cdot\exp(\boldsymbol{\psi})+2\nabla^{\mathrm{s}}\boldsymbol{u},\boldsymbol{\chi}\right)=\frac{1}{2\lambda_0}\langle\boldsymbol{I},\boldsymbol{\chi}\rangle, \tag{21}$$

for all $\boldsymbol{V}=[\boldsymbol{v},q,\boldsymbol{\chi}]\in\boldsymbol{\mathcal{X}}$. We could also have used a test function for the constitutive equation of the form $\frac{\eta_p}{\lambda_0}\exp(\boldsymbol{\chi})$, where now $\boldsymbol{\chi}$ would be dimensionless. This would simplify the analysis but, as commented above, complicate significantly the FE approximations described later on.

In compact form, problems (16)-(18) and (19)-(21) can be written as:

$$(\mathcal{D}_t(\boldsymbol{U}),\boldsymbol{V})+B(\boldsymbol{u};\boldsymbol{U},\boldsymbol{V})=\langle\boldsymbol{F},\boldsymbol{V}\rangle, \tag{22}$$

where the semi-linear form $B(\boldsymbol{u};\boldsymbol{U},\boldsymbol{V})$ (linear in the last two arguments) should be defined for the standard and for the logarithmic case. For the standard case it reads as

$$B(\hat{\boldsymbol{u}};\boldsymbol{U},\boldsymbol{V})=(\boldsymbol{\sigma},\nabla^{\mathrm{s}}\boldsymbol{v})+2\left(\eta_s\nabla^{\mathrm{s}}\boldsymbol{u},\nabla^{\mathrm{s}}\boldsymbol{v}\right)+\langle\rho\hat{\boldsymbol{u}}\cdot\nabla\boldsymbol{u},\boldsymbol{v}\rangle-(p,\nabla\cdot\boldsymbol{v})$$
$$+(\nabla\cdot\boldsymbol{u},q)+\frac{1}{2\eta_p}\left(\boldsymbol{\sigma},\boldsymbol{\chi}\right)-(\nabla^{\mathrm{s}}\boldsymbol{u},\boldsymbol{\chi})+\frac{\lambda}{2\eta_p}\left(\hat{\boldsymbol{u}}\cdot\nabla\boldsymbol{\sigma},\boldsymbol{\chi}\right)$$
$$-\frac{\lambda}{2\eta_p}\left(\boldsymbol{\sigma}\cdot\nabla\hat{\boldsymbol{u}}+(\nabla\hat{\boldsymbol{u}})^T\cdot\boldsymbol{\sigma},\boldsymbol{\chi}\right), \tag{23}$$

while in the logarithmic case it reads as

$$B(\hat{\boldsymbol{u}};\boldsymbol{U},\boldsymbol{V}) = \frac{\eta_p}{\lambda_0}(\exp(\boldsymbol{\psi}),\nabla^{\mathrm{s}}\boldsymbol{v}) + 2(\eta_s\nabla^{\mathrm{s}}\boldsymbol{u},\nabla^{\mathrm{s}}\boldsymbol{v}) + \langle\rho\hat{\boldsymbol{u}}\cdot\nabla\boldsymbol{u},\boldsymbol{v}\rangle$$
$$- (p,\nabla\cdot\boldsymbol{v}) + (\nabla\cdot\boldsymbol{u},q) + \frac{1}{2\lambda_0}(\exp(\boldsymbol{\psi}),\boldsymbol{\chi}) - (\nabla^{\mathrm{s}}\boldsymbol{u},\boldsymbol{\chi})$$
$$+ \frac{\lambda}{2\lambda_0}\left(\hat{\boldsymbol{u}}\cdot\nabla\exp(\boldsymbol{\psi}) - \exp(\boldsymbol{\psi})\cdot\nabla\hat{\boldsymbol{u}} - (\nabla\hat{\boldsymbol{u}})^T\cdot\exp(\boldsymbol{\psi}) + 2\nabla^{\mathrm{s}}\boldsymbol{u},\boldsymbol{\chi}\right). \quad (24)$$

Note that using this compact notation we can refer to each problem in a general way. In fact, we shall not distinguish between the spaces $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{X}}_{\boldsymbol{\psi}}$, being understood that if the log-conformation formulation the unknowns belong to the latter.

4 Galerkin finite element discretisation and time discretisation

4.1 Space approximation

The standard Galerkin approximation for the variational problem defined by (22) can be performed by considering a FE partition $\mathcal{T}_h$ of the domain Ω. The diameter of an element domain $K \in \mathcal{T}_h$ is denoted by h_K and the diameter of the element partition is defined by $h = \max\{h_K \mid K \in \mathcal{T}_h\}$. Under the above considerations, we can construct conforming finite element spaces, $\boldsymbol{\mathcal{V}}_{0,h} \subset \boldsymbol{\mathcal{V}}_0$ and $\mathcal{Q}_h \subset \mathcal{Q}$ in the usual manner. As said before, abusing of the notation we will use the same symbol for the space of the stresses and the logarithm of the conformation tensor, $\boldsymbol{\Upsilon}$, being $\boldsymbol{\Upsilon}_h \subset \boldsymbol{\Upsilon}$ the FE space. If $\boldsymbol{\mathcal{X}}_h = \boldsymbol{\mathcal{V}}_{0,h} \times \mathcal{Q}_h \times \boldsymbol{\Upsilon}_h$ and $\boldsymbol{U}_h = [\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h \,\text{or}\, \boldsymbol{\psi}_h]$, the Galerkin FE approximation consists in finding $\boldsymbol{U}_h : (0,t_{\mathrm{f}}) \longrightarrow \boldsymbol{\mathcal{X}}_h$ such that

$$(\mathcal{D}_t(\boldsymbol{U}_h),\boldsymbol{V}_h) + B(\boldsymbol{u}_h;\boldsymbol{U}_h,\boldsymbol{V}_h) = \langle\boldsymbol{F},\boldsymbol{V}_h\rangle, \quad (25)$$

for all $\boldsymbol{V}_h = [\boldsymbol{v}_h, q_h, \boldsymbol{\chi}_h] \in \boldsymbol{\mathcal{X}}_h$.

4.2 Time discretisation

There are a lot of possibilities for the discretisation in time using finite differences, but we will restrict ourselves to the classical backward difference (BDF) approximations. Consider a partition of the interval $[0,t_{\mathrm{f}}]$ into m subintervals of constant size δt, and let $f(t)$ be a generic time-dependent function. We will denote as f^n the approximation to $f(t^n)$, with $t^n = n\delta t$, $n = 0,1,2,\ldots,m$. A BDF approximation to the time derivative of the function f order $k = 1,2,\ldots,$ is given by $\frac{\delta_k f^{n+1}}{\delta t}$, where $\delta_k f^{n+1}$ is defined as

$$\delta_k f^{n+1} = \frac{1}{\gamma_k}\left(f^{n+1} - \sum_{i=0}^{i-1}\varphi_k^i f^{n-k}\right),$$

and where γ_k and φ_k^i are parameters.

4.3 Compatibility conditions for the viscoelastic problem

Until now, we have posed no restrictions on the choice of the FE spaces. However, some restrictions must be satisfied explicitly in a three-field formulation (even for non elastic-flows) [56,99,55,100,13]. For example, it is readily checked for the standard approximation that

$$B\left(\boldsymbol{U}_h,[\boldsymbol{u}_h,p_h,\boldsymbol{\sigma}_h]\right)=2\eta_s\left\|\nabla^{\mathrm{s}}\boldsymbol{u}_h\right\|^2+\frac{1}{2\eta_p}\left\|\boldsymbol{\sigma}_h\right\|^2 - \frac{\lambda}{2\eta_p}\left(\boldsymbol{\sigma}_h\cdot\nabla\boldsymbol{u}_h+\left(\nabla\boldsymbol{u}_h\right)^T\cdot\boldsymbol{\sigma}_h,\boldsymbol{\sigma}_h\right).$$

Assuming $\lambda\nabla\hat{\boldsymbol{u}}_h$ to be small enough, this expression provides only control on $\left\|\boldsymbol{\sigma}_h\right\|^2$ for all $\beta\in[0,1]$. To control the other two fields one has then to make use of the two inf-sup conditions that restrict the possible interpolations:

$$\inf_{q_h\in\mathcal{Q}_h}\sup_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{V}}_h}\frac{\left(q_h,\nabla\cdot\boldsymbol{v}_h\right)}{\left\|\boldsymbol{v}_h\right\|_{\boldsymbol{\mathcal{V}}_h}\left\|q_h\right\|_{\mathcal{Q}_h}}\geq C_1, \tag{26}$$

to control p_h, and

$$\inf_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{V}}_h}\sup_{\boldsymbol{\chi}_h\in\boldsymbol{\Upsilon}_h}\frac{\left(\boldsymbol{\chi}_h,\nabla^{\mathrm{s}}\boldsymbol{v}_h\right)}{\left\|\boldsymbol{\chi}_h\right\|_{\boldsymbol{\Upsilon}_h}\left\|\boldsymbol{v}_h\right\|_{\boldsymbol{\mathcal{V}}_h}}\geq C_2, \tag{27}$$

to control $\nabla^{\mathrm{s}}\boldsymbol{u}_h$, where C_1 and C_2 are positive constants. It is therefore required that the FE spaces satisfy (26)-(27). These two conditions pose stringent requirements on the choice of the FE spaces both in the standard and in the logarithmic case. Our intention in this work is to summarise some stabilised FE formulations that avoid the need for such conditions and, in particular, allow one to use equal interpolation for all the unknowns, including the possibility to use arbitrary interpolations. In general, a stabilised formulation consists of replacing the bilinear form of the problem (Eq. (25)) by another bilinear form B_h, possibly mesh dependent, with enhanced stability properties. In the viscoelastic problem, even for inf-sup stable elements, the convective terms in both the momentum and constitutive equations need to be stabilised.

The objective of the next section is to present a general approach to design stabilised FE formulations under the VMS framework, highlighting only the most essential aspects, avoiding technicalities due to the extension of the work and referring to the review [101] for more details.

5 Variational multi-scale formulation for an abstract problem

The basic idea of a stabilised formulation in a VMS framework is to approximate the effect of the components of the solution of the continuous problem that cannot be resolved by the FE mesh. Here we present the general idea in

the case of a general nonlinear evolution problem and then particularise it to the problem of interest both in the standard as in the logarithmic case.

Consider a differential equation of the form:

$$\boldsymbol{M}\left(\boldsymbol{U}\right)\frac{\partial \boldsymbol{U}}{\partial t}+\boldsymbol{\mathcal{L}}\left(\boldsymbol{U},\boldsymbol{U}\right)=\boldsymbol{F}, \tag{28}$$

where $\boldsymbol{U}$ contains the unknowns of the problem (which in our particular case are $[\boldsymbol{u}, p, \boldsymbol{\sigma}\,\text{or}\,\boldsymbol{\psi}]$), $\boldsymbol{\mathcal{L}}(\boldsymbol{U},\cdot)$ corresponds to an operator (the spatial operator) associated with the specific problem at hand, assumed to be linear in the second argument, $\boldsymbol{M}(\boldsymbol{U})$ is a mass matrix, and $\boldsymbol{F}$ a force vector.

The weak form of the generic problem (28) can be formally written as

$$\left(\boldsymbol{M}\left(\boldsymbol{U}\right)\frac{\partial \boldsymbol{U}}{\partial t},\boldsymbol{V}\right)+\left\langle\boldsymbol{\mathcal{L}}\left(\boldsymbol{U},\boldsymbol{U}\right),\boldsymbol{V}\right\rangle=\left\langle\boldsymbol{F},\boldsymbol{V}\right\rangle, \tag{29}$$

for an appropriate "duality" $\langle\cdot,\cdot\rangle$, and considering $\boldsymbol{V}$ as a test function. The boundary conditions of the problem have to be taken into account, and are understood to be incorporated in the duality $\langle\cdot,\cdot\rangle$. Likewise, when considering FE functions the duality has to be understood as the element-wise integral with appropriate inter-element jumps.

5.1 The sub-scale concept

The basic idea of the VMS formulation applied to the generic problem is to split the unknown as $\boldsymbol{U}=\boldsymbol{U}_h+\tilde{\boldsymbol{U}}$, where $\boldsymbol{U}_h\in\boldsymbol{\mathcal{X}}_h$ is the component of the solution that belongs to the FE space (the Galerkin component) and $\tilde{\boldsymbol{U}}\in\tilde{\boldsymbol{\mathcal{X}}}$ is the remainder, referred to as sub-grid scale or sub-scale. The spaces $\boldsymbol{\mathcal{X}}_h$ and $\tilde{\boldsymbol{\mathcal{X}}}$ are such that $\boldsymbol{\mathcal{X}}=\boldsymbol{\mathcal{X}}_h\oplus\tilde{\boldsymbol{\mathcal{X}}}$. Under the above considerations, the original problem (29) is exactly equivalent to:

$$\left(\boldsymbol{M}\left(\boldsymbol{U}\right)\frac{\partial \boldsymbol{U}}{\partial t},\boldsymbol{V}_h\right)+\left\langle\boldsymbol{\mathcal{L}}\left(\boldsymbol{U},\boldsymbol{U}\right),\boldsymbol{V}_h\right\rangle=\left\langle\boldsymbol{F},\boldsymbol{V}_h\right\rangle\quad\forall\,\boldsymbol{V}_h\in\boldsymbol{\mathcal{X}}_h, \tag{30}$$

$$\left(\boldsymbol{M}\left(\boldsymbol{U}\right)\frac{\partial \boldsymbol{U}}{\partial t},\tilde{\boldsymbol{V}}\right)+\left\langle\boldsymbol{\mathcal{L}}\left(\boldsymbol{U},\boldsymbol{U}\right),\tilde{\boldsymbol{V}}\right\rangle=\left\langle\boldsymbol{F},\tilde{\boldsymbol{V}}\right\rangle\quad\forall\,\tilde{\boldsymbol{V}}\in\tilde{\boldsymbol{\mathcal{X}}}. \tag{31}$$

In essence, the goal of all sub-scale methods is to approximate $\tilde{\boldsymbol{U}}$ in one way or another and end up with a problem for $\boldsymbol{U}_h$ alone.

5.2 General approach of the sub-scale stabilised formulation

There are different approximations for the sub-scale component $\tilde{\boldsymbol{U}}$ that allow writing it as a function of the FE component, $\boldsymbol{U}_h$.

We call the sub-scales *dynamic* if their temporal derivative is taken into account, whereas if it is neglected, we call them *quasi-static*. Another possible simplification is to consider that they vanish on the inter-element boundaries,

as it is done, for example, when bubble functions are used to approximate the sub-scales. We will assume this, although this restriction could be relaxed [102,103]. A third possible simplification explained in more detail below, is to neglect the sub-scale effect on the non-linear terms of the equation [104].

Apart from the simplifications described, another important ingredient in the construction of the stabilised formulation is the choice of the space where the sub-scales belong. The most common choice is to take it equal to the space generated by the operator associated with the problem, applied to the FE space [105]. Another possibility is to take it orthogonal to the finite element space, that is, $\tilde{\boldsymbol{\mathcal{X}}} = \boldsymbol{\mathcal{X}}_h^{\perp}$, resulting in the so called orthogonal sub-scales (OSS) method [106].

Very briefly, the procedure described below allows one to approximate the sub-scale component $\tilde{\boldsymbol{U}}$ in terms of the FE unknown $\boldsymbol{U}_h$; this is achieved after manipulating the sub-scale equation (31). The final problem can be therefore understood as an *enriched* problem with the same numbers of degrees of freedom as in the Galerkin case. A key point in the approximation of the sub-scales in terms of the FE component is to avoid derivatives of the variable that we want to approximate or model (the sub-scale) in equation (30). After an appropriate integration by parts, equations (30)-(31), can be written as follows:

$$\left(\boldsymbol{M}\left(\boldsymbol{U}\right)\frac{\partial \boldsymbol{U}}{\partial t},\boldsymbol{V}_h\right)+\left\langle \mathcal{L}\left(\boldsymbol{U},\boldsymbol{U}_h\right),\boldsymbol{V}_h\right\rangle + \left\langle \tilde{\boldsymbol{U}},\mathcal{L}^{*}\left(\boldsymbol{U},\boldsymbol{V}_h\right)\right\rangle = \left\langle \boldsymbol{F},\boldsymbol{V}_h\right\rangle \quad \forall\, \boldsymbol{V}_h \in \boldsymbol{\mathcal{X}}_h, \tag{32}$$

$$\left(\boldsymbol{M}\left(\boldsymbol{U}\right)\frac{\partial \boldsymbol{U}}{\partial t},\tilde{\boldsymbol{V}}\right)+\left\langle \mathcal{L}\left(\boldsymbol{U},\boldsymbol{U}_h\right),\tilde{\boldsymbol{V}}\right\rangle + \left\langle \mathcal{L}(\boldsymbol{U},\tilde{\boldsymbol{U}}),\tilde{\boldsymbol{V}}\right\rangle = \left\langle \boldsymbol{F},\tilde{\boldsymbol{V}}\right\rangle \quad \forall\, \tilde{\boldsymbol{V}} \in \tilde{\boldsymbol{\mathcal{X}}}, \tag{33}$$

where we have introduced the formal adjoint $\mathcal{L}^{*}(\boldsymbol{U},\cdot)$ of operator $\mathcal{L}(\boldsymbol{U},\cdot)$, which is defined through the relationship

$$\left\langle \mathcal{L}\left(\boldsymbol{U},\boldsymbol{W}\right),\boldsymbol{V}\right\rangle = \left\langle \boldsymbol{W},\mathcal{L}^{*}\left(\boldsymbol{U},\boldsymbol{V}\right)\right\rangle,$$

for all $\boldsymbol{U}$, $\boldsymbol{W}$, $\boldsymbol{V} \in \boldsymbol{\mathcal{X}}$. Note that this operator appears after transferring the space derivatives to the test function. Again, the duality might involve inter-element jump terms when FE functions are considered. However, if these inter-element terms are neglected and $\tilde{P}$ denotes the L^2 projection onto the space of sub-scales, Eq. (33) can be formally written as

$$\tilde{P}\left[\boldsymbol{M}\left(\boldsymbol{U}\right)\frac{\partial \tilde{\boldsymbol{U}}}{\partial t}+\mathcal{L}\left(\boldsymbol{U},\tilde{\boldsymbol{U}}\right)\right]=\tilde{P}\left[\boldsymbol{R}_{\boldsymbol{U}}\right],$$

where $\boldsymbol{R}_{\boldsymbol{U}}$ represents the residual of the FE approximation, defined as

$$\boldsymbol{R}_{\boldsymbol{U}}=\boldsymbol{F}-\boldsymbol{M}\left(\boldsymbol{U}\right)\frac{\partial \boldsymbol{U}_h}{\partial t}-\mathcal{L}\left(\boldsymbol{U},\boldsymbol{U}_h\right).$$

At this point, additional approximations are required, and different methods may be devised according to the approximations chosen. A particular case is an algebraic approximation of the sub-scales [107]. The approximation of the problem in this case is:

$$\mathcal{L}\left(\boldsymbol{U},\tilde{\boldsymbol{U}}\right)\approx\boldsymbol{\alpha}^{-1}\left(\boldsymbol{U}\right)\tilde{\boldsymbol{U}},$$

where $\boldsymbol{\alpha}^{-1}\left(\boldsymbol{U}\right)$ is a matrix defined within each element domain that has to be determined. The standard option is to define $\boldsymbol{\alpha}\left(\boldsymbol{U}\right)$ as a diagonal matrix; however, this is not the only option as seen in [108]. Throughout this work, we will refer to $\boldsymbol{\alpha}\left(\boldsymbol{U}\right)$ as the matrix of stabilisation parameters. From the physical point of view, this matrix ensures dimensional consistency in the stabilisation terms that finally are added to the Galerkin problem, and from the numerical point of view it ensures stability and optimal order of convergence.

With the above considerations and for an adequate projection onto the sub-scale space $\tilde{P}$, $\tilde{\boldsymbol{U}}$ needs to be calculated from the following ordinary nonlinear differential equation in each element:

$$\boldsymbol{M}\left(\boldsymbol{U}\right)\frac{\partial\tilde{\boldsymbol{U}}}{\partial t}+\boldsymbol{\alpha}^{-1}\left(\boldsymbol{U}\right)\tilde{\boldsymbol{U}}=\tilde{P}\left[\boldsymbol{F}-\boldsymbol{M}\left(\boldsymbol{U}\right)\frac{\partial\boldsymbol{U}_h}{\partial t}-\mathcal{L}\left(\boldsymbol{U}_h+\tilde{\boldsymbol{U}},\boldsymbol{U}_h\right)\right],\quad(34)$$

where we have assumed for simplicity that the term on the left belongs to the sub-scale space. It is important to note that the calculation of $\tilde{\boldsymbol{U}}$ needs to be made at the integration points, that is to say, Eq. (34) is, in fact, a nonlinear ordinary differential equation.

Concerning the approximation of the sub-scale from Eq. (34), some remarks can be done:

Remark 1 For any definition of $\tilde{P}$, the final formulation is residual-based, that is, the sub-scales depends on the residual of the FE approximation and, therefore, the method derived under this approach is *consistent* by construction.

Remark 2 Neglecting the time derivative in Eq. (34) could be understood as considering that the sub-scales adapt automatically to the FE residual. The sub-scales obtained from this assumption were defined in [106] as *quasi-static*.

Remark 3 If the time derivative in Eq. (34) is considered, the obtained method is called *dynamic*. The benefits of a dynamic formulation can be found in [109]. Among other things, a dynamic formulation allows one to use an anisotropic space-time discretisation, avoiding the small time step instability encountered in the quasi-static option.

Remark 4 Observe that (34) is a nonlinear equation, due to the dependence of $\boldsymbol{\alpha}$ and $\boldsymbol{R}_{\boldsymbol{U}}$ on the sub-scale $\tilde{\boldsymbol{U}}$. Note that this nonlinearity is independent on whether the sub-scales vary in time or not. The final non-linear problem associated to the sub-scale can be linearised and solved iteratively. If this is the case, the sub-scale is called *nonlinear* (see [104] for more details).

Replacing the sub-scale obtained from Eq. (34) after discretising in time in Eq. (32), we obtain a residual-based VMS stabilised formulation that permits equal order interpolation between the variables of the problem. It remains only to define the space of sub-scales or, what is equivalent, the projection $\tilde{P}$. As mentioned before, the two possibilities that we consider are to take $\tilde{P}$ as the adequate identity I when applied to FE residuals or to take it as the L^2 orthogonal (global) projection to the FE space. We refer to the first option as ASGS (Algebraic Sub-Grid Scale) and to the second as OSS (Orthogonal Sub-scale stabilisation).

5.3 Two particular aspects

In the following, we describe in more detail two aspects of the general approach presented above. First, we elaborate on the time-dependent sub-scales, and next, we consider the possibility of using non-residual-based formulations.

5.3.1 Time dependent sub-scales

Stabilised FE methods were initially motivated for the steady-state case and later extended to the transient case (see for example [110] for an early space-time approach). In a VMS method, the exact problem for the sub-grid scales involves inverting a differential operator; the approximation to this operator leads to the stabilisation parameters on which the formulation depends. A classical way to proceed is to consider the time derivative discretised by finite differences, and therefore to consider its effect as a source with coefficient δt^{-1}, and to neglect the time derivative of the sub-scales. Thus, the stabilisation parameters depend on δt^{-1}. This fact, in particular, implies that the steady state solution, if it exists, depends on the value of δt used to step in time. Apart from this inconsistency, as it has been mentioned this approach is unstable for anisotropic space-time discretisations, as pointed out in [111,73].

The use of time-dependent sub-scales was introduced in [106], and its importance elaborated in [72,112,73]. Here we summarise the key ideas of a dynamic formulation for the sub-scales, which for the viscoelastic case was proposed and tested numerically in [17].

Using a first order backward difference scheme to discretise in time the sub-scale equation (34) and taking n as the time integrator counter, we arrive at:

$$\tilde{\boldsymbol{U}}^{n+1} = \left(\frac{1}{\delta t} \boldsymbol{M}\left(\boldsymbol{U}\right) + \boldsymbol{\alpha}\left(\boldsymbol{U}\right)^{-1} \right)^{-1} \tilde{P} \left(\frac{1}{\delta t} \tilde{\boldsymbol{U}}^{n} + \left[\boldsymbol{R}_{\boldsymbol{U}}\right] \right), \tag{35}$$

where it is understood that $\boldsymbol{\alpha}\left(\boldsymbol{U}\right)$ needs to be computed with the sub-scale at time step $n+1$. From these expressions we see that the sub-scales depend on $\boldsymbol{\alpha}\left(\boldsymbol{U}\right)_{\mathrm{dyn}} = \left(\delta t^{-1} \boldsymbol{I} + \boldsymbol{\alpha}\left(\boldsymbol{U}\right)^{-1} \right)^{-1}$. Expressions with asymptotic behaviour similar to that of $\boldsymbol{\alpha}\left(\boldsymbol{U}\right)_{\mathrm{dyn}}$ have been proposed for the Navier-Stokes problem,

for example in [110,113]. In [111], there is a study of the instability encountered when the ASGS method *and quasi-static sub-scales* are used. It is shown in that reference that even for the Stokes time continuous problem the Schur complement matrix for the pressure is not uniformly invertible, and this property is inherited as $\delta t \to 0$ if h remains fixed. The same arguments can be applied to the viscoelastic case, where in addition we have a time dependent constitutive equation.

It is not difficult to prove that this discrete space-time instability disappears if, for a specific matrix norm $|\cdot|$,

$$|\boldsymbol{M}(\boldsymbol{U})\,\delta t^{-1}| \leq C|\boldsymbol{\alpha}(\boldsymbol{U})^{-1}|, \tag{36}$$

where C represents a positive constant. From this condition, it seems clear that the stabilisation parameters and the time step size must be related in classical (quasi-static) stabilised FE methods. It is important to note that, if (36) holds, it is irrelevant from the analysis point of view whether the sub-scales are considered dynamic or quasi-static. The importance of considering dynamic sub-scales is that (36) is not needed for stability, which could become crucial when we have two evolutionary equations with a different dynamic nature.

5.3.2 Non-residual-based stabilised formulations

A stabilised FE formulation designed following the steps presented above is globally stable and for smooth enough solutions should display optimal order of convergence. Let us suppose, to simplify the following discussion, that quasi-static sub-scales are used. In this case Eq. (35) reduces to $\tilde{\boldsymbol{U}} = \boldsymbol{\alpha}(\boldsymbol{U})\tilde{P}\left[\boldsymbol{R}_{\boldsymbol{U}}\right]$ at each time step. Replacing this in (32), yields:

$$\begin{aligned}\left(\boldsymbol{M}(\boldsymbol{U})\frac{\partial \boldsymbol{U}}{\partial t}, \boldsymbol{V}_h\right) + \left\langle \mathcal{L}(\boldsymbol{U}, \boldsymbol{U}_h), \boldsymbol{V}_h \right\rangle \\ + \left\langle \boldsymbol{\alpha}(\boldsymbol{U})\tilde{P}\left[\boldsymbol{R}_{\boldsymbol{U}}\right], \mathcal{L}^*(\boldsymbol{U}, \boldsymbol{V}_h)\right\rangle = \langle \boldsymbol{F}, \boldsymbol{V}_h \rangle \quad \forall\, \boldsymbol{V}_h \in \boldsymbol{\mathcal{X}}_h,\end{aligned} \tag{37}$$

where we can see that the stabilisation term added to the original weak form is a function of the FE residual and the adjoint operator applied to the test functions. The number of stabilisation terms that arise from this inner product can be very large, this is the case of the viscoelastic flow problem. The key idea of non-residual-based formulations is that not all the resulting terms provide stability to the formulation.

If we consider the case $\tilde{P} = P_h^{\perp}$, from (37) we can design *a simplified method*, which consists in *neglecting the cross local inner-product terms* that arise from the stabilisation, as well as some other terms that do not contribute to stability. The key aspect that allows one to do that even for high order elements is the orthogonal projection $P_h^{\perp}$.

The price to pay when using a non-residual method is that the terms added to the Galerkin problem are not zero when the FE solution is replaced by the

continuous solution, and therefore the method is not consistent in the sense used in the FE context. However, *the consistency error is of optimal order*, since for any smooth enough function f, $P_h(f)$ is an optimal approximation to f in the FE space, and therefore $P_h^{\perp}(f)$ goes to zero with h at the optimal rate permitted by the FE interpolation (and the smoothness of f).

In the next section, the general VMS ideas presented heretofore will be applied to the viscoelastic problem, firstly in the standard case and then for the logarithmic reformulation of the viscoelastic problem. Let us remark that the non-residual term-by-term stabilisation, first proposed in [114] and analysed in [115], was found to be particularly well suited to the viscoelastic flow problem, showing improved accuracy and robustness in classical benchmarks where the problem presents geometric singularities like a non-convex corner, or in general when the gradients of the elastic stresses are important [10,16].

6 Stabilised finite element formulations for the viscoelastic problem

In this section we finally present the stabilised FE formulations for the viscoelastic flow problem we wish to review, all based on the VMS concept and applied to the standard formulation and the logarithmic conformation reformulation. In both cases we use a similar scheme: we start considering quasi-static sub-scales with continuous interpolations for all fields and a residual-based formulation, then we describe the non-residual-based approach and finally we describe the modifications caused by considering the sub-scales dynamic. In the case of the standard formulation, we consider two more issues, that could be easily extended to the logarithmic conformation reformulation case: the introduction of some inter-element terms that are needed when pressures and stresses are discontinuous and the convenience of using a discontinuity-capturing technique.

6.1 The standard viscoelastic problem

6.1.1 Residual-based stabilised finite element formulations

Let us assume for the moment that we use quasi-static sub-scales and that the FE interpolation for all fields is continuous. A stabilised residual-based FE formulation designed following the ideas presented above consists in adding to the Galerkin terms of the problem the term $\left\langle \boldsymbol{\alpha} \tilde{P}\left[\boldsymbol{R}_{\boldsymbol{U}}\right], \mathcal{L}^*\left(\boldsymbol{U}, \boldsymbol{V}_h\right)\right\rangle$. Thus, we need to specify all terms of this expression for the viscoelastic problem. In particular, the adjoint operator is given by

$$\mathcal{L}^*\left(\hat{\boldsymbol{u}}; \boldsymbol{V}\right) = \begin{pmatrix} \nabla \cdot \boldsymbol{\chi} - 2\eta_s \nabla \cdot \nabla^{\mathrm{s}} \boldsymbol{v} - \rho \hat{\boldsymbol{u}} \cdot \nabla \boldsymbol{v} - \nabla q \\ -\nabla \cdot \boldsymbol{v} \\ \dfrac{1}{2\eta_p} \boldsymbol{\chi} + \nabla^{\mathrm{s}} \boldsymbol{v} - \dfrac{\lambda}{2\eta_p} \left(\hat{\boldsymbol{u}} \cdot \nabla \boldsymbol{\chi} + \boldsymbol{\chi} \cdot \left(\nabla \hat{\boldsymbol{u}}\right)^T + \nabla \hat{\boldsymbol{u}} \cdot \boldsymbol{\chi} \right) \end{pmatrix}. \quad (38)$$

The matrix of stabilisation parameters ($\boldsymbol{\alpha}$) is taken as diagonal [100], of the form:

$$\boldsymbol{\alpha} = \mathrm{diag}\left(\alpha_1 \boldsymbol{I}_d, \alpha_2, \alpha_3 \boldsymbol{I}_{d\times d}\right), \tag{39}$$

with $\boldsymbol{I}_d$ the identity on vectors of $\mathbb{R}^d$, and $\boldsymbol{I}_{d\times d}$ the identity on second order tensors. The parameters α_i, $i = 1, 2, 3$, are computed following [100] as

$$\alpha_1 = \left[c_1 \frac{\eta_s}{h_1^2} + c_2 \frac{\rho\,|\hat{\boldsymbol{u}}_h|}{h_2}\right]^{-1}, \tag{40}$$

$$\alpha_2 = \frac{h_1^2}{c_1 \alpha_1}, \tag{41}$$

$$\alpha_3 = \left[c_3 \frac{1}{2\eta_p} + c_4 \left(\frac{\lambda}{2\eta_p}\frac{|\hat{\boldsymbol{u}}_h|}{h_2} + \frac{\lambda}{\eta_p}\left|\nabla \hat{\boldsymbol{u}}_h\right|\right)\right]^{-1}. \tag{42}$$

In these expressions, h_1 corresponds to a characteristic length calculated as the square root of the element area in the 2D case and the cubic root of the element volume in 3D, and h_2 corresponds to another characteristic length calculated as the element length in the streamline direction. The term $|\boldsymbol{u}_h|$ is the Euclidean norm of the velocity, and $|\nabla \boldsymbol{u}_h|$ the Frobenius norm of the velocity gradient (another possibility for this last term is to take the maximum eigenvalue of the velocity gradient matrix). The constants c_i, $i = 1, 4$ are algorithmic parameters in the formulation. The values proposed in [10] for linear elements were $c_1 = 4.0$, $c_2 = 2.0$, $c_3 = 4.0$ and $c_4 = 0.25$. For higher order elements, the characteristic lengths h_1 and h_2 are respectively divided by k^2 and k, k being the order of the FE interpolation, and we keep the value of the constants used for linear elements. In [116], the convection-diffusion-reaction equation problem was solved using elements of order one up to four, highlighting the importance of considering the dependence of the constants that appear in the stabilising terms with k.

Having introduced the matrix of the stabilisation parameters and the adjoint operator, the stabilisation terms can be written as:

$$\begin{aligned}\left\langle \boldsymbol{\alpha}\tilde{P}\left[\boldsymbol{R}_{\boldsymbol{U}}\right], \mathcal{L}^*\left(\boldsymbol{U}_h, \boldsymbol{V}_h\right)\right\rangle &= \sum_K \alpha_1 S_1\left(\boldsymbol{u}_h; \boldsymbol{U}_h, \boldsymbol{V}_h\right)|_K \\ &+ \sum_K \alpha_2 S_2\left(\boldsymbol{U}_h, \boldsymbol{V}_h\right)|_K + \sum_K \alpha_3 S_3\left(\boldsymbol{u}_h; \boldsymbol{U}_h, \boldsymbol{V}_h\right)|_K.\end{aligned} \tag{43}$$

where $\sum_K$ represents the summation over all the elements of $\mathcal{T}_h$. In this expression, we have not introduced any inter-element boundary terms, which are unnecessary when all FE unknowns are continuous. The need to introduce them is commented below.

In (43), we have the following terms that we need to add to the Galerkin problem, computed in each element as:

$$S_1\left(\hat{\boldsymbol{u}}_h; \boldsymbol{U}_h, \boldsymbol{V}_h\right)|_K = \left\langle \tilde{P}\left[\boldsymbol{R}_{\boldsymbol{u}}\right], \mathcal{L}^*_{\boldsymbol{u}}(\hat{\boldsymbol{u}}_h; \boldsymbol{U}_h, \boldsymbol{V}_h)\right\rangle_K, \tag{44}$$

$$S_2\left(\boldsymbol{U}_h, \boldsymbol{V}_h\right)|_K = \left\langle \tilde{P}\left[R_p\right], \mathcal{L}^*_p(\boldsymbol{U}_h, \boldsymbol{V}_h)\right\rangle_K, \tag{45}$$

$$S_3\left(\hat{\boldsymbol{u}}_h;\boldsymbol{U}_h,\boldsymbol{V}_h\right)|_K = \left\langle \tilde{P}\left[\boldsymbol{R}_{\boldsymbol{\sigma}}\right], \mathcal{L}^*_{\boldsymbol{\sigma}}(\hat{\boldsymbol{u}}_h;\boldsymbol{U}_h,\boldsymbol{V}_h)\right\rangle_K, \tag{46}$$

where $\mathcal{L}^*_{\boldsymbol{u}}(\hat{\boldsymbol{u}}_h;\boldsymbol{U}_h,\boldsymbol{V}_h)$, $\mathcal{L}^*_p(\boldsymbol{U}_h,\boldsymbol{V}_h)$ and $\mathcal{L}^*_{\boldsymbol{\sigma}}(\hat{\boldsymbol{u}}_h;\boldsymbol{U}_h,\boldsymbol{V}_h)$ are the three components of the adjoint operator and $\boldsymbol{R}_{\boldsymbol{u}}$, R_p and $\boldsymbol{R}_{\boldsymbol{\sigma}}$ represent the residual of the momentum, continuity, and constitutive equations, respectively, which are defined as:

$$\boldsymbol{R}_{\boldsymbol{u}} = \boldsymbol{f} - \left(\rho\frac{\partial \boldsymbol{u}_h}{\partial t} - \nabla\cdot\boldsymbol{\sigma}_h - 2\eta_s\nabla\cdot\nabla^{\mathrm{s}}\boldsymbol{u}_h + \rho\hat{\boldsymbol{u}}_h\cdot\nabla\boldsymbol{u}_h + \nabla p_h\right), \tag{47}$$

$$R_p = -\nabla\cdot\boldsymbol{u}_h, \tag{48}$$

$$\boldsymbol{R}_{\boldsymbol{\sigma}} = -\frac{1}{2\eta_p}\boldsymbol{\sigma}_h + \nabla^{\mathrm{s}}\boldsymbol{u}_h - \frac{\lambda}{2\eta_p}\left(\frac{\partial\boldsymbol{\sigma}_h}{\partial t} + \hat{\boldsymbol{u}}_h\cdot\nabla\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h\cdot\nabla\hat{\boldsymbol{u}}_h - (\nabla\hat{\boldsymbol{u}}_h)^T\cdot\boldsymbol{\sigma}_h\right). \tag{49}$$

With respect to the definition of $\tilde{P}$, recall that two possible choices are considered, as in the general case. The first is the ASGS formulation, in which $\tilde{P} = I$ (the identity) when applied to FE residuals, whereas the second is $\tilde{P} = P_h^{\perp} = I - P_h$, where P_h is the L^2 projection onto the appropriate FE space.

Remark 5 Independently of the selection of $\tilde{P}$, either $\tilde{P} = I$ or $\tilde{P} = P_h^{\perp}$, the obtained method should display optimal order of convergence for smooth enough solutions. The above residual-based stabilisation method was proposed in [10] for the steady state case.

6.1.2 Non-residual-based stabilised finite element formulations

In [10,11,14], the authors found that a residual-based formulation in viscoelastic fluid flows is not always the best option from the point of view of accuracy and robustness when solving classical viscoelastic benchmarks, such as the 4:1 planar contraction problem and the flow over a confined cylinder problem. *If we consider the case* $\tilde{P} = P_h^{\perp}$, from (43) we can design *a simplified method*, which consists in *neglecting the cross local inner-product terms* in (43), as well as some other terms that do not contribute to stability. In particular, the term $S_1(\hat{\boldsymbol{u}}_h;\boldsymbol{U}_h,\boldsymbol{V}_h)|_K$ in (44) can in principle be replaced by

$$\left\langle P_h^{\perp}\left[\nabla\cdot\boldsymbol{\sigma}_h\right], \nabla\cdot\boldsymbol{\chi}_h\right\rangle_K + \rho^2\left\langle P_h^{\perp}\left[\hat{\boldsymbol{u}}_h\cdot\nabla\boldsymbol{u}_h\right], \hat{\boldsymbol{u}}_h\cdot\nabla\boldsymbol{v}_h\right\rangle_K + \left\langle P_h^{\perp}\left[\nabla p_h\right], \nabla q_h\right\rangle_K - 4\beta^2\eta_0^2\left\langle P_h^{\perp}\left[\nabla\cdot(\nabla^{\mathrm{s}}\boldsymbol{u}_h)\right], \nabla\cdot(\nabla^{\mathrm{s}}\boldsymbol{v}_h)\right\rangle_K. \tag{50}$$

The three first terms in (50) help to improve stability, the first giving control on the divergence of the viscoelastic stress, the second on the convective term, and the third one the pressure gradient. The last term can be neglected, as it does not contribute to stability.

Remark 6 As already mentioned, the method associated to (43) is consistent for any projection $\tilde{P}$, and therefore we might consider $\tilde{P} = I - P_{h,0}$ for the projection applied to the residual of the momentum equation, where $P_{h,0}$ is the L^2 projection onto the velocity FE space *incorporating* boundary conditions, i.e., the L^2 projection onto $\boldsymbol{\mathcal{V}}_{h,0}$. However, in (50) we need to take $P_h^\perp = I - P_h$, P_h being the L^2 projection *without* boundary conditions, since otherwise $P_h^\perp(f)$ would not converge to zero at the optimal order mentioned before. The price to be paid is that the FE mesh needs to satisfy a mild compatibility condition, as explained in [117], which is easily fulfilled for most meshes.

It should be noted that, apart from $P_h^\perp$, other projections can be used in (50), as long as one can guarantee that they provide enough stability and that the consistency error introduced has optimal order. Among these, let us mention those associated to the Local Projection Stabilisation. For details about this method, see for example [118,119] in the context of Newtonian fluids or the work of [70] for viscoelastic fluids. These local projections avoid the global projection P_h, which requires to solve a linear system; however, the bottleneck of these methods is that they all increase the sparsity of the final matrix. This is usually unaffordable, and therefore an iterative strategy is needed anyway. For another term-by-term approach, only applicable to low order elements, see [120].

Similar considerations could be applied to modify the term $S_3(\hat{\boldsymbol{u}}_h; \boldsymbol{U}_h, \boldsymbol{V}_h)$ in (46), now taking into account that $P_h^\perp(\boldsymbol{\sigma}_h) = \boldsymbol{0}$ ($P_h^\perp$ being now the orthogonal projection to the space of viscoelastic stresses), giving place to a *fully* non-residual stabilisation method. Under these considerations, this modified method will be called *fully split OSS method*, and reads as: find $\boldsymbol{U}_h \in \boldsymbol{\mathcal{X}}_h$ such that

$$\begin{aligned}
&(\mathcal{D}_t(\boldsymbol{U}_h), \boldsymbol{V}_h) + B(\boldsymbol{u}_h; \boldsymbol{U}_h, \boldsymbol{V}_h) + \sum_K \alpha_1 S_1^\perp\left(\boldsymbol{u}_h; \boldsymbol{U}_h, \boldsymbol{V}_h\right)|_K \\
&\quad + \sum_K \alpha_2 S_2^\perp\left(\boldsymbol{U}_h, \boldsymbol{V}_h\right)|_K + \sum_K \alpha_3 S_3^\perp\left(\boldsymbol{u}_h; \boldsymbol{U}_h, \boldsymbol{V}_h\right)|_K = \langle \boldsymbol{F}, \boldsymbol{V}_h \rangle, \qquad (51)
\end{aligned}$$

for all $\boldsymbol{V}_h \in \boldsymbol{\mathcal{X}}_h$, where

$$\begin{aligned}
&S_1^\perp\left(\hat{\boldsymbol{u}}_h; \boldsymbol{U}_h, \boldsymbol{V}_h\right)|_K = \left\langle P_h^\perp\left[\nabla \cdot \boldsymbol{\sigma}_h\right], \nabla \cdot \boldsymbol{\chi}_h \right\rangle_K + \left\langle P_h^\perp\left[\nabla p_h\right], \nabla q_h \right\rangle_K \\
&\quad + \left\langle P_h^\perp\left[\rho \hat{\boldsymbol{u}}_h \cdot \nabla \boldsymbol{u}_h\right], \rho \hat{\boldsymbol{u}}_h \cdot \nabla \boldsymbol{v}_h \right\rangle_K, \qquad (52) \\
&S_2^\perp\left(\boldsymbol{U}_h, \boldsymbol{V}_h\right)|_K = \left\langle P_h^\perp\left[\nabla \cdot \boldsymbol{u}_h\right], \nabla \cdot \boldsymbol{v}_h \right\rangle_K, \qquad (53) \\
&S_3^\perp\left(\boldsymbol{u}_h; \boldsymbol{U}_h, \boldsymbol{V}_h\right)|_K = \left\langle P_h^\perp\left[\nabla^{\mathrm{s}} \boldsymbol{u}_h\right], \nabla^{\mathrm{s}} \boldsymbol{v}_h \right\rangle_K \\
&+ \left(\frac{\lambda}{2\eta_p}\right)^2 \left\langle P_h^\perp\left[\hat{\boldsymbol{u}}_h \cdot \nabla \boldsymbol{\sigma}_h\right], \hat{\boldsymbol{u}}_h \cdot \nabla \boldsymbol{\chi}_h \right\rangle_K \\
&- \left(\frac{\lambda}{2\eta_p}\right)^2 \left\langle P_h^\perp\left[\boldsymbol{\sigma}_h \cdot \nabla \hat{\boldsymbol{u}}_h + (\nabla \hat{\boldsymbol{u}}_h)^T \cdot \boldsymbol{\sigma}_h\right], \boldsymbol{\chi}_h \cdot (\nabla \hat{\boldsymbol{u}}_h)^T + \nabla \hat{\boldsymbol{u}}_h \cdot \boldsymbol{\chi}_h \right\rangle_K. \qquad (54)
\end{aligned}$$

The stabilising mechanism introduced by the different terms in these expressions is clear. Term $S_1^{\perp}$ has already been discussed. Regarding term $S_3^{\perp}$ in (54) it is observed that the first part has an EVSS-like structure (see [121]), that now has been derived from a simplification of the OSS method, whereas the second part has a streamline-upwind structure, but the introduction of the orthogonal projection $P_h^{\perp}$ makes it have an optimal consistency error.

In [10], both the residual and the fully split stabilised FE formulations presented above were tested exhaustively, concluding that the best option is an *amphibian* between a residual-based structure for the constitutive equation and a non-residual-based structure for the momentum equation. The reason why it is better to use a residual-based method for the constitutive equation is explained in [17] in the context of the LCR, and is related to the convergence of iterative schemes. These turn out to behave much better using the residual-based formulation because while the residual of the constitutive equation may be small, each term separately is not, causing difficulties in convergence of the split version.

From what has been explained, expression (52) is not just a simplification of (44). In [10], it is emphasised that the "problematic" terms in the residual formulation that disable it for the calculation of highly elastic fluids are the cross-terms $\alpha_1 \langle \nabla p_h, -\nabla \cdot \boldsymbol{\chi}_h \rangle_K$ and $\alpha_1 \langle -\nabla \cdot \boldsymbol{\sigma}_h, \nabla q_h \rangle_K$. In the presence of high gradients of pressure and stresses, such as in the corner of the 4:1 contraction, they lead to convergence difficulties for high Weissenberg numbers and to inaccurate localization of pressure and stress peaks.

With these idea in mind, the stabilised method we recommended is: find $\boldsymbol{U}_h : [0, t_{\mathrm{f}}] \longrightarrow \boldsymbol{\mathcal{X}}_h$ such that

$$\begin{aligned}(\mathcal{D}_t(\boldsymbol{U}_h), \boldsymbol{V}_h) + B(\boldsymbol{u}_h; \boldsymbol{U}_h, \boldsymbol{V}_h) + \sum_K \alpha_1 S_1^{\perp}(\boldsymbol{u}_h; \boldsymbol{U}_h, \boldsymbol{V}_h)\,|_K \\ + \sum_K \alpha_2 S_2(\boldsymbol{U}_h, \boldsymbol{V}_h)\,|_K + \sum_K \alpha_3 S_3(\boldsymbol{u}_h; \boldsymbol{U}_h, \boldsymbol{V}_h)\,|_K = \langle \boldsymbol{F}, \boldsymbol{V}_h \rangle\,, \end{aligned} \tag{55}$$

for all $\boldsymbol{V}_h \in \boldsymbol{\mathcal{X}}_h$, and satisfying the initial conditions in a weak sense. The projection involved in S_2 and S_3 may be either $\tilde{P} = I$ or $\tilde{P} = P_h^{\perp}$.

6.1.3 Discontinuity-capturing technique

The stabilised FE formulation presented above yields a *globally* stable solution, i.e., norms of the unknowns over the whole domain Ω are bounded. However, if the solution displays very high gradients, local oscillations may still remain. This is a known issue in compressible flows, where the high gradients of the velocity and pressure fields make evident the need for some numerical treatment additional to the stabilised methods. In this context, discontinuity-capturing (DC) or shock-capturing techniques can be introduced in the numerical formulation. In general, the main idea of any DC technique is to increase the amount of numerical dissipation in the neighbourhood of layers (see [78, 122, 123], for example).

In viscoelastic flow problems, we can find local instabilities or very high gradients in the pressure and in the viscoelastic stress components when the fluid flow finds an abrupt change in the geometry. This gradient can be especially strong when the amount of elastic component in the fluid is important, giving place to boundary layers even in creeping flows.

Due to the fact that the constitutive equation is of convective-reactive nature, the general ideas developed for classical methods can be used in the viscoelastic problem. At the discrete level, the advection velocity is $\boldsymbol{u}_h$ and the reactive term is proportional to the velocity gradient $\nabla \boldsymbol{u}_h$. Therefore, if a non-consistent artificial diffusion is to be introduced, it must be of the form

$$k_\sigma = c_a h_1 \left|\boldsymbol{u}_h\right| + c_b h_1^2 \left|\nabla \boldsymbol{u}_h\right|, \tag{56}$$

where characteristic lengths give dimensional consistency to this numerical diffusion, and the parameters c_a and c_b are algorithmic constants.

Remark 7 If an artificial diffusion term is added to problem (55), with the diffusion given by (56), the consistency error would make the method converge at most with an order $\mathcal{O}(h^{1/2})$.

In order to design a method that can yield optimal convergence, at least when the solution is smooth, a possible option is to multiply the diffusion in (56) by a term proportional to the residual of the equation being solved properly normalised (see [78] and references therein). In this case, the residual would be that of the constitutive equation, as in [77]. We have used another switch to activate the numerical diffusion, which is to make it proportional to the component of the viscoelastic stress orthogonal to the FE space, that is to say, we take as diffusion coefficient

$$k_\sigma = \left(c_a h_1 \left|\boldsymbol{u}_h\right| + c_b h_1^2 \left|\nabla \boldsymbol{u}_h\right|\right) \frac{\left|P_h^\perp \left[\nabla \boldsymbol{\sigma}_h\right]\right|}{\left|\nabla \boldsymbol{\sigma}_h\right|}, \tag{57}$$

instead of (56). The way to introduce the DC dissipation is to add the term

$$\sum_K k_\sigma \left\langle \nabla \boldsymbol{\sigma}_h, \nabla \boldsymbol{\chi}_h \right\rangle_K,$$

to the left-hand-side of (55), where $\boldsymbol{\chi}_h$ is the test function of the constitutive equation. The algorithmic constants suggested in [10] are $c_a = 0.1$ and $c_b = 0.5$.

6.1.4 Interpolation with discontinuous pressures and stresses

The stabilised FE formulations described above permits the use of equal order interpolation for all the unknowns. The extension to arbitrary interpolations can be done adding a single extra term to (51), as was proposed and analysed in [13].

Let us introduce some notation. The collection of all edges (faces, for $d = 3$) of the elements will be denoted by $\mathcal{E}_h = \{E\}$ and, as for the elements, summation over all these edges will be indicated as $\sum_E$. Suppose now that elements

K_1 and K_2 share an edge E, and let $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ be the normals to E exterior to K_1 and K_2, respectively. For a scalar function g, possibly discontinuous across E, we define its jump as $[\![\boldsymbol{n} g]\!]_E := \boldsymbol{n}_1 g|_{\partial K_1 \cap E} + \boldsymbol{n}_2 g|_{\partial K_2 \cap E}$, and for a vector or tensor $\boldsymbol{v}$, $[\![\boldsymbol{n} \cdot \boldsymbol{v}]\!]_E := \boldsymbol{n}_1 \cdot \boldsymbol{v}|_{\partial K_1 \cap E} + \boldsymbol{n}_2 \cdot \boldsymbol{v}|_{\partial K_2 \cap E}$. When $E \subset \partial\Omega$ and $\boldsymbol{n}$ is the external normal, these definitions reduce to $[\![\boldsymbol{n} g]\!]_E := \boldsymbol{n} g|_E$ and $[\![\boldsymbol{n} \cdot \boldsymbol{v}]\!]_E := \boldsymbol{n} \cdot \boldsymbol{v}|_E$.

Since velocity components need to belong to a subspace of $H^1(\Omega)$, conforming approximations require continuity. However, pressures and stress components need to belong to a subspace of $L^2(\Omega)$, and therefore one might use discontinuous interpolations and still be conforming. If this is the case, formulation (51) is not necessary stable. The terms to stabilise it can be motivated again by the VMS concept, as explained in [103,55]. In the present context, the following term needs to be added to the left-hand-side of problem (51):

$$\sum_E \alpha_{[\boldsymbol{u}]} \langle [\![(\boldsymbol{n} q_h - \boldsymbol{n} \cdot \boldsymbol{\chi}_h) + 2\eta_s \boldsymbol{n} \cdot \nabla^{\mathrm{s}} \boldsymbol{v}_h]\!] , \\ [\![(\boldsymbol{n} p_h - \boldsymbol{n} \cdot \boldsymbol{\sigma}_h) - 2\eta_s \boldsymbol{n} \cdot \nabla^{\mathrm{s}} \boldsymbol{u}_h]\!] \rangle_E , \tag{58}$$

where $\alpha_{[\boldsymbol{u}]}$ represents an additional stabilisation parameter defined as $\alpha_{[\boldsymbol{u}]} = \frac{\delta_0 h}{2\eta_0}$, where δ_0 is an algorithmic parameter that can be taken as $\delta_0 = \frac{1}{10}$ [103, 55]. The inclusion of (58) in the stabilisation method allows the use of discontinuous pressure and stress interpolations. The numerical analysis of this method was done in [13] for a linearised version of the Navier-Stokes/Oldroyd-B case, where it was proved that the method is stable and optimally convergent for small Weissenberg numbers, independently of the interpolation used.

6.1.5 Time-dependent sub-scales

As mentioned earlier, a VMS stabilised method can be dynamic or quasi-static. Following the ideas proposed in [17], for residual-based stabilised methods, we can design a dynamic formulation using Eq. (34), which in this case reads:

$$\rho \frac{\partial \tilde{\boldsymbol{u}}}{\partial t} + \alpha_1^{-1} \tilde{\boldsymbol{u}} = \tilde{P} [\boldsymbol{R}_{\boldsymbol{u}}] , \tag{59}$$

$$\alpha_2^{-1} \tilde{p} = \tilde{P} [R_p] , \tag{60}$$

$$\frac{\lambda}{2\eta_p} \frac{\partial \tilde{\boldsymbol{\sigma}}}{\partial t} + \alpha_3^{-1} \tilde{\boldsymbol{\sigma}} = \tilde{P} [\boldsymbol{R}_{\boldsymbol{\sigma}}] . \tag{61}$$

Using a first order BDF scheme, the velocity and stress sub-scales are obtained from:

$$\tilde{\boldsymbol{u}}^{n+1} = \left(\frac{\rho}{\delta t} + \frac{1}{\alpha_1} \right)^{-1} \tilde{P} \left[\frac{1}{\delta t} \tilde{\boldsymbol{u}}^n + \boldsymbol{R}_{\boldsymbol{u}} \right] , \tag{62}$$

$$\tilde{\boldsymbol{\sigma}}^{n+1} = \left(\frac{\lambda}{2\eta_p \delta t} + \frac{1}{\alpha_3} \right)^{-1} \tilde{P} \left[\frac{1}{\delta t} \tilde{\boldsymbol{\sigma}}^n + \boldsymbol{R}_{\boldsymbol{\sigma}} \right] , \tag{63}$$

where $\boldsymbol{R_u}$ and $\boldsymbol{R_\sigma}$ are the FE residual of the momentum and constitutive equations defined in Eqs. (47) and (49). Note that the continuity equation does not have time derivative, and therefore, does not play any role in the dynamic nature of the method. The final dynamic residual-based stabilised method is obtained after replacing expressions (62) and (63) in the stabilising term $\left\langle \tilde{\boldsymbol{U}}, \mathcal{L}^* \left(\boldsymbol{U}, \boldsymbol{V}_h\right) \right\rangle$ of Eq. (32). The problem to be solved is thus: find $\boldsymbol{U}_h : [0, t_\mathrm{f}] \longrightarrow \boldsymbol{\mathcal{X}}_h$ such that

$$
\begin{aligned}
&(\mathcal{D}_t(\boldsymbol{U}_h), \boldsymbol{V}_h) + B(\boldsymbol{u}_h; \boldsymbol{U}_h, \boldsymbol{V}_h) \\
&\quad + \sum_K \langle \tilde{\boldsymbol{u}}, \nabla \cdot \boldsymbol{\chi}_h - 2\eta_s \nabla \cdot \nabla^\mathrm{s} \boldsymbol{v}_h - \rho \boldsymbol{u}_h \cdot \nabla \boldsymbol{v}_h - \nabla q_h \rangle_K \\
&\quad + \sum_K \langle \tilde{p}, -\nabla \cdot \boldsymbol{v}_h \rangle_K \\
&\quad + \sum_K \left\langle \tilde{\boldsymbol{\sigma}}, \frac{1}{2\eta_p} \boldsymbol{\chi}_h + \nabla^\mathrm{s} \boldsymbol{v} - \frac{\lambda}{2\eta_p} \left(\boldsymbol{u}_h \cdot \nabla \boldsymbol{\chi}_h + \boldsymbol{\chi}_h \cdot (\nabla \boldsymbol{u}_h)^T + \nabla \boldsymbol{u}_h \cdot \boldsymbol{\chi}_h \right) \right\rangle_K \\
&\quad = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle \,, \qquad (64)
\end{aligned}
$$

for all $\boldsymbol{V}_h \in \boldsymbol{\mathcal{X}}_h$.

In [74] and [17], the idea of non-residual-based dynamic VMS methods was proposed for the two fields Navier-Stokes problem and for the viscoelastic case. In those works, contrary to residual-based formulations, it was necessary to introduce two and three velocity sub-scales, respectively, instead of a single one that is a function of the FE residual of the momentum equation.

Let us consider expressions (59) and (61), taking into account that $\tilde{P} = P_h^\perp$, $P_h^\perp [\boldsymbol{f}] \approx \boldsymbol{0}$, $P_h^\perp \left[\dfrac{\partial \boldsymbol{u}_h}{\partial t} \right] = 0$ and $P_h^\perp \left[\dfrac{\partial \boldsymbol{\sigma}_h}{\partial t} \right] = 0$. Therefore we can rewrite them as follows:

$$
\begin{aligned}
\rho \frac{\partial \tilde{\boldsymbol{u}}}{\partial t} + \alpha_1^{-1} \tilde{\boldsymbol{u}} =& P_h^\perp [\nabla \cdot \boldsymbol{\sigma}_h] + P_h^\perp [2\eta_s \nabla \cdot (\nabla^\mathrm{s} \boldsymbol{u}_h)] \\
& - P_h^\perp [\rho \boldsymbol{u}_h \cdot \nabla \boldsymbol{u}_h] - P_h^\perp [\nabla p_h] \,, \qquad (65) \\
\frac{\lambda}{2\eta_p} \frac{\partial \tilde{\boldsymbol{\sigma}}}{\partial t} + \alpha_3^{-1} \tilde{\boldsymbol{\sigma}} =& - P_h^\perp \left[\frac{1}{2\eta_p} \boldsymbol{\sigma}_h \right] + P_h^\perp [\nabla^\mathrm{s} \boldsymbol{u}_h] - P_h^\perp \left[\frac{\lambda}{2\eta_p} \boldsymbol{u}_h \cdot \nabla \boldsymbol{\sigma}_h \right] \\
& + P_h^\perp \left[\frac{\lambda}{2\eta_p} \left(\boldsymbol{\sigma}_h \cdot \nabla \boldsymbol{u}_h + (\nabla \boldsymbol{u}_h)^T \cdot \boldsymbol{\sigma}_h \right) \right] . \qquad (66)
\end{aligned}
$$

The key ingredient that allows us to consider *only* any of these terms is the orthogonal projection $P_h^\perp$. The right-hand-side (RHS) of (65) and (66) is not zero when the FE solution is replaced by the continuous solution, and consequently the method is not consistent. Nevertheless, the consistency error is optimal [117]. Additionally, some of the terms in the RHS of (65) and (66) can be neglected, like the second term of (65), because they do not contribute to stability. The three remaining terms help to improve stability, the first one giving control of the divergence of the viscoelastic stresses, the the third one on the convective term and the fourth one on the pressure gradient. Similar

considerations can be applied to modify equation (66), now considering that $P_h^{\perp}(\boldsymbol{\sigma}_h) = \mathbf{0}$.

As explained earlier, the previous splitting and simplification technique results in an a priori weakly consistent method. However, when the splitting approach is used in (66), the method fails to converge when applied to simple numerical tests as explained before in the context of quasi-static sub-scales.

In view of these remarks, we can split $\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}_1 + \tilde{\boldsymbol{u}}_2 + \tilde{\boldsymbol{u}}_3$, while the stress sub-grid scale remains as $\tilde{\boldsymbol{\sigma}}$. The sub-grid scales $\tilde{\boldsymbol{u}}_1$, $\tilde{\boldsymbol{u}}_2$, $\tilde{\boldsymbol{u}}_3$, $\tilde{p}$, are the solution of the evolution problems:

$$\rho\frac{\partial\tilde{\boldsymbol{u}}_1}{\partial t} + \alpha_1^{-1}\tilde{\boldsymbol{u}}_1 = -P_h^{\perp}\left[\rho\boldsymbol{u}_h\cdot\nabla\boldsymbol{u}_h\right], \tag{67}$$

$$\rho\frac{\partial\tilde{\boldsymbol{u}}_2}{\partial t} + \alpha_1^{-1}\tilde{\boldsymbol{u}}_2 = -P_h^{\perp}\left[\nabla p_h\right], \tag{68}$$

$$\rho\frac{\partial\tilde{\boldsymbol{u}}_3}{\partial t} + \alpha_1^{-1}\tilde{\boldsymbol{u}}_3 = P_h^{\perp}\left[\nabla\cdot\boldsymbol{\sigma}_h\right], \tag{69}$$

while the sub-grid scale $\tilde{\boldsymbol{\sigma}}$ is the solution of (66).

Therefore, the term-by-term FE formulation proposed consists in finding $\boldsymbol{U}_h : [0, t_{\mathrm{f}}] \longrightarrow \boldsymbol{\mathcal{X}}_h$ such that

$$\begin{aligned}
&(\mathcal{D}_t(\boldsymbol{U}_h), \boldsymbol{V}_h) + B(\boldsymbol{u}_h; \boldsymbol{U}_h, \boldsymbol{V}_h) + \sum_K \langle\tilde{\boldsymbol{u}}_1, -\rho\boldsymbol{u}_h\cdot\nabla\boldsymbol{v}_h\rangle_K \\
&\quad + \sum_K \langle\tilde{\boldsymbol{u}}_2, -\nabla q_h\rangle_K + \sum_K \langle\tilde{\boldsymbol{u}}_3, \nabla\cdot\boldsymbol{\chi}_h\rangle_K + \sum_K \langle\tilde{p}, -\nabla\cdot\boldsymbol{v}_h\rangle_K \\
&\quad + \sum_K \left\langle\tilde{\boldsymbol{\sigma}}, \frac{1}{2\eta_p}\boldsymbol{\chi}_h + \nabla^{\mathrm{s}}\boldsymbol{v} - \frac{\lambda}{2\eta_p}\left(\boldsymbol{u}_h\cdot\nabla\boldsymbol{\chi}_h + \boldsymbol{\chi}_h\cdot(\nabla\boldsymbol{u}_h)^T + \nabla\boldsymbol{u}_h\cdot\boldsymbol{\chi}_h\right)\right\rangle_K \\
&\quad = \langle\boldsymbol{f}, \boldsymbol{v}_h\rangle,
\end{aligned} \tag{70}$$

for all $\boldsymbol{V}_h \in \boldsymbol{\mathcal{X}}_h$ and satisfying the initial conditions in a weak sense. This problem is alternative to (64).

As for the quasi-static sub-scale version, the proposed method is not residual-based, and therefore, it is not consistent in the FE sense, although it has an optimal consistency error. Problem (70) is the dynamic counterpart of the formulation we propose in the quasi-static case, given in Eq. (55).

6.2 The logarithmic viscoelastic problem

Following the same nomenclature and structure used in the standard viscoelastic problem, we present now stabilised VMS FE formulations for the logarithmic reformulated problem. In essence, the difference is the change of variables in the stress unknown. Recall that in the variational formulation we have *not* made this change of variables in the stress test function.

We will not discuss in this section the issues related neither to discontinuous pressure or stress interpolation nor to the convenience of using discontinuity-capturing techniques. Their use in the LCR would be straightforward.

6.2.1 Residual-based stabilised finite element formulations

As for the standard case, let us start considering quasi-static sub-scales. A residual-based VMS method for the LCR case consists in adding to the Galerkin problem (Eq. (25)) a stabilised term of the form given again by Eq. (43), but now with different terms involved. Observe that this implies that we have also taken matrix $\boldsymbol{\alpha}$ as diagonal; in fact, we compute it for the LCR exactly as for the standard case, i.e., the diagonal components of $\boldsymbol{\alpha}$ are given by Eqs. (40)-(42). This is because the change of variables made does not affect the terms from where the stabilisation parameters come.

Since for the LCR we have made a change of variables only in the stress unknown and not in the stress test function, we can use the same adjoint operator for a given velocity $\hat{\boldsymbol{u}}$, given by Eq. (38).

Formally, the terms S_1, S_2, and S_3 are again given by Eqs. (44)-(46), again with the components of the adjoint operator given in Eq. (38). What is different with respect to the standard formulation is the expression of the residuals, which are now given by:

$$\begin{aligned}\boldsymbol{R_u} =& \boldsymbol{f} + \frac{1}{2\lambda_0}\boldsymbol{I} - \Big(\rho\frac{\partial \boldsymbol{u}_h}{\partial t} - \frac{\eta_p}{\lambda_0}\nabla\cdot\exp(\boldsymbol{\psi}_h) - 2\eta_s\nabla\cdot(\nabla^{\mathrm{s}}\boldsymbol{u}_h)\\ &+ \rho\hat{\boldsymbol{u}}_h\cdot\nabla\boldsymbol{u}_h + \nabla p_h\Big), \qquad (71)\\ R_p =& -\nabla\cdot\boldsymbol{u}_h, \qquad (72)\\ \boldsymbol{R_\sigma} =& -\frac{1}{2\lambda_0}\exp(\boldsymbol{\psi}_h) + \nabla^{\mathrm{s}}\boldsymbol{u}_h - \frac{\lambda}{2\lambda_0}\left(\frac{\partial\exp(\boldsymbol{\psi}_h)}{\partial t} + \hat{\boldsymbol{u}}_h\cdot\nabla\exp(\boldsymbol{\psi}_h)\right)\\ &+ \frac{1}{2\lambda_0}\left(\exp(\boldsymbol{\psi}_h)\cdot\nabla\hat{\boldsymbol{u}}_h + (\nabla\hat{\boldsymbol{u}}_h)^T\cdot\exp(\boldsymbol{\psi}_h) - 2\nabla^{\mathrm{s}}\boldsymbol{u}_h\right). \qquad (73)\end{aligned}$$

Again, for the definition of $\tilde{P}$, two natural options are $\tilde{P} = P_h^{\perp} = I - P_h$ and $P_h = I$, leading respectively to the OSS and the ASGS formulations. The formulation is complete with the linearisation of the exponential described later.

6.2.2 Non-residual-based stabilised finite element formulations

One of the key ingredients in stabilised FE methods proposed by our group in the context of viscoelastic fluids and summarised in thxis article is the non-residual-based alternative, which in the VMS framework is not common. In the original articles [10,16], hybrid methods, that mix a residual approach for the continuity and constitutive equations and a term-by-term structure for the momentum equation, were presented as the most robust option after solving classical benchmarks problems. This conclusion was the same both for the standard and the LCR approaches.

The formulation we favour has again the general form of Eq. (55) introduced for the standard method, but now with:

$$S_1^{\perp}(\hat{\boldsymbol{u}}_h;\boldsymbol{U}_h,\boldsymbol{V}_h) = \left\langle P_h^{\perp}\left[\frac{\eta_p}{\lambda_0}\nabla\cdot\exp(\boldsymbol{\psi})\right], \nabla\cdot\boldsymbol{\chi}_h\right\rangle + \left\langle P_h^{\perp}[\nabla p_h], \nabla q_h\right\rangle + \left\langle P_h^{\perp}[\rho\hat{\boldsymbol{u}}_h\cdot\nabla\boldsymbol{u}_h], \rho\hat{\boldsymbol{u}}_h\cdot\nabla\boldsymbol{v}_h\right\rangle, \tag{74}$$

using $\tilde{P} = P_h^{\perp}$. This change of structure in the stabilisation method that comes from taking $S_1^{\perp}$ instead of S_1 is not just a simplification. For smooth solutions, both have an optimal convergence rate in h. However, in problems where the solution has strong gradients, we have found the term-by-term one more robust, similarly to what it is explained in [14].

6.2.3 Time-dependent sub-scales

In the same way as described for the standard case, a dynamic formulation that takes into account the evolutionary component of the sub-scale can be designed for the LCR problem. Again, residual-based or term-by-term methods can be obtained.

In the case of the residual-based option, the formulation with dynamic sub-scales has exactly the same expression as for the standard formulation, i.e., that given by Eq. (64), but now with the sub-scales being the solution of:

$$\begin{aligned}
\rho\frac{\partial\tilde{\boldsymbol{u}}}{\partial t} + \alpha_1^{-1}\tilde{\boldsymbol{u}} =& \tilde{P}\left[\boldsymbol{f} - \left(\rho\frac{\partial\boldsymbol{u}_h}{\partial t} - \frac{\eta_p}{\lambda_0}\nabla\cdot\exp(\boldsymbol{\psi}_h)\right)\right] \\
&+ \tilde{P}\left[-2\eta_s\nabla\cdot(\nabla^s\boldsymbol{u}_h) + \rho\boldsymbol{u}_h\cdot\nabla\boldsymbol{u}_h + \nabla p_h\right], \\
\alpha_2^{-1}\tilde{p} =& \tilde{P}\left(-\nabla\cdot\boldsymbol{u}_h\right), \\
\frac{\lambda}{2\eta_p}\frac{\partial\tilde{\boldsymbol{\sigma}}}{\partial t} + \alpha_3^{-1}\tilde{\boldsymbol{\sigma}} =& \tilde{P}\left[-\frac{1}{2\lambda_0}\exp(\boldsymbol{\psi}_h) + \nabla^s\boldsymbol{u}_h - \frac{\lambda}{2\lambda_0}\frac{\partial\exp(\boldsymbol{\psi}_h)}{\partial t}\right] \\
&- \frac{\lambda}{2\lambda_0}\tilde{P}\left[\boldsymbol{u}_h\cdot\nabla\exp\left(\boldsymbol{\psi}_h\right) - \exp(\boldsymbol{\psi}_h)\cdot\nabla\boldsymbol{u}_h\right] \\
&- \frac{\lambda}{2\lambda_0}\tilde{P}\left[-(\nabla\boldsymbol{u}_h)^T\cdot\exp(\boldsymbol{\psi}_h) + 2\nabla^s\boldsymbol{u}_h\right],
\end{aligned}$$

and considering $\tilde{\boldsymbol{\sigma}} = \frac{\eta_p}{\lambda_0}\left(\exp\left(\tilde{\boldsymbol{\psi}}\right) - \boldsymbol{I}\right)$.

For the term-by-term formulation, the same reasoning as for the standard case can be repeated. In this way we arrive at the same problem (70), the only difference being that now the sub-scale $\tilde{\boldsymbol{u}}_3$ is the solution of:

$$\rho\frac{\partial\tilde{\boldsymbol{u}}_3}{\partial t} + \alpha_1^{-1}\tilde{\boldsymbol{u}}_3 = P_h^{\perp}\left[\frac{\eta_p}{\lambda_0}\nabla\cdot\exp(\boldsymbol{\psi}_h)\right],$$

instead of Eq. (69).

7 Computational aspects

7.1 Fully discrete and linearised problem: standard case

The numerical approximation of incompressible viscoelastic flows involves dealing with several non-linear terms. On the one hand, the momentum equation has the convective term, and on the other, the constitutive equation has another convective term related to the stresses and, additionally, the rotational terms arising from the objective derivative. Furthermore, at the discrete level and using any of the stabilisation methods reviewed in this article, the stabilisation terms depend also on the velocity, introducing, therefore, additional nonlinearities.

Based on the ideas proposed in [11,12], the following remarks need to be taken into account:

- The nonlinear term in the momentum equation can be linearised indistinctly with the fixed point scheme or with Newton-Raphson's method.
- Only a fixed point scheme is considered for the stabilisation terms of the constitutive equation.
- The stabilisation parameters are computed with values of the unknowns at the previous iterations.
- The orthogonal projection of any function f has been approximated iteratively as $P_h^{\perp}(f^i) \approx f^i - P_h(f^{i-1})$, where i represents the non-linearity iteration counter. Note that the iterative treatment of the orthogonal projection is thus coupled to the linearisation of the whole system.
- The discontinuity-capturing dissipation, when used, is linearised using a fixed point strategy.

Details of the linearisation strategies can be found in the original references.

7.2 Linearisation of the exponential

In the logarithmic reformulation case, apart from the well-known non-linearities associated with the standard viscoelastic problem, a treatment for the linearisation of the exponential function of a tensor needs to be taken into account. Following the ideas proposed in [16], we can write:

$$\exp(\boldsymbol{\psi}) = \exp(\hat{\boldsymbol{\psi}} + \delta\boldsymbol{\psi}) = \exp(\hat{\boldsymbol{\psi}}) \cdot \exp(\delta\boldsymbol{\psi}),$$

where $\delta\boldsymbol{\psi} = \boldsymbol{\psi} - \hat{\boldsymbol{\psi}}$ and $\hat{\boldsymbol{\psi}}$ is a known tensor, which in practice will be calculated at the previous iteration in the linearisation scheme. Assuming $\delta\boldsymbol{\psi}$ to be small (in a certain sense), the term $\exp(\delta\boldsymbol{\psi})$ can been linearised through a Taylor expansion with a truncation error of order $(\delta\boldsymbol{\psi})^2$:

$$\exp(\delta\boldsymbol{\psi}) \approx \boldsymbol{I} + \delta\boldsymbol{\psi}.$$

Consequently,

$$\exp(\boldsymbol{\psi}) \approx \exp(\hat{\boldsymbol{\psi}}) \cdot (\boldsymbol{I} + \delta\boldsymbol{\psi}) = \exp(\hat{\boldsymbol{\psi}}) \cdot \boldsymbol{\psi} + \exp(\hat{\boldsymbol{\psi}}) \cdot (\boldsymbol{I} - \hat{\boldsymbol{\psi}}). \tag{75}$$

So, inserting this approximation into system (10)-(12), this system is linearised around $\hat{\boldsymbol{\psi}}$ as

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} - \frac{\eta_p}{\lambda_0}\nabla\cdot(\exp(\hat{\boldsymbol{\psi}})\cdot\boldsymbol{\psi}) - 2\eta_s\nabla\cdot(\nabla^{\mathrm{s}}\boldsymbol{u}) + \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f} + \frac{\eta_p}{\lambda_0}\nabla\cdot(-\exp(\hat{\boldsymbol{\psi}})\cdot\hat{\boldsymbol{\psi}} + \exp(\hat{\boldsymbol{\psi}})),$$

$$\nabla\cdot\boldsymbol{u} = 0,$$

$$\frac{1}{2\lambda_0}\exp(\hat{\boldsymbol{\psi}})\cdot\boldsymbol{\psi} - \nabla^{\mathrm{s}}\boldsymbol{u} + \frac{\lambda}{2\lambda_0}\left(\frac{\partial(\exp(\hat{\boldsymbol{\psi}})\cdot\boldsymbol{\psi})}{\partial t} + \boldsymbol{u}\cdot\nabla\left(\exp(\hat{\boldsymbol{\psi}})\cdot\boldsymbol{\psi}\right)\right) - \frac{\lambda}{2\lambda_0}\left(\exp(\hat{\boldsymbol{\psi}})\cdot\boldsymbol{\psi}\cdot\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\cdot\exp(\hat{\boldsymbol{\psi}})\cdot\boldsymbol{\psi} - 2\nabla^{\mathrm{s}}\boldsymbol{u}\right)$$
$$= \frac{1}{2\lambda_0}(\boldsymbol{I} - \exp(\hat{\boldsymbol{\psi}}) + \exp(\hat{\boldsymbol{\psi}})\cdot\hat{\boldsymbol{\psi}}) + \frac{\lambda}{2\lambda_0}\left(\frac{\partial(\exp(\hat{\boldsymbol{\psi}})\cdot\hat{\boldsymbol{\psi}})}{\partial t} - \frac{\partial\exp(\hat{\boldsymbol{\psi}})}{\partial t}\right)$$
$$+\frac{\lambda}{2\lambda_0}\left(\boldsymbol{u}\cdot\nabla\left(\exp(\hat{\boldsymbol{\psi}})\cdot\hat{\boldsymbol{\psi}} - \exp(\hat{\boldsymbol{\psi}})\right) + \left(-\exp(\hat{\boldsymbol{\psi}})\cdot\hat{\boldsymbol{\psi}} + \exp(\hat{\boldsymbol{\psi}})\right)\cdot\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\cdot\left(-\exp(\hat{\boldsymbol{\psi}})\cdot\hat{\boldsymbol{\psi}} + \exp(\hat{\boldsymbol{\psi}})\right)\right).$$

The variational formulation of this problem is straightforward.

A remark is needed for the time derivative of the exponential. Using approximation (75), it is easily shown that the operations "linearisation" and "time approximation" commute if we identify $\hat{\boldsymbol{\psi}}^n = \boldsymbol{\psi}^n$, being n the previous time step. Indeed, in both cases we obtain:

$$\left.\frac{\partial(\exp(\boldsymbol{\psi}))}{\partial t}\right|_{t^{n+1}} = \frac{1}{\delta t}\Big[\exp(\hat{\boldsymbol{\psi}}^{n+1})\cdot\boldsymbol{\psi}^{n+1} + \exp(\hat{\boldsymbol{\psi}}^{n+1}) - \exp(\hat{\boldsymbol{\psi}}^{n+1})\cdot\hat{\boldsymbol{\psi}}^{n+1} - \exp(\boldsymbol{\psi}^n)\Big] + \mathcal{O}(\delta t) + \mathcal{O}((\delta\boldsymbol{\psi}^{n+1})^2).$$

7.3 Fully discrete and linearised problem: logarithmic case

It is clear that the nonlinearities of the standard formulation are maintained with the logarithmic reformulation, and new non-linearities arising from the exponential function appear, as it has been exposed when the continuous problem has been described.

For the convective term of the momentum equation we can use a fixed point scheme or Newton-Raphson's scheme. However, for the nonlinear terms in the constitutive equation, we always use a Newton-Raphson linearisation, and it has been decisive to be able to compute some high Weissemberg cases and get the optimal convergence of the method.

Let us make the following remarks about the algorithm used:

– The nonlinear term in the momentum equation can be linearised with the fixed point scheme or with Newton-Raphson's method, although for the

LCR we favour the latter, particularly for the linearisation of the exponential, as described above. A close enough initial guess can be obtained using the continuation in the Weissemberg and Reynolds numbers.
- The exponential terms that appear both in the momentum equation and in the constitutive equation have been linearised as described in the previous subsection, taking tensor $\hat{\boldsymbol{\psi}}$ as the one obtained from the previous iteration of the current time step.
- All non-linear terms belonging to the constitutive equation, both in the Galerkin terms and in the stabilisation, have been linearised using Newton-Raphson's method described above.
- The stabilisation parameters are computed with the values of the unknowns at the previous iterations.
- The iterative treatment of the orthogonal projection is coupled to the linearisation of the total system. The same treatment used for the orthogonal projection in the standard formulation can be used.

Apart from the linearisation carried out in some terms just explained, we have found extremely useful the application of other techniques that lead to better convergence. One of them is under-relaxation, taking as a relaxation parameter $\epsilon = 0.5$, which has been found useful in most of the cases; the second tool employed is the continuation method in terms of the relaxation time λ, which consists in N_λ continuation steps of equal size $\delta\lambda = \lambda/N_\lambda$. Note that continuation techniques can be employed because of the modification of the log-conformation formulation; if the original logarithmic formulation had been taken, they could not be used. Besides, the continuation loop and the linearisation loop can be coupled in the non-linear loop as it is explained in [16].

7.4 Solution strategies for the stabilised viscoelastic flow problem

The solution of the viscoelastic flow problem has several difficulties, as explained in the previous sections. One of them is precisely the lack of convergence of the non-linear iterative scheme, which leads to the need of using formulations such as the log-conformation reformulation presented in Section 3.2. This is however not the only numerical difficulty faced when dealing with the viscoelastic flow problem: another important aspect is the need of solving the linear systems of equations arising at each non-linear iteration of the discretised FE problem, either for the standard or the log-conformation formulations.

A linear system solution strategy needs to be devised, specifically for the viscoelastic flow problem, which allows for its efficient solution. This is so, because apart from the several heavily non-linear and element-by-element stabilisation terms that might spoil the condition number of the problem, the nature of the viscoelastic equations make it impossible to eliminate or reduce the stress unknown. This causes that if, for instance, an equal order interpolation FE discretisation is used, the number of unknowns per node rises

from four (three for the velocity, and one addition unknown for the pressure), for the standard incompressible Navier-Stokes equations, to ten (including the same unknowns plus six extra unknowns for the stress tensor), for either the standard and the log-conformation viscoelastic flow problem. Moreover, the resulting problem is of saddle-point type, with possibly null terms (before applying stabilisation) both in the velocity and pressure diagonal blocks of the linear system.

Direct linear system solvers are usually not affordable due to its high computational cost. As the discretisation accuracy increases and the number of unknowns becomes large, the condition number of the resulting linear system of equations worsens, which makes it difficult for iterative solvers to deal with the problem. Apart from the usage of general available efficient solvers such as multigrid preconditioners, one can rely on specific solvers or preconditioners for the viscoelastic flow problem. In this sense, the usage of a fractional step scheme which allows one to separate the solution of the velocity, pressure and stress unknowns is very convenient.

There have been several proposals of fractional step methods for viscoelastic flow problems. The Θ-method, originally proposed in [124] for Newtonian fluids and later extended to viscoelastic flows in [125, 126] is one of the most popular ones. It consists in splitting the viscoelastic flow problem into a Stokes system where the stress is treated explicitly, followed by a transport problem for the constitutive equation and an additional Stokes problem which is used to improve the stability of the global system. The method is reported to be second order accurate for proper Θ values, and it has been extensively studied (see for instance the a priori error estimates presented in [127]). A second order in time two-step decoupling method for modelling inertialess viscoelastic fluids was presented in [128]. This method was extended to solve fluid flows with inertia in [129]. In this method, the first step consists of the computation of the elastic stress by using the velocity of the previous time step. Then the elastic stress is used explicitly as a force term in the momentum equation. Second order accuracy is reported for the method. In [130], an extension is presented of the fourth step fractional step method proposed in [131] for the standard Navier-Stokes problem, now using the Oldroyd-B constitutive model and the DEVSS-G/DG stabilisation technique.

In our group, we have been working on several fractional step methods (see [11]) with first, second and third order splitting errors. We summarise now the general approach pursued in these strategies. The methods are going to be explained for the standard formulation, although they can be extended to the log-conformation reformulation.

For the exposition of the fractional step approach, it is convenient to write the problem (16)-(18) after its time discretisation in algebraic form. For simplicity, we will not include stabilisation terms, although they need to be included if arbitrary interpolation spaces for the unknowns are used, or if convection is dominant. Matrices will be written with light case italic characters. In the case of elastic stresses, the Voight notation will be employed. Having

this notation in mind, the algebraic structure of the problem is:

$$M_{\boldsymbol{u}}\frac{\delta_k}{\delta t}\mathbf{U}^{n+1} + K_{\boldsymbol{u}}\left(\mathbf{U}^{n+1}\right)\mathbf{U}^{n+1} + G\mathrm{P}^{n+1} - D_{\boldsymbol{\sigma}}\boldsymbol{\Sigma}^{n+1} = \mathbf{F}^{n+1}, \tag{76}$$

$$D\mathbf{U}^{n+1} = \mathbf{0}, \tag{77}$$

$$M_{\boldsymbol{\sigma}}\frac{\delta_k}{\delta t}\boldsymbol{\Sigma}^{n+1} + K_{\boldsymbol{\sigma}}\left(\mathbf{U}^{n+1}\right)\boldsymbol{\Sigma}^{n+1} - S\mathbf{U}^{n+1} = \mathbf{0}. \tag{78}$$

The dependence of matrices $K_{\boldsymbol{u}}$ and $K_{\boldsymbol{\sigma}}$ on $\mathbf{U}$ has been explicitly displayed. The identification of the different terms that contribute to the matrices in these expressions is straightforward.

System (76)-(78) can be written as

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & 0 \\ A_{31} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}^{n+1} \\ \boldsymbol{\Sigma}^{n+1} \\ \mathrm{P}^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1^{n+1} \\ \mathbf{F}_2^{n+1} \\ \mathrm{F}_3^{n+1} \end{bmatrix}, \tag{79}$$

where

$$A_{11} = \frac{1}{\gamma_k \delta t}M_{\boldsymbol{u}} + K_{\boldsymbol{u}}\left(\mathbf{U}^{n+1}\right), \quad A_{12} = -D_{\sigma}, \quad A_{13} = G,$$

$$A_{21} = -S, \quad A_{22} = \frac{1}{\gamma_k \delta t}M_{\boldsymbol{\sigma}} + K_{\boldsymbol{\sigma}}\left(\mathbf{U}^{n+1}\right), \quad A_{31} = D,$$

$$\mathbf{F}_1 = \mathbf{F}^{n+1} + \frac{1}{\delta t \gamma_k}\left(\sum_{i=0}^{k-1}\varphi_k^i \mathbf{U}^{n-i}\right), \quad \mathbf{F}_2 = \frac{1}{\delta t \gamma_k}\left(\sum_{i=0}^{k-1}\varphi_k^i \boldsymbol{\Sigma}^{n-i}\right), \quad \mathrm{F}_3 = 0.$$

The ordering of the equations chosen is consistent with the steps of the splitting algorithm described below.

The fractional step method presented next can be viewed as an extension of pressure-segregation or pressure-correction schemes applied to the viscoelastic fluid flow problem. Instead of calculating the velocity with a guess of the pressure only, now we will need a guess of the elastic stresses as well. After computing these two fields, velocities will be corrected.

To derive the method, equations (76)-(78) need to be rewritten in the equivalent form

$$M_{\boldsymbol{u}}\frac{\delta_k}{\delta t}\tilde{\mathbf{U}}^{n+1} + K_{\boldsymbol{u}}(\tilde{\mathbf{U}}^{n+1})\tilde{\mathbf{U}}^{n+1} + G\hat{\mathrm{P}}_{k'-1}^{n+1} - D_{\boldsymbol{\sigma}}\hat{\boldsymbol{\Sigma}}_{k'-1}^{n+1} = \mathbf{F}^{n+1}, \tag{80}$$

$$M_{\boldsymbol{u}}\frac{1}{\gamma_k \delta t}(\mathbf{U}^{n+1} - \tilde{\mathbf{U}}^{n+1})$$

$$+\mathbf{N}_{\boldsymbol{u}}^{n+1} + G(\mathrm{P}^{n+1} - \hat{\mathrm{P}}_{k'-1}^{n+1}) - D_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma}^{n+1} - \hat{\boldsymbol{\Sigma}}_{k'-1}^{n+1}) = \mathbf{0}, \tag{81}$$

$$M_{\boldsymbol{\sigma}}\frac{\delta_k}{\delta t}\tilde{\boldsymbol{\Sigma}}^{n+1} + K_{\boldsymbol{\sigma}}(\tilde{\mathbf{U}}^{n+1})\tilde{\boldsymbol{\Sigma}}^{n+1} - S\tilde{\mathbf{U}}^{n+1} = \mathbf{0}, \tag{82}$$

$$M_{\boldsymbol{\sigma}}\frac{1}{\gamma_k \delta t}(\boldsymbol{\Sigma}^{n+1} - \tilde{\boldsymbol{\Sigma}}^{n+1}) + \mathbf{N}_{\boldsymbol{\sigma}}^{n+1} - S(\mathbf{U}^{n+1} - \tilde{\mathbf{U}}^{n+1}) = \mathbf{0}, \tag{83}$$

$$-D\tilde{\mathbf{U}}^{n+1} + \gamma_k \delta t D M_{\boldsymbol{u}}^{-1}\mathbf{N}_{\boldsymbol{u}}^{n+1} + \gamma_k \delta t D M_{\boldsymbol{u}}^{-1} G(\mathrm{P}^{n+1} - \hat{\mathrm{P}}_{k'-1}^{n+1})$$

$$-\gamma_k \delta t D M_{\boldsymbol{u}}^{-1} D_{\boldsymbol{\sigma}} (\boldsymbol{\Sigma}^{n+1} - \hat{\boldsymbol{\Sigma}}_{k'-1}^{n+1}) = \mathbf{0}, \tag{84}$$

where

$$\mathbf{N}_{\boldsymbol{u}}^{n+1} = K_{\boldsymbol{u}}(\mathbf{U}^{n+1})\mathbf{U}^{n+1} - K_{\boldsymbol{u}}(\tilde{\mathbf{U}}^{n+1})\tilde{\mathbf{U}}^{n+1},$$
$$\mathbf{N}_{\boldsymbol{\sigma}}^{n+1} = K_{\boldsymbol{\sigma}}(\mathbf{U}^{n+1})\boldsymbol{\Sigma}^{n+1} - K_{\boldsymbol{\sigma}}(\tilde{\mathbf{U}}^{n+1})\tilde{\boldsymbol{\Sigma}}^{n+1}.$$

In these equations, $\tilde{\mathbf{U}}^{n+1}$ and $\tilde{\boldsymbol{\Sigma}}^{n+1}$ are auxiliary variables, $\hat{g}_{k'-1}^{n+1}$ is the extrapolation of order $k'-1$ at t^{n+1} of a function g, and it is understood that the difference $\delta_k \tilde{g}^{n+1}$ is computed from $\tilde{g}^{n+1}$ and g^m, for time steps m previous to $n+1$, g now being either $\mathbf{U}$ or $\boldsymbol{\Sigma}$.

In the previous equations, k determines the order of the time integration scheme and k' the order of the extrapolations to uncouple variables. They can be chosen independently, but need to be adequately balanced to achieve a certain order of the final approximation.

The final algorithm for first and second order schemes is displayed in Algorithm 1. Details on how to extend the fractional step algorithm to third order accuracy can be found in [11]. The extension is not straightforward due to stability issues appearing if Algorithm 1 is applied with third order extrapolations.

The scheme resulting from Algorithm 1 can be used to solve the viscoelastic flow problem directly, or it can be used as a preconditioner for the original monolithic system if splitting errors want to be avoided. In this sense, another possibility is to try to obtain a pseudo algebraic fractional step method directly from the assembled system of equations for the monolithic viscoelastic problem, as done for instance for the incompressible Navier-Stokes equations in [132]. In our experience, however, a tailored fractional step approach such as Algorithm 1 turns out in a more stable algorithmic process.

8 Some numerical analysis results

In this section, some results of the numerical analysis of the formulations discussed are presented. In the first and second subsections, the linearised standard and the linearised logarithmic formulation are analysed in a mesh-dependent norm and in a natural norm, both following the same methodology; the results to be presented are proved in [13,98], respectively. The last section presents the results of the full nonlinear problem discretised only in space, which are proved in [15].

8.1 The stationary standard linearised viscoelastic problem

We shall consider first the stationary an linearised form of the problem presented in Subsection 3.1. Calling $\boldsymbol{a}$ a given velocity field resulting from the (fixed point) linearisation, the equations to be solved are

$$\rho \boldsymbol{a} \cdot \nabla \boldsymbol{u} - \nabla \cdot (2\eta_s \nabla^{\mathrm{s}} \boldsymbol{u} + \boldsymbol{\sigma}) + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{85}$$

Algorithm 1 First and second order fractional step schemes ($k = 1, 2$)

1. Intermediate velocity using the pressure and the elastic stress values extrapolated:

$$M_{\boldsymbol{u}}\frac{\delta_k}{\delta t}\tilde{\mathbf{U}}^{n+1} + K_{\boldsymbol{u}}(\tilde{\mathbf{U}}^{n+1})\tilde{\mathbf{U}}^{n+1} + G\hat{\mathrm{P}}_{k-1}^{n+1} - D_{\boldsymbol{\sigma}}\hat{\boldsymbol{\Sigma}}_{k-1}^{n+1} = \mathbf{F}^{n+1} \rightarrow \tilde{\mathbf{U}}^{n+1}$$

2. Intermediate elastic stress values using the intermediate velocity:

$$M_{\boldsymbol{\sigma}}\frac{\delta_k}{\delta t}\tilde{\boldsymbol{\Sigma}}^{n+1} + K_{\boldsymbol{\sigma}}(\tilde{\mathbf{U}}^{n+1})\tilde{\boldsymbol{\Sigma}}^{n+1} - S\tilde{\mathbf{U}}^{n+1} = \mathbf{0} \rightarrow \tilde{\boldsymbol{\Sigma}}^{n+1}$$

3. Intermediate pressure calculation using the intermediate velocity and elastic stress:

$$-D\tilde{\mathbf{U}}^{n+1} + \gamma_k\delta t D M_{\boldsymbol{u}}^{-1}G(\tilde{\mathrm{P}}^{n+1} - \hat{\mathrm{P}}_{k-1}^{n+1}) - \gamma_k\delta t D M_{\boldsymbol{u}}^{-1}D_{\sigma}(\tilde{\boldsymbol{\Sigma}}^{n+1} - \hat{\boldsymbol{\Sigma}}_{k-1}^{n+1}) = \mathbf{0} \rightarrow \tilde{\mathrm{P}}^{n+1}$$

4. Velocity correction:

$$\frac{1}{\gamma_k\delta t}M_{\boldsymbol{u}}(\mathbf{U}^{n+1} - \tilde{\mathbf{U}}^{n+1}) + G(\tilde{\mathrm{P}}^{n+1} - \hat{\mathrm{P}}_{k-1}^{n+1}) - D_{\sigma}(\tilde{\boldsymbol{\Sigma}}^{n+1} - \hat{\boldsymbol{\Sigma}}_{k-1}^{n+1}) = \mathbf{0} \rightarrow \mathbf{U}^{n+1}$$

5. Elastic stress correction:

$$\frac{1}{\gamma_k\delta t}M_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma}^{n+1} - \tilde{\boldsymbol{\Sigma}}^{n+1}) - S(\mathbf{U}^{n+1} - \tilde{\mathbf{U}}^{n+1}) = \mathbf{0} \rightarrow \boldsymbol{\Sigma}^{n+1}$$

6. Pressure correction: $\mathrm{P}^{n+1} = \tilde{\mathrm{P}}^{n+1} \rightarrow \mathrm{P}^{n+1}$.

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{86}$$

$$\frac{1}{2\eta_p}\boldsymbol{\sigma} - \nabla^{\mathrm{s}}\boldsymbol{u} + \frac{\lambda}{2\eta_p}\left(\boldsymbol{a}\cdot\nabla\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot\nabla\boldsymbol{a} - (\nabla\boldsymbol{a})^T\cdot\boldsymbol{\sigma}\right) = \mathbf{0} \quad \text{in } \Omega. \tag{87}$$

Setting the variational form of this problem and the Galerkin approximation is straightforward, following what has been presented in Subsection 3.1. In the FE approximation, $\boldsymbol{a}$ needs to be approximated by a FE function (its interpolant, for example), that we shall denote by $\boldsymbol{a}_h$.

8.1.1 Stabilised finite element method

The stabilised FE approximation we consider now is based on the formulation detailed in Subection 6.1.2, considering now the problem stationary and using the residual form of the stabilisation term of the constitutive equation. Moreover, we shall also introduce interelement boundary terms that allow one to use discontinuous pressure and stress interpolations. The discrete problem

consists in finding $\boldsymbol{U}_h \in \boldsymbol{\mathcal{X}}_h$ such that

$$\begin{aligned} B_{\text{stab}}\left(\boldsymbol{U}_h, \boldsymbol{V}_h\right) =& B\left(\boldsymbol{U}_h, \boldsymbol{V}_h\right) + S_1^{\perp}\left(\boldsymbol{U}_h, \boldsymbol{V}_h\right) \\ &+ S_2^{\perp}\left(\boldsymbol{U}_h, \boldsymbol{V}_h\right) + S_3^{\perp}\left(\boldsymbol{U}_h, \boldsymbol{V}_h\right) = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle \end{aligned} \tag{88}$$

for all $\boldsymbol{V}_h \in \boldsymbol{\mathcal{X}}_h$ where the stabilisation terms of each equation are defined as

$$\begin{aligned} S_1^{\perp}\left(\boldsymbol{U}_h, \boldsymbol{V}_h\right) =& \sum_K \alpha_1 \left\langle P_u^{\perp}\left[\rho \boldsymbol{a}_h \cdot \nabla \boldsymbol{u}_h\right], P_h^{\perp}\left[\rho \boldsymbol{a}_h \cdot \nabla \boldsymbol{v}_h\right]\right\rangle_K \\ &+ \sum_K \alpha_1 \left\langle P_u^{\perp}\left[\nabla p_h\right], P_h^{\perp}\left[\nabla q_h\right]\right\rangle_K \\ &+ \sum_K \alpha_1 \left\langle P_u^{\perp}\left[\nabla \cdot \boldsymbol{\sigma}_h\right], P_u^{\perp}\left[\nabla \cdot \boldsymbol{\chi}_h\right]\right\rangle_K \\ &+ \sum_E \alpha_{[1]} \langle [\![\left(\boldsymbol{n} q_h - \boldsymbol{n} \cdot \boldsymbol{\chi}_h\right) + 2\eta_s \boldsymbol{n} \cdot \nabla^{\mathrm{s}} \boldsymbol{v}_h]\!], \\ &\quad [\![\left(\boldsymbol{n} p_h - \boldsymbol{n} \cdot \boldsymbol{\sigma}_h\right) - 2\eta_s \boldsymbol{n} \cdot \nabla^{\mathrm{s}} \boldsymbol{u}_h]\!] \rangle_E \end{aligned} \tag{89}$$

$$S_2^{\perp}\left(\boldsymbol{U}_h, \boldsymbol{V}_h\right) = \sum_K \alpha_2 \left\langle P_p^{\perp}\left[\nabla \cdot \boldsymbol{u}_h\right], P_p^{\perp}\left[\nabla \cdot \boldsymbol{v}_h\right]\right\rangle_K, \tag{90}$$

$$\begin{aligned} S_3^{\perp}\left(\boldsymbol{U}_h, \boldsymbol{V}_h\right) =& \sum_K \alpha_3 \left\langle P_{\sigma}^{\perp}\left[\boldsymbol{\mathcal{R}}_{\sigma}\right], \right. \\ & \left. P_{\sigma}^{\perp}\left[\nabla^{\mathrm{s}} \boldsymbol{v}_h - \frac{\lambda}{2\eta_p}\left(\boldsymbol{a}_h \cdot \nabla \boldsymbol{\chi}_h + \boldsymbol{\chi}_h \cdot \left(\nabla \boldsymbol{a}_h\right)^T + \nabla \boldsymbol{a}_h \cdot \boldsymbol{\chi}_h\right)\right]\right\rangle_K \end{aligned} \tag{91}$$

In the last expression, $\boldsymbol{\mathcal{R}}_{\sigma}$ represents the residual of the constitutive equation without the stress, given by

$$\boldsymbol{\mathcal{R}}_{\sigma} = \nabla^{\mathrm{s}} \boldsymbol{u}_h - \frac{\lambda}{2\eta_p}\left(\boldsymbol{a}_h \cdot \nabla \boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h \cdot \nabla \boldsymbol{a}_h - \left(\nabla \boldsymbol{a}_h\right)^T \cdot \boldsymbol{\sigma}_h\right)$$

In the numerical analysis below we will also use the notation

$$\boldsymbol{\sigma}_h \cdot \nabla \boldsymbol{a}_h + \left(\nabla \boldsymbol{a}_h\right)^T \cdot \boldsymbol{\sigma}_h = \dot{\boldsymbol{\sigma}}_h^{*} + \dot{\boldsymbol{\sigma}}_h^{**}, \quad \boldsymbol{\sigma}_h \cdot \left(\nabla \boldsymbol{a}_h\right)^T + \nabla \boldsymbol{a}_h \cdot \boldsymbol{\sigma}_h = \dot{\boldsymbol{\sigma}}_h^{*} - \dot{\boldsymbol{\sigma}}_h^{**},$$

where $\dot{\boldsymbol{\sigma}}_h^{*} = \boldsymbol{\sigma}_h \cdot \nabla^{\mathrm{s}} \boldsymbol{a}_h + \nabla^{\mathrm{s}} \boldsymbol{a}_h \cdot \boldsymbol{\sigma}_h$ and $\dot{\boldsymbol{\sigma}}_h^{**} = \boldsymbol{\sigma}_h \cdot \nabla^{\mathrm{as}} \boldsymbol{a}_h - \nabla^{\mathrm{as}} \boldsymbol{a}_h \cdot \boldsymbol{\sigma}_h$. In these expressions, $\nabla^{\mathrm{as}} \boldsymbol{a}_h$ represents the skew-symmetric part of the velocity gradient, given by

$$\nabla^{\mathrm{as}} \boldsymbol{a}_h = \frac{1}{2}[\nabla \boldsymbol{a}_h - (\nabla \boldsymbol{a}_h)^T]. \tag{92}$$

In (89)-(91), as explained in Remark 6 in Section 6.1.2, $P_u^{\perp}$ represents the projection L^2-orthogonal to the velocity space without boundary conditions, $P_p^{\perp}$ the projection L^2-orthogonal to the pressure space and $P_{\sigma}^{\perp}$ the projection L^2-orthogonal to the stress space. We will also need to use the L^2-projection onto the velocity space with boundary conditions, $\boldsymbol{\mathcal{V}}_h$, that we will denote by $P_{u,0}$. The last term in $S_1^{\perp}$ is an approximation to the subscales on the

element boundaries and allows us to consider discontinuous interpolations for the pressure and the stress.

Lastly, the stabilisation parameters α_i, $i = 1, 2, 3$ are computed within each element K as defined in Section 6.1.1. For the linearisation of the problem the velocity $\boldsymbol{u}_h$ is replaced by the advection velocity $\boldsymbol{a}_h$ in the definition of these parameters. Regarding the boundary stabilisation parameter in $S_1^{\perp}$, it is defined as $\alpha_{[1]} = \frac{\delta_0 h}{2\mu}$, as in [103,55], where δ_0 is an algorithmic parameter that can be taken as $\delta_0 = 0.1$.

8.1.2 Stability and convergence in a mesh-dependent norm

In this subsection, we first state stability in the form of an inf-sup condition in a mesh dependent norm that depends on the stabilised formulation used, and next we state convergence using the same norm.

To state these results, we will need a condition on the interpolating spaces that holds in the case of equal order interpolations. It can be written as:

Assumption H1 Given $\boldsymbol{a}_h, \boldsymbol{v}_h \in \boldsymbol{\mathcal{V}}_h, q_h \in \mathcal{Q}_h, \boldsymbol{\chi}_h \in \boldsymbol{\Upsilon}_h$ and $\boldsymbol{z}_h \coloneqq \rho \boldsymbol{a}_h \cdot \nabla \boldsymbol{v}_h + \nabla q_h - \nabla \cdot \boldsymbol{\chi}_h$, there holds $\|\boldsymbol{z}_h\| \leq c_{\mathrm{m}} \left(\|P_{u,0}\left[\boldsymbol{z}_h\right]\| + \left\|P_u^{\perp}\left[\boldsymbol{z}_h\right]\right\| \right)$, for a constant $c_{\mathrm{m}} > 0$.

According to this condition, the component of $P_u\left(\boldsymbol{z}_h\right)$ that corresponds to the boundary of Ω can be bounded in terms of the right-hand-side of the inequality in **H1**. For a piecewise linear velocity $\boldsymbol{a}_h$ this assumption is known to hold; here we assume that $\boldsymbol{a}_h$ is such that it is satisfied. Note that c_{m} may depend on the different components of $\boldsymbol{z}_h$, but not on its Euclidean norm.

The norm employed to obtain the results is as follows:

$$
\begin{aligned}
\|\boldsymbol{V}_h\|_W^2 = {} & 2\eta_s \left\|\nabla^s \boldsymbol{v}_h\right\|^2 + \frac{1}{\eta_p} \left\|\boldsymbol{\chi}_h\right\|^2 \\
& + \sum_K \alpha_1 \left\|\rho \boldsymbol{a} \cdot \nabla \boldsymbol{v}_h + \nabla q_h - \nabla \cdot \boldsymbol{\chi}_h\right\|_K^2 \\
& + \sum_K \alpha_1 \left\|P_u^{\perp}[\rho \boldsymbol{a} \cdot \nabla \boldsymbol{v}_h]\right\|_K^2 + \sum_K \alpha_1 \left\|P_u^{\perp}[\nabla q_h]\right\|_K^2 \\
& + \sum_K \alpha_1 \left\|P_u^{\perp}\left[\nabla \cdot \boldsymbol{\chi}_h\right]\right\|_K^2 + \sum_K \alpha_2 \left\|\nabla \cdot \boldsymbol{v}_h\right\|_K^2 \\
& + \sum_K \alpha_3 \left\| -\nabla^s \boldsymbol{v}_h + \frac{\lambda}{2\eta_p} \left(\boldsymbol{a} \cdot \nabla \boldsymbol{\chi}_h - \dot{\boldsymbol{\chi}}_h^{**}\right) \right\|_K^2 \\
& + \sum_E \alpha_{[1]} \left\|[\![\boldsymbol{n} q_h - \boldsymbol{n} \cdot \boldsymbol{\chi}_h]\!]\right\|_E^2 ,
\end{aligned}
\tag{93}
$$

If β is very small (or $\beta = 0$), control on the velocity gradient can be obtained from the term multiplied by α_3. However, to simplify the analysis, we will

consider $\beta > 0$, and that the stability provided by the first term in the right hand side of (93) is sufficient.

Then, considering the working norm (93) and supposing that **H1** holds, for λ small enough, there is a constant $C > 0$ such that

$$\inf_{\boldsymbol{U}_h \in \boldsymbol{\mathcal{X}}_h} \sup_{\boldsymbol{V}_h \in \boldsymbol{\mathcal{X}}_h} \frac{B_{\text{stab}}(\boldsymbol{U}_h, \boldsymbol{V}_h)}{\|\boldsymbol{U}_h\|_W \|\boldsymbol{V}_h\|_W} \geq C,$$

provided the constants $c_i, i = 1, .., 4$ defined in (40)-(42) are large enough. Therefore this result ensures *stability* of the stabilised method. The proof of this result is detailed in [13].

On the other hand, the convergence of the method is also proved in [13]. For this, we have to properly define the interpolation errors $\varepsilon_i(v)$. Considering a FE space $\mathcal{W}_h$, made of piecewise continuous polynomial functions of degree k_v; given a function $v \in H^{k'_v+1}(\Omega)$, for $i = 0, 1$ the interpolation errors $\varepsilon_i(v)$ are defined as

$$\inf_{v_h \in \mathcal{W}_h} \sum_K \|v - v_h\|_{H^i(K)} \leq C h^{k''_v+1-i} \sum_K \|v\|_{H^{k''_v+1}(K)} =: \sum_K \varepsilon_{i,K}(v) =: \varepsilon_i(v) \tag{94}$$

where $k''_v = \min(k_v, k'_v)$. We will denote from this point by $\tilde{v}_h$ the best approximation of v in $\mathcal{W}_h$. Note that $\varepsilon_0(v) = h\varepsilon_1(v)$.

The main result we obtained (see [13]) is that the error function of the method we consider is:

$$\begin{aligned}\mathcal{E}(h) \coloneqq & \sqrt{\eta_0}\varepsilon_1(\boldsymbol{u}) + \sqrt{\eta_0}\sum_K \sqrt{\text{Re}_K}\varepsilon_{1,K}(\boldsymbol{u}) + \frac{1}{\sqrt{\eta_0}}\varepsilon_0(\boldsymbol{\sigma}) \\ & + \frac{1}{\sqrt{\eta_0}}\sum_K \sqrt{\text{We}_K}\varepsilon_{0,K}(\boldsymbol{\sigma}) + \frac{1}{\sqrt{\eta_0}}\varepsilon_0(p),\end{aligned} \tag{95}$$

where

$$\text{Re}_K \coloneqq \frac{\rho \|\boldsymbol{a}\|_{L^\infty(K)} h}{\eta_0}, \qquad \text{We}_K \coloneqq \frac{\lambda \|\boldsymbol{a}\|_{L^\infty(K)}}{h} \tag{96}$$

are the element (or cell) Reynolds and Weissenberg numbers, respectively. More precisely, we proved that there exists a constant $C > 0$ such that

$$\|\boldsymbol{U} - \boldsymbol{U}_h\|_W \leq C\mathcal{E}(h). \tag{97}$$

8.1.3 Stability and convergence in a natural norm

Stability and convergence can be proved not only for the mesh-dependent norm, but also for a natural norm (or a norm in the space where the continuous problem is defined). We have to remark that since the natural norm does not include any control on the convective terms, nor does stability and convergence in this norm. These results will only be meaningful in the case of small cell Reynolds numbers and small cell Weissenberg numbers. Under the

same assumptions considered for the stability in a mesh-dependent norm, the solution of the discrete problem $\boldsymbol{U}_h = [\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h] \in \boldsymbol{\mathcal{X}}_h$ can be bounded as follows:

$$\sqrt{\eta_0}\,\|\boldsymbol{u}_h\|_{H^1(\Omega)} + \frac{1}{\sqrt{\eta_0}}\,\|\boldsymbol{\sigma}_h\| + \frac{1}{\sqrt{\eta_0}}\,\|p_h\| \leq C\frac{1}{\sqrt{\eta_0}}\,\|\boldsymbol{f}\|_{H^{-1}(\Omega)}\,.$$

Also, if the solution of the continuous problem is regular enough, the next *error estimate* can be proved:

$$\sqrt{\eta_0}\,\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1(\Omega)} + \frac{1}{\sqrt{\eta_0}}\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| + \frac{1}{\sqrt{\eta_0}}\,\|p - p_h\| \leq C\mathcal{E}(h).$$

Lastly, to complete the analysis of the problem, an L^2-error estimate for the velocity field can be obtained. Assuming the same hypothesis used in the stability and convergence for natural norms and supposing that an elliptic regularity condition is satisfied by the continuous problem, we have

$$\sqrt{\eta_0}\,\|\boldsymbol{u} - \boldsymbol{u}_h\| \leq Ch\left(\sqrt{\eta_0}\,\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1(\Omega)} + \frac{1}{\sqrt{\eta_0}}\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| + \frac{1}{\sqrt{\eta_0}}\,\|p - p_h\|\right).$$

8.2 The stationary logarithmic linearised viscoelastic problem

All the details concerning the reformulation of the standard equations into the logarithmic formulation can be found in Section 3.2. In this case, the same methodology as in the standard case detailed in the previous sections can be followed, and similar results can be proved.

8.2.1 Linearised problem and Galerkin FE discretisation

To motivate the linearised problem we analysed (see [98]), let us consider the Newton-Raphson linearisation explained in Section 7.2. Since we consider $\exp(\hat{\boldsymbol{\psi}})$ and $\hat{\boldsymbol{\psi}}$ known, we can denote these tensors as $\boldsymbol{E} = \exp(\hat{\boldsymbol{\psi}})$ and $\boldsymbol{S} = \hat{\boldsymbol{\psi}}$, respectively, and introduce $\boldsymbol{R} = \boldsymbol{E}\cdot\boldsymbol{S} - \boldsymbol{E}$. The linearised equations of the log-conformation formulation are now expressed as follows:

$$\begin{aligned}
-\frac{\eta_p}{\lambda_0}\nabla\cdot(\boldsymbol{E}\cdot\boldsymbol{\psi} - \boldsymbol{R}) - 2\eta_s\nabla\cdot(\nabla^{\mathrm{s}}\boldsymbol{u}) + \rho\boldsymbol{a}\cdot\nabla\boldsymbol{u} + \nabla p &= \boldsymbol{f},\\
\nabla\cdot\boldsymbol{u} &= 0,\\
\frac{1}{2\lambda_0}(\boldsymbol{E}\cdot\boldsymbol{\psi} - \boldsymbol{R} - \boldsymbol{I}) - \nabla^{\mathrm{s}}\boldsymbol{u} + \frac{\lambda}{2\lambda_0}\,(\boldsymbol{a}\cdot\nabla\,(\boldsymbol{E}\cdot\boldsymbol{\psi} - \boldsymbol{R})&\\
-(\boldsymbol{E}\cdot\boldsymbol{\psi} - \boldsymbol{R})\cdot\nabla\boldsymbol{a} - (\nabla\boldsymbol{a})^T\cdot(\boldsymbol{E}\cdot\boldsymbol{\psi} - \boldsymbol{R}) + 2\nabla^{\mathrm{s}}\boldsymbol{u}\big) &= \boldsymbol{0},
\end{aligned}$$

where the unknowns are the velocity, the pressure, and tensor $\boldsymbol{\psi}$. Again, $\boldsymbol{a}$ is the velocity of advection which is known, approximated by $\boldsymbol{a}_h$ in the FE approximations. Note the presence of the last term $2\nabla^{\mathrm{s}}\boldsymbol{u}$, which has a crucial

role in the dependence of the error estimate to be stated with the Weissenberg number.

Regarding the variational formulation for the logarithmic reformulation in the linearised form, the weak form consists in finding $\boldsymbol{U} = [\boldsymbol{u}, p, \boldsymbol{\psi}] \in \boldsymbol{\mathcal{X}}_{\boldsymbol{\psi}} := \boldsymbol{\mathcal{V}} \times \mathcal{Q} \times \boldsymbol{\Upsilon}_{\boldsymbol{\psi}}$ such that

$$\begin{aligned}
\frac{\eta_p}{\lambda_0}(\boldsymbol{E}\cdot\boldsymbol{\psi}, \nabla^{\mathrm{s}}\boldsymbol{v}) + 2(\eta_s\nabla^{\mathrm{s}}\boldsymbol{u}, \nabla^{\mathrm{s}}\boldsymbol{v}) + \langle\rho\boldsymbol{a}\cdot\nabla\boldsymbol{u}, \boldsymbol{v}\rangle - (p, \nabla\cdot\boldsymbol{v}) \\
= \langle\boldsymbol{f}, \boldsymbol{v}\rangle + \frac{\eta_p}{\lambda_0}(\boldsymbol{R}, \nabla^{\mathrm{s}}\boldsymbol{v}), \\
(q, \nabla\cdot\boldsymbol{u}) = 0, \\
\frac{1}{2\lambda_0}(\boldsymbol{E}\cdot\boldsymbol{\psi}, \boldsymbol{\chi}) - (\nabla^{\mathrm{s}}\boldsymbol{u}, \boldsymbol{\chi}) + \frac{\lambda}{2\lambda_0}\left(\boldsymbol{a}\cdot\nabla\left(\boldsymbol{E}\cdot\boldsymbol{\psi}\right) - \boldsymbol{E}\cdot\boldsymbol{\psi}\cdot\nabla\boldsymbol{a}\right. \\
\left. -(\nabla\boldsymbol{a})^T\cdot\boldsymbol{E}\cdot\boldsymbol{\psi} + 2\nabla^{\mathrm{s}}\boldsymbol{u}, \boldsymbol{\chi}\right) = \frac{1}{2\lambda_0}\left(\boldsymbol{I} + \boldsymbol{R}, \boldsymbol{\chi}\right) \\
+\frac{\lambda}{2\lambda_0}\left(\boldsymbol{a}\cdot\nabla\boldsymbol{R}, \boldsymbol{\chi}\right) + \frac{\lambda}{2\lambda_0}\left(-\boldsymbol{R}\cdot\nabla\boldsymbol{a} - (\nabla\boldsymbol{a})^T\cdot\boldsymbol{R}, \boldsymbol{\chi}\right),
\end{aligned}$$

for all $\boldsymbol{V} = [\boldsymbol{v}, q, \boldsymbol{\chi}] \in \boldsymbol{\mathcal{X}}$, where it is assumed that $\boldsymbol{f}$, $\boldsymbol{R}$ and $\boldsymbol{E}$ are such that the known terms are well defined. The details about spaces where variables are defined can be found in Section 3.3. Similarly to the standard case, the problem can be written in compact form as:

$$B(\boldsymbol{U}, \boldsymbol{V}) = L(\boldsymbol{V}),$$

with the obvious identification of B and L.

As explained in Section 3.3, the test function $\boldsymbol{\chi}$ is a stress, whereas the dimensionless unknown $\boldsymbol{\psi}$ is the logarithm of the conformation tensor. We could also have used a test function for the constitutive equation of the form $\frac{\eta_p}{\lambda_0}\exp(\boldsymbol{\chi})$, where now $\boldsymbol{\chi}$ would be dimensionless. This would simplify the analysis (some stability would follow taking $\boldsymbol{\chi} = \boldsymbol{\psi}$), but complicate significantly the FE approximations described below. Note that, strictly speaking, the space of stress test functions could be taken as the L^2 projection onto $L^2(\Omega)^{d\times d}$ of functions of the form $\exp(\boldsymbol{\psi})\cdot\boldsymbol{\varphi}$ properly scaled, for example by a factor $\frac{\eta_p}{\lambda_0}$, with $\boldsymbol{\varphi}$ belonging to the space of trial solutions.

The following condition will be needed:

Assumption H2 $\boldsymbol{E}$ and $\boldsymbol{R}$ have components in $L^\infty(\Omega)$.
$\boldsymbol{E}$ is invertible with a bounded inverse.

Concerning the Galerkin FE discretisation, the condition that the convective derivative of the stress be square integrable will follow from **H2** and choosing the stresses continuous. Calling $\boldsymbol{\mathcal{X}}_h := \boldsymbol{\mathcal{V}}_h \times \mathcal{Q}_h \times \boldsymbol{\Upsilon}_h$, the Galerkin FE approximation of the problem consists in finding $\boldsymbol{U}_h \in \boldsymbol{\mathcal{X}}_h$, such that:

$$B_\psi(\boldsymbol{U}_h, \boldsymbol{V}_h) = L(\boldsymbol{V}_h), \tag{98}$$

for all $\boldsymbol{V}_h = [\boldsymbol{v}_h, q_h, \boldsymbol{\chi}_h] \in \boldsymbol{\mathcal{X}}_h$, where B_ψ is obtained from B replacing $\boldsymbol{E}\cdot\boldsymbol{\psi}_h$ by $P_\psi(\boldsymbol{E}\cdot\boldsymbol{\psi}_h)$, where P_ψ is the L^2 projection onto $\boldsymbol{\Upsilon}_h$.

As in the standard formulation, problem (98) lacks stability unless appropriate inf-sup conditions hold. Likewise, convective terms are not bounded, and these may dominate those that can be controlled.

8.2.2 Stabilised finite element method

As in the linearised standard case, the numerical method we analysed is a non-residual type stabilisation. This method has been detailed in Section 6.2.2. It reads: find $\boldsymbol{U}_h \in \boldsymbol{\mathcal{X}}_h$ such that

$$\begin{aligned} B_{\text{stab}}(\boldsymbol{U}_h, \boldsymbol{V}_h) = B_\psi(\boldsymbol{U}_h, \boldsymbol{V}_h) + S_1^\perp(\boldsymbol{U}_h, \boldsymbol{V}_h) \\ + S_2^\perp(\boldsymbol{U}_h, \boldsymbol{V}_h) + S_3^\perp(\boldsymbol{u}_h; \boldsymbol{U}_h, \boldsymbol{V}_h) = L(\boldsymbol{V}_h) \end{aligned}$$

for all $\boldsymbol{V}_h \in \boldsymbol{\mathcal{X}}_h$, where now the linearisation terms are:

$$\begin{aligned} S_1^\perp(\boldsymbol{U}_h, \boldsymbol{V}_h) = \sum_K \alpha_u \Big\langle P_u^\perp \Big[-\frac{\eta_p}{\lambda_0} \nabla\cdot P_\psi[\boldsymbol{E}\cdot\boldsymbol{\psi}_h]\Big], -\nabla\cdot\boldsymbol{\chi}_h \Big\rangle_K \\ + \sum_K \alpha_1 \Big\langle P_u^\perp[\nabla p_h], \nabla q_h \Big\rangle_K + \sum_K \alpha_1 \Big\langle P_u^\perp[\rho \boldsymbol{a}_h\cdot\nabla\boldsymbol{u}_h], \rho\boldsymbol{a}_h\cdot\nabla\boldsymbol{v}_h \Big\rangle_K, \end{aligned} \tag{99}$$

$$S_2^\perp(\boldsymbol{U}_h, \boldsymbol{V}_h) = \sum_K \alpha_2 \Big\langle P_p^\perp[\nabla\cdot\boldsymbol{u}_h], \nabla\cdot\boldsymbol{v}_h \Big\rangle_K, \tag{100}$$

$$\begin{aligned} S_3^\perp(\boldsymbol{U}_h, \boldsymbol{V}_h) = \sum_K \alpha_3 \big\langle P_\psi^\perp[\boldsymbol{R}_\psi], \\ -\nabla^{\text{s}}\boldsymbol{v}_h + \frac{\lambda}{2\eta_p}\left(\boldsymbol{a}_h\cdot\nabla\boldsymbol{\chi}_h + \boldsymbol{\chi}_h\cdot(\nabla\boldsymbol{a}_h)^T + \nabla\boldsymbol{a}_h\cdot\boldsymbol{\chi}_h\right)\Big\rangle_K, \end{aligned} \tag{101}$$

where $\boldsymbol{R}_\psi$ is the residual of the constitutive equation

$$\begin{aligned} \boldsymbol{R}_\psi = -\nabla^{\text{s}}\boldsymbol{u}_h + \frac{\lambda}{2\lambda_0}\left(\boldsymbol{a}_h\cdot\nabla P_\psi[\boldsymbol{E}\cdot\boldsymbol{\psi}_h] - P_\psi[\boldsymbol{E}\cdot\boldsymbol{\psi}_h]\cdot\nabla\boldsymbol{a}_h\right. \\ \left. -(\nabla\boldsymbol{a}_h)^T\cdot P_\psi[\boldsymbol{E}\cdot\boldsymbol{\psi}_h] + 2\nabla^{\text{s}}\boldsymbol{u}_h\right). \end{aligned}$$

Again, the L^2 projections onto the FE spaces for the velocity (without boundary conditions), pressure and stress have respectively been denoted by P_u, P_p and, as already mentioned, P_ψ. In the following results, we will also use the notation

$$P_\psi[\boldsymbol{E}\cdot\boldsymbol{\psi}_h]\cdot\nabla\boldsymbol{a}_h + (\nabla\boldsymbol{a}_h)^T\cdot P_\psi[\boldsymbol{E}\cdot\boldsymbol{\psi}_h] = \dot{\boldsymbol{\psi}}_h^* + \dot{\boldsymbol{\psi}}_h^{**},$$

and

$$P_\psi[\boldsymbol{E}\cdot\boldsymbol{\psi}_h]\cdot(\nabla\boldsymbol{a}_h)^T + \nabla\boldsymbol{a}_h\cdot P_\psi[\boldsymbol{E}\cdot\boldsymbol{\psi}_h] = \dot{\boldsymbol{\psi}}_h^* - \dot{\boldsymbol{\psi}}_h^{**},$$

where $\dot{\boldsymbol{\psi}}_h^* = P_\psi[\boldsymbol{E}\cdot\boldsymbol{\psi}_h]\cdot\nabla^{\text{s}}\boldsymbol{a}_h + \nabla^{\text{s}}\boldsymbol{a}_h\cdot P_\psi[\boldsymbol{E}\cdot\boldsymbol{\psi}_h]$ and $\dot{\boldsymbol{\psi}}_h^{**} = P_\psi[\boldsymbol{E}\cdot\boldsymbol{\psi}_h]\cdot\nabla^{\text{as}}\boldsymbol{a}_h - \nabla^{\text{as}}\boldsymbol{a}_h\cdot P_\psi[\boldsymbol{E}\cdot\boldsymbol{\psi}_h]$. In these expressions, $\nabla^{\text{as}}\boldsymbol{a}_h$ is defined by (92).

8.2.3 Stability and convergence in a mesh-dependent norm

We first state stability in the form of an inf-sup condition in a mesh dependent norm that depends on the stabilised formulation used, and next we state convergence using the same norm.

Analogously to assumption **H1** defined in Section 8.1.2, we now assume that the following holds:

Assumption H3 Given $\boldsymbol{a}_h, \boldsymbol{v}_h \in \boldsymbol{\mathcal{V}}_h, q_h \in \mathcal{Q}_h, \boldsymbol{\psi}_h \in \boldsymbol{\Upsilon}_h$ and

$$\boldsymbol{z}_h \coloneqq \rho \boldsymbol{a}_h \cdot \nabla \boldsymbol{v}_h + \nabla q_h - \frac{\eta_p}{\lambda_0} \nabla \cdot P_\psi[\boldsymbol{E} \cdot \boldsymbol{\psi}_h],$$

$$\text{there holds } \|\boldsymbol{z}_h\| \leq c_{\mathrm{m}} \left(\|P_{u,0}\left[\boldsymbol{z}_h\right]\| + \left\|P_u^{\perp}\left[\boldsymbol{z}_h\right]\right\| \right),$$

$$\text{for a constant } c_{\mathrm{m}} > 0.$$

The norm in which the results will be first presented is

$$\begin{aligned}
\|\boldsymbol{V}_h\|_W^2 &= 2\eta_s \|\nabla^s \boldsymbol{v}_h\|^2 + \frac{\eta_p}{\lambda_0^2} \|P_\psi[\boldsymbol{E} \cdot \boldsymbol{\varphi}_h]\|^2 \\
&+ \sum_K \alpha_u \left\| \rho \boldsymbol{a}_h \cdot \nabla \boldsymbol{v}_h + \nabla q_h - \frac{\eta_p}{\lambda_0} \nabla \cdot P_\psi[\boldsymbol{E} \cdot \boldsymbol{\varphi}_h] \right\|_K^2 \\
&+ \sum_K \alpha_u \left\| P_u^{\perp}[\rho \boldsymbol{a}_h \cdot \nabla \boldsymbol{v}_h] \right\|_K^2 + \sum_K \alpha_u \left\| P_u^{\perp}[\nabla q_h] \right\|_K^2 \\
&+ \sum_K \alpha_u \left\| P_u^{\perp} \left[\frac{\eta_p}{\lambda_0} \nabla \cdot P_\psi[\boldsymbol{E} \cdot \boldsymbol{\varphi}_h] \right] \right\|_K^2 \\
&+ \sum_K \alpha_p \|\nabla \cdot \boldsymbol{v}_h\|_K^2 + \sum_K \alpha_\psi \left\| \frac{\lambda}{2\lambda_0} \left(\boldsymbol{a}_h \cdot \nabla P_\psi[\boldsymbol{E} \cdot \boldsymbol{\varphi}_h] - \dot{\boldsymbol{\varphi}}_h^{**} \right) \right\|_K^2, \qquad (102)
\end{aligned}$$

considering $\boldsymbol{V}_h = [\boldsymbol{v}_h, q_h, \boldsymbol{\varphi}_h] \in \boldsymbol{\mathcal{X}}_h$ (note again that $\boldsymbol{\varphi}_h$ is dimensionless). This is clearly a norm for the homogeneous velocity boundary conditions considered, since if $\|\boldsymbol{V}_h\|_W = 0$, $\boldsymbol{v}_h = \boldsymbol{0}$ because of the first term (using Körn's inequality), $P_\psi[\boldsymbol{E} \cdot \boldsymbol{\varphi}_h] = \boldsymbol{0}$ for $\eta_p > 0$ because of the second term (and, in fact, $\boldsymbol{\varphi}_h = \boldsymbol{0}$ because of Assumption **H5** stated later), and, finally, $q_h = 0$ because of the third term and the definition of $\mathcal{Q}$.

Our the main *stability* result, which implies existence and uniqueness of discrete solutions supposing that assumptions **H2** and **H3** hold, and for a λ small enough compared to the rest of physical parameters, can be stated as follows: under the hypothesis above, there is a constant $C > 0$ such that

$$\inf_{\boldsymbol{U}_h \in \boldsymbol{\mathcal{X}}_h} \sup_{\boldsymbol{V}_h \in \boldsymbol{\mathcal{X}}_h} \frac{B_{\mathrm{stab}}(\boldsymbol{U}_h, \boldsymbol{V}_h)}{\|\boldsymbol{U}_h\|_W \|\boldsymbol{V}_h\|_W} \geq C,$$

provided the constants $c_i, i = 1, .., 4$ defined in (40)-(42) are large enough.

The interpolation errors $\varepsilon_i(v)$ are defined by (94) . In the case of $v = \boldsymbol{\psi}$, it is understood that $\varepsilon_i(\boldsymbol{\psi}) \coloneqq \inf_{\boldsymbol{\psi}_h \in \boldsymbol{\Upsilon}_h} \sum_K \|\boldsymbol{E} \cdot \boldsymbol{\psi} - \boldsymbol{E} \cdot \boldsymbol{\psi}_h\|_{H^i(K)}$.

Our main convergence result (see [98]) is that the error function of the method is:

$$\mathcal{E}(h) \coloneqq \sqrt{\eta_0}\varepsilon_1(\boldsymbol{u}) + \sqrt{\eta_0}\sum_K \sqrt{\mathrm{Re}_K}\varepsilon_{1,K}(\boldsymbol{u}) + \frac{\sqrt{\eta_0}}{\lambda_0}\varepsilon_0(\boldsymbol{\psi}) + \frac{\sqrt{\eta_0}}{\lambda_0}\sum_K \sqrt{\mathrm{We}_K}\varepsilon_{0,K}(\boldsymbol{\psi}) + \frac{1}{\sqrt{\eta_0}}\varepsilon_0(p), \tag{103}$$

where Re_K and We_K defined in (96) are the element (or cell) Reynolds and Weissenberg numbers, respectively. More precisely, we proved that

$$\|\boldsymbol{U} - \boldsymbol{U}_h\|_W \leq C\mathcal{E}(h). \tag{104}$$

At this point, a very important remark is needed. In Section 8.1 it is proved that the FE method proposed for the *standard* formulation of the viscoelastic flow problem is stable and has an error function similar to (103) (See (95)). However, a *a major difference* needs to be highlighted: now the term that accounts for the error of the logarithm of the conformation tensor has a factor λ_0^{-1} in front. This is a very important improvement, as the growth of the error with the elasticity of the flow will be significantly reduced in the log-conformation formulation with respect to the standard one.

In the way to prove (104), one has to prove (weak) consistency and that the interpolation error of the method is also (103). In the prove of the second result, a technical condition is required, which is worth to point out. It reads:

Assumption H5 For all $\boldsymbol{\chi}_h \in \boldsymbol{\Upsilon}_h$ if $\mathcal{M}$ is a bounded linear operator of $\boldsymbol{\chi}_h$ and $\nabla\boldsymbol{\chi}_h$, there holds
$\|\mathcal{M}(\boldsymbol{\chi}_h, \nabla\boldsymbol{\chi}_h)\|_K \leq C\,\|\mathcal{M}(P_\psi[\boldsymbol{E}\cdot\boldsymbol{\chi}_h], \nabla P_\psi[\boldsymbol{E}\cdot\boldsymbol{\chi}_h])\|_K\,,$
$K \in \mathcal{T}_h$.

If $\tilde{\boldsymbol{U}}_h$ is the best FE approximation of $\boldsymbol{U}$, this assumption allows us to obtain, for λ small enough, the interpolation estimates

$$B_{\mathrm{stab}}\left(\boldsymbol{U} - \tilde{\boldsymbol{U}}_h, \boldsymbol{V}_h\right) \leq C\mathcal{E}\,(h)\,\|\boldsymbol{V}_h\|_W\,,$$
$$\left\|\boldsymbol{U} - \tilde{\boldsymbol{U}}_h\right\|_W \leq C\mathcal{E}\,(h)\,,$$

which are needed to prove (104).

8.2.4 Stability and convergence in a natural norm

The next results aim at proving stability and convergence in a natural norm. As remarked in Section 8.1.3, since this natural norm does not include any control on the convective terms, stability and convergence in this norm is only meaningful in the case of small cell Reynolds and Weissenberg numbers. In the following, and contrary to what we have been considering up to this

point, generic constants C may depend on these numbers and explode as they increase.

We proved that the solution of the discrete problem $\boldsymbol{U}_h = [\boldsymbol{u}_h, p_h, \boldsymbol{\psi}_h] \in \boldsymbol{\mathcal{X}}_h$ can be bounded as

$$\begin{aligned}\sqrt{\eta_0}\,\|\boldsymbol{u}_h\|_{H^1(\Omega)} + \frac{\sqrt{\eta_0}}{\lambda_0}\,\|P_\psi[\boldsymbol{E}\cdot\boldsymbol{\psi}_h]\| + \frac{1}{\sqrt{\eta_0}}\,\|p_h\| \\ \leq C\left(\frac{1}{\sqrt{\eta_0}}\,\|\boldsymbol{f}_u\|_{H^{-1}(\Omega)} + \frac{\lambda_0}{\sqrt{\eta_0}}\,\|\boldsymbol{f}_\psi\|\right).\end{aligned}$$

Moreover, under the assumptions of convergence in the mesh-dependent case and if the solution of the continuous problem $\boldsymbol{U} = [\boldsymbol{u}, p, \boldsymbol{\psi}] \in \boldsymbol{\mathcal{X}}$ is regular enough, the following error estimate holds:

$$\begin{aligned}\sqrt{\eta_0}\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{H^1(\Omega)} + \frac{\sqrt{\eta_0}}{\lambda_0}\,\|\exp(\boldsymbol{\psi})\cdot\boldsymbol{\psi} - P_\psi[\boldsymbol{E}\cdot\boldsymbol{\psi}_h]\| + \frac{1}{\sqrt{\eta_0}}\,\|p-p_h\| \\ \leq C\mathcal{E}(h).\end{aligned}$$

Finally, we proved a L^2 error estimate for the velocity, supposing the same assumptions of stability for the natural norm and that the continuous problem satisfies the following elliptic regularity condition

$$\sqrt{\eta_0}\,\|\boldsymbol{u}\|_{H^2(\Omega)} + \frac{\sqrt{\eta_0}}{\lambda_0}\,\|\exp(\boldsymbol{\psi})\cdot\boldsymbol{\psi}\|_{H^1(\Omega)} + \frac{1}{\sqrt{\eta_0}}\,\|p\|_{H^1(\Omega)} \leq C\frac{1}{\sqrt{\eta_0}}\,\|\boldsymbol{f}_u\|.$$

Then

$$\begin{aligned}&\sqrt{\eta_0}\,\|\boldsymbol{u}-\boldsymbol{u}_h\| \\ &\quad\leq Ch\left(\sqrt{\eta_0}\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{H^1(\Omega)} + \frac{\sqrt{\eta_0}}{\lambda_0}\,\|\boldsymbol{E}\cdot\boldsymbol{\psi} - P_\psi[\boldsymbol{E}\cdot\boldsymbol{\psi}_h]\| + \frac{1}{\sqrt{\eta_0}}\,\|p-p_h\|\right).\end{aligned}$$

8.3 The time-dependent semi-discrete analysis of the standard viscoelastic problem

8.3.1 Problem statement and finite element discretisation

The boundary problem considered in this section corresponds to the one described in Section 3.1. To equations (1)-(4) we add and initial condition for the velocity and for the stress, $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$, respectively, as well as homogeneous Dirichlet conditions for the velocity (and no boundary conditions for the stress field). For a complete description of the mathematical structure of the problem we refer to [85,19].

The weak form of the problem is the one detailed in Section 3.3 for the standard formulation, in particular in equations (16)-(18).

Finally, the Galerkin FE discretisation is given in (25), and the two compatibility conditions that do not allow the use of an arbitrary interpolation

because the scheme may become unstable are explained in Section 4.3, inequalities (26) and (27).

The stabilisation we analysed in [15] is non-residual and is explained in Section 6.1.2, with the stabilisation parameters (40)-(42) given in Section 6.1.1.

8.3.2 Analysis of the linearised time-dependent case

We start considering the transient linearised problem. In this case, the advection velocity $\hat{\boldsymbol{u}}$ in the expressions of the stabilisation parameters needs to be replaced by the linearisation velocity $\boldsymbol{a}$ that we are using in this section.

The transient linearised problem consists in finding $\boldsymbol{U}_h : (0, t_\mathrm{f}) \to \boldsymbol{\mathcal{X}}_h$ such that

$$(\mathcal{D}_t(\boldsymbol{U}_h), \boldsymbol{V}_h) + B_\mathrm{stab}\left(\boldsymbol{U}_h, \boldsymbol{V}_h\right) = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle\,, \tag{105}$$

for all $\boldsymbol{V}_h \in \boldsymbol{\mathcal{X}}_h$, where $\mathcal{D}_t$ is defined in (5), and B_stab, defined in (88), is the sum of the bilinear form of the problem and the stabilisation terms:

$$\begin{aligned} B_\mathrm{stab}\left(\boldsymbol{U}_h, \boldsymbol{V}_h\right) =& B\left(\boldsymbol{U}_h, \boldsymbol{V}_h\right) + S_1^\perp\left(\boldsymbol{U}_h, \boldsymbol{V}_h\right) \\ &+ S_2^\perp\left(\boldsymbol{U}_h, \boldsymbol{V}_h\right) + S_3^\perp\left(\boldsymbol{U}_h, \boldsymbol{V}_h\right). \end{aligned}$$

The notation employed here is the one used in the stationary linearised problem (Section 8.1).

Moreover, assumption **H1** detailed in Section 8.1.2 is a condition on the interpolating spaces to have a well-posed problem that holds in the case of equal order interpolations.

The following existence and uniqueness result for the solution of (105) was motivated by the procedure followed in [133] for the two-field Navier-Stokes problem. Let us consider the following pressure and velocity subspaces:

$$\begin{aligned} Q_h^\star &= \left\{ q_h \in Q_h |\ \left(P_{u,0}^\perp\left[\nabla q_h\right], P_{u,0}^\perp\left[\nabla q_h\right]\right) = 0 \right\}, \\ \mathcal{V}_h^\mathrm{div} &= \left\{ \boldsymbol{v}_h \in \mathcal{V}_h |\ \left(q_h, \nabla\cdot\boldsymbol{v}_h\right) = 0,\ \forall q_h \in Q_h^\star \right\}. \end{aligned}$$

In addition, $Q_h \backslash Q_h^\star$ will stand for the supplementary of $Q_h^\star$ in Q_h, i.e., $Q_h = (Q_h \backslash Q_h^\star) \oplus Q_h^\star$.

To ensure that $\mathcal{V}_h^\mathrm{div}$ is not trivial, we use the following result: there exists a constant $\gamma > 0$, independent of h, such that

$$\inf_{q_h \in Q_h^\star} \sup_{\boldsymbol{v}_h \in \mathcal{V}_h} \frac{\left(q_h, \nabla\cdot\boldsymbol{v}_h\right)}{\left\|\boldsymbol{v}_h\right\|_1 \left\|q_h\right\|} \geq \gamma.$$

Using this, the existence and uniqueness of the semi-discrete problem (105) can be proved.

On the other hand, a stability estimate can be obtained for a slightly different linearised formulation, obtained by evaluating all the rotational terms in the constitutive equation in a previous iteration. Thus, let $\hat{\boldsymbol{\sigma}}_h$ be a given stress and define $g\left(\boldsymbol{a}_h, \hat{\boldsymbol{\sigma}}_h\right) := \hat{\boldsymbol{\sigma}}_h \cdot \nabla \boldsymbol{a}_h + \left(\nabla \boldsymbol{a}_h\right)^T \cdot \hat{\boldsymbol{\sigma}}_h$. The working norm to

consider is similar to (93) used in Section 8.1.2; in the transient case, we need to consider:

$$\begin{aligned}\|\boldsymbol{V}_h\|_W^2 =& 2\eta_s \|\nabla^s \boldsymbol{v}_h\|^2 + \frac{1}{2\mu}\|\boldsymbol{\chi}_h\|^2 + \alpha_1 \left\|P_u^\perp\left[\rho \boldsymbol{a}_h \cdot \nabla \boldsymbol{v}_h\right]\right\|^2 \\ &+ \alpha_1 \|\rho \partial_t \boldsymbol{v}_h + \rho \boldsymbol{a}_h \cdot \nabla \boldsymbol{v}_h + \nabla q_h - \nabla \cdot \boldsymbol{\chi}_h\|^2 \\ &+ \alpha_2 \|\nabla \cdot \boldsymbol{v}_h\|^2 + \alpha_1 \left\|P_u^\perp\left[\nabla q_h\right]\right\|^2 + \alpha_1 \left\|P_u^\perp\left[\nabla \cdot \boldsymbol{\chi}_h\right]\right\|^2 \\ &+ \alpha_3 \left\|\frac{\lambda}{2\eta_p}\partial_t \boldsymbol{\chi}_h - \nabla^s \boldsymbol{v}_h + \frac{\lambda}{2\eta_p}\boldsymbol{a}_h \cdot \nabla \boldsymbol{\chi}_h\right\|^2.\end{aligned}$$

The next result states the stability of the proposed semi-discrete linearised formulation: for almost all $t \in (0, t_\mathrm{f})$ there holds

$$\begin{aligned}&\frac{\rho}{2}\|\boldsymbol{u}_h(t)\|^2 + \frac{\lambda}{4\eta_p}\|\boldsymbol{\sigma}_h(t)\|^2 + \int_0^t \|[\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h]\|_W^2 \,\mathrm{d}t \\ &\quad\lesssim \frac{c_\mathrm{K}^2}{2\eta_p}\int_0^t \|\boldsymbol{f}\|_{H^{-1}}^2 \,\mathrm{d}t + \frac{\lambda^2}{2\eta_p}\int_0^t \|g(\boldsymbol{a}_h, \hat{\boldsymbol{\sigma}}_h)\|^2 \,\mathrm{d}t + \frac{\rho}{2}\|\boldsymbol{u}_0\|^2 + \frac{\lambda}{4\eta_p}\|\boldsymbol{\sigma}_0\|^2.\end{aligned}$$

8.3.3 Analysis of the non-linear problem

We finally consider the nonlinear problem (55), discretised in space using the stabilised FE we propose and still continuous in time. Using the same procedure as that proposed in [96], the existence of a solution under suitable conditions can be proved. The result stated in this section yields existence of a semi-discrete solution, as well as stability and convergence. As usual in the case of the fully nonlinear Navier-Stokes equations, due to the hypotheses needed in the proof of this result, the norm in which it is presented is weaker than the norm used in the linearised problem; this is proved in [15].

In essence, we will have $L^\infty(0, t_\mathrm{f})$-control for both the $L^2(\Omega)$-norm of velocity and stresses and $L^2(0, t_\mathrm{f})$-control for the $H^1(\Omega)$-norm of the velocity. The pressure, on the other hand, is controlled only in a norm involving the stabilisation term, and not the natural $L^2(\Omega)$-norm.

The precise assumptions required on the continuous solution are the following:

Assumption H6 System (7) has a solution $[\boldsymbol{u}, p, \boldsymbol{\sigma}]$ continuous in time and satisfying

$$\begin{aligned}&\sup_{0\le t\le t_\mathrm{f}} \|\boldsymbol{u}\|_\infty \le D_1, \quad \sup_{0\le t\le t_\mathrm{f}} \|\nabla \boldsymbol{u}\|_\infty \le D_2, \quad \sup_{0\le t\le t_\mathrm{f}} \|\boldsymbol{u}\|_{k+1} \le D_3, \\ &\sup_{0\le t\le t_\mathrm{f}} \|\boldsymbol{\sigma}\|_\infty \le D_4, \quad \sup_{0\le t\le t_\mathrm{f}} \|\nabla \boldsymbol{\sigma}\|_\infty \le D_5, \quad \sup_{0\le t\le t_\mathrm{f}} \|\boldsymbol{\sigma}\|_{k+1} \le D_6, \\ &\sup_{0\le t\le t_\mathrm{f}} \|p\|_k \le D_7, \quad \sup_{0\le t\le t_\mathrm{f}} \|\partial_t \boldsymbol{u}\|_k \le D_8, \quad \sup_{0\le t\le t_\mathrm{f}} \|\partial_t \boldsymbol{\sigma}\|_k \le D_9,\end{aligned}$$

for certain positive constants D_i $i = 1, ..., 9$ which are supposed to be small enough. In these inequalities, k is the order of the FE interpolation, assumed to be the same for all the unknowns.

For the time-discrete problem, if δt is the time step size, one usually needs a condition of the form $\delta t \geq C\alpha_1$ for a positive constant C, which is encountered in most stabilised FE methods; see [134,109] and references therein for a description of the problem and a way to avoid this restriction. However, in the time continuous case, the boundedness in time of $\|p\|_k$ *and the assumption that* t_f *is large enough* allows us to prove convergence. We do not pretend however to consider the long-term behaviour of the solution, which would require the modification of the stabilised formulation.

In order to write all estimates in dimensionless form, let L_d be a characteristic length of the problem and t_d a characteristic time scale. These parameters may explode with the viscosity, and therefore the following estimate is useless for high Reynolds numbers.

Assume that $k \geq d/2$ and that α_u is constant (for example computed with zero velocity). Suppose also that Assumption **H6** holds, that t_f is sufficiently large and that the $L^2(0,t_\mathrm{f};(H^{-1}(\Omega))^d)$-norm of $\boldsymbol{f}$ is bounded. Then, if the viscosity is sufficiently large, there exists a solution to (51) satisfying

$$\sup_{0\leq t\leq t_\mathrm{f}} \|\boldsymbol{u}-\boldsymbol{u}_h\|^2 + \frac{1}{t_\mathrm{d}}\int_0^{t_\mathrm{f}} \left(L_\mathrm{d}\,\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|\right)^2 \mathrm{d}t \leq u_\star^2 h^{2k},$$

$$\sup_{0\leq t\leq T} \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|^2 \leq \sigma_\star^2 h^{2k},$$

$$\int_0^{t_\mathrm{f}} \alpha_u\,\|\nabla(p-p_h)\|^2 \,\mathrm{d}t \leq p_\star^2 h^{2k},$$

where u_*, p_* and σ_* are appropriate dimensional factors that render the estimates dimensionally consistent. Up to the assumptions required, this estimate is optimal.

9 Conclusions and outlook

In this paper, we have described the stabilised FE formulation we have developed over the years to approximate the incompressible flow of viscoelastic fluids, both using the standard and the logarithmic versions of the problem. For the former approach, the final formulation we favour is given by (55), together with the introduction of interelement boundary and shock-capturing terms and the dynamic approximation of the sub-grid scales.

This formulation satisfies all the requirements that one may *a priori* pose to a *stabilised* FE formulation: it allows one to use arbitrary interpolations for all variables and it provides stability (and convergence) in highly convective flows. Moreover, it has proven to be very robust in the examples of application, both in 2D and in 3D; the success in this respect is due in a significant deal to the non-residual-based structure of the formulation and the introduction of dynamic sub-grid scales, particularly when the time step is small. The use of discontinuity-capturing techniques has also been crucial in some applications.

Concerning the design of the formulation, it is obviously possible to introduce improvements, but we believe it is pretty complete. As in all flow

problems, one of these improvements could be the design of new discontinuity-capturing techniques. The approaches available so far, one of which we have followed, are not as fundamental in nature as the design of the stabilisation terms, and more basic concepts to design them would be welcome. Other improvements could go in the direction of obtaining better approximations to the sub-grid scales, in particular of the stabilisation parameters, but we do not expect significant changes in the numerical behaviour of the formulation.

There is room for improvement in the design of time integration schemes, either based on finite differences or in FE. Computational efficiency is a major target, and has to be considered in the design of these schemes. In particular, fractional step schemes will be surely essential in the applications. FE methods in time, not touched in this paper, may also become important, due to their flexibility and possibility to use adaptivity.

Adaptivity, both in space and time, is an extremely important topic in computational mechanics, and also in the approximation of viscoelastic flows. Mesh refinement techniques need to be based on *a posteriori* error estimates, and these have to be proposed for the type of methods we have described. In particular, sub-grid scales may be used to design such *a posteriori* error estimates.

Another issue that we have not touched and that is extremely important in the approximation of viscoelastic flows is the design of iterative schemes. Even when the solution is stable at the continuous level, it is very hard to converge in many applications, particularly at high Reynolds and Weissenberg numbers. All the machinery used in nonlinear problems (continuation, relaxation, tangent operators of some terms) need to be adapted to the problem at hand.

The analysis of the formulation we have presented has still some gaps that need to be filled. Stability and convergence estimates robust in terms of the Reynolds and Weissenberg numbers for the fully nonlinear problem will require time, as they are still not available for Newtonian flows (with zero Weissenberg number). However, issues such as the long time behaviour, stability in stronger norms, fully discrete schemes or different constitutive laws could be analysed.

Obviously, all the emerging techniques in computational fluid mechanics can also be applied to the numerical simulation of viscoelastic fluids in combination with the FE formulation proposed, such as reduced order models, machine learning tools or optimisation strategies. This last point is particularly relevant to applications, which are too numerous to list. In particular, a promising field of application of viscoelastic flows is that related to the exploitation of *elastic turbulence*; understanding the mechanisms of drag reduction or the turbulent energy cascade is a topic which can benefit from a numerical tool as the one we have reviewed in this paper.

Acknowledgments

E. Castillo acknowledges the funding received from the Chilean Council for Scientific and Technological Research (CONICYT- FONDECYT 11160160). L. Moreno acknowledges the support received from the Spanish Government through a predoctoral FPI Grant. J. Baiges gratefully acknowledges the support of the Spanish Government through the Ramón y Cajal grant RYC-2015-17367. R. Codina acknowledges the support received through the ICREA Acadèmia Research Program, from the Catalan government.

Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. R. C. J. Richardson, *Non-Newtonian Flow and Applied Rheology (Second Edition)*. Butterworth-Heinemann, second edition ed., 2008.
2. K. Ueda, H. Kanai, and T. Amari, "Viscoelastic properties of paint films and formability in deep drawing of pre-painted steel sheets," *Progress in Organic Coatings*, vol. 45, no. 1, pp. 15 – 21, 2002.
3. T. Chatterjee, M. Linsen, V. V. Ginzburg, D. A. Saucy, A. I. Nakatani, and A. K. V. Dyk, "Influence of the first normal stress differences on model hydrophobically modified ethoxylated urethane-thickened waterborne paints brush drag," *Progress in Organic Coatings*, vol. 135, pp. 582 – 590, 2019.
4. D. Santiago-Ramos, J. de Dios Figueroa-Cárdenas, and J. J. Véles-Medina, "Viscoelastic behaviour of masa from corn flours obtained by nixtamalization with different calcium sources," *Food Chemistry*, vol. 248, pp. 21 – 28, 2018.
5. J. Guillemenet, S. Bistac, and J. Schultz, "Relationship between polymer viscoelastic properties and adhesive behaviour," *International Journal of Adhesion and Adhesives*, vol. 22, no. 1, pp. 1 – 5, 2002.
6. C. Duty, C. Ajinjeru, V. Kishore, B. Compton, N. Hmeidat, X. Chen, P. Liu, A. A. Hassen, J. Lindahl, and V. Kunc, "What makes a material printable? a viscoelastic model for extrusion-based 3d printing of polymers," *Journal of Manufacturing Processes*, vol. 35, pp. 526 – 537, 2018.
7. K. Bootsma, M. M. Fitzgerald, B. Free, E. Dimbath, J. Conjerti, G. Reese, D. Konkolewicz, J. A. Berberich, and J. L. Sparks, "3d printing of an interpenetrating network hydrogel material with tunable viscoelastic properties," *Journal of the Mechanical Behavior of Biomedical Materials*, vol. 70, pp. 84 – 94, 2017. Mechanics of additively manufactured biomaterials and implants.
8. X. Zhuang, J. Ouyang, Y. Li, C. Jiang, and L. Wang, "A three-dimensional thermal model for viscoelastic polymer melt packing process in injection molding," *Applied Thermal Engineering*, vol. 128, pp. 1391 – 1403, 2018.
9. Q. Liao, I. Noda, and C. W. Frank, "Melt viscoelasticity of biodegradable poly(3-hydroxybutyrate-co-3-hydroxyhexanoate) copolymers," *Polymer*, vol. 50, no. 25, pp. 6139 – 6148, 2009.
10. E. Castillo and R. Codina, "Variational multi-scale stabilized formulations for the stationary three-field incompressible viscoelastic flow problem," *Computer Methods in Applied Mechanics and Engineering*, vol. 279, pp. 579–605, 2014.
11. E. Castillo and R. Codina, "First, second and third order fractional step methods for the three-field viscoelastic flow problem," *Journal of Computational Physics*, vol. 296, pp. 113–137, 2015.

12. E. Castillo, J. Baiges, and R. Codina, "Approximation of the two-fluid flow problem for viscoelastic fluids using the level set method and pressure enriched finite element shape functions," *Journal of Non-Newtonian Fluid Mechanics*, vol. 225, pp. 37–53, 2015.
13. E. Castillo and R. Codina, "Numerical analysis of a stabilized finite element approximation for the three-field linearized viscoelastic fluid problem using arbitrary interpolations," *ESAIM: Mathematical Modelling and Numerical Analysis*, vol. 51, pp. 1407–1427, 2017.
14. E. Castillo and R. Codina, "Finite element approximation of the viscoelastic flow problem: A non-residual based stabilized formulation," *Computers & Fluids*, vol. 142, pp. 72–78, 2017. Selected papers of 18th International Conference on Finite Elements in Flow Problems.
15. G. R. Barrenechea, E. Castillo, and R. Codina, "Time-dependent semidiscrete analysis of the viscoelastic fluid flow problem using a variational multiscale stabilized formulation," *IMA Journal of Numerical Analysis*, vol. 39, pp. 792–819, 2019.
16. L. Moreno, R. Codina, J. Baiges, and E. Castillo, "Logarithmic conformation reformulation in viscoelastic flow problems approximated by a vms-type stabilized finite element formulation," *Computer Methods in Applied Mechanics and Engineering*, vol. 354, pp. 706 – 731, 2019.
17. L. Moreno, R. Codina, J. Baiges, and E. Castillo, "Solution of transient viscoelastic flow problems approximated by a term-by-term VMS stabilized finite element formulation using time-dependent subgrid-scales," *Computer Methods in Applied Mechanics and Engineering*, vol. 354, pp. 706 – 731, 2019.
18. R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*. Wiley, New York, 2002.
19. M. Renardy, *Mathematical Analysis of Viscoelastic Flows*. CBMS-NSF Regional Conference Series in Applied Mathematics, 1989.
20. D. D. Joseph, *Fluid Dynamics of Viscoelastic Liquids*. Applied Mathematical Sciences, Springer-Verlag New York, 1 ed., 1990.
21. R. B. Bird, R. C. Amstrong, and O. Hassager, *Dynamics of Polymeric Liquids, vol.1: Fluid Mechanics*. Wiley, New York, 2nd ed., 1987.
22. R. B. Bird, R. C. Amstrong, and O. Hassager, *Dynamics of Polymeric Liquids, vol.2: Kinetic Theory*. Wiley, New York, 2nd ed., 1987.
23. M. Benhamou, "Polymer viscoelasticity – basics, molecular theories, experiments and simulations, 2nd edn, by yn-hwang lin," *Contemporary Physics*, vol. 53, no. 6, pp. 524–524, 2012.
24. A. F. Domingues, R. J. Poole, and D. J. C. Dennis, "Inertial instabilities in a microfluidic mixing-separating device," *Physics of Fluids*, vol. 31, no. 7, p. 074101, 2019.
25. C. M. White and M. G. Mungal, "Mechanics and prediction of turbulent drag reduction with polymer additives," *Annual Review of Fluid Mechanics*, vol. 40, no. 1, pp. 235–256, 2008.
26. P. C. Sousa, F. T. Pinho, and M. A. Alves, "Purely-elastic flow instabilities and elastic turbulence in microfluidic cross-slot devices," *Soft Matter*, vol. 14, pp. 1344–1354, 2018.
27. R. J. Poole, "Three-dimensional viscoelastic instabilities in microchannels," *Journal of Fluid Mechanics*, vol. 870, p. 1–4, 2019.
28. L. Ducloué, L. Casanellas, S. J. Haward, R. J. Poole, M. A. Alves, S. Lerouge, A. Q. Shen, and A. Lindner, "Secondary flows of viscoelastic fluids in serpentine microchannels," *Microfluidics and Nanofluidics*, vol. 23, p. 33, Feb 2019.
29. P. Sousa, P. Coelho, M. Oliveira, and M. Alves, "Effect of the contraction ratio upon viscoelastic fluid flow in three-dimensional square-square contractions," *Chemical Engineering Science*, vol. 66, no. 5, pp. 998–1009, 2011.
30. Y. Kwon, "Numerical aspects in modeling high Deborah number flow and elastic instability," *Journal of Computational Physics*, vol. 265, pp. 128–144, 2014.
31. R. G. Owens and T. Phillips in *Computational Rheology*, vol. 14, World Scientific, 2002.
32. R. Fattal and R. Kupferman, "Time-dependent simulation of viscoelastic flows at high Weissenberg number using the log-conformation representation," *Journal of Non-Newtonian Fluid Mechanics*, vol. 126, no. 1, pp. 23–37, 2005.

33. A. Leonov, "Analysis of simple constitutive equations for viscoelastic liquids," *Journal of Non-Newtonian Fluid Mechanics*, vol. 42, no. 3, pp. 323 – 350, 1992.
34. Y. Kwon, "Recent results on the analysis of viscoelastic constitutive equations," *Korea-Australia Rheology Journal*, vol. 14, no. 1, pp. 33 – 45, 2002.
35. E. Voyiatzis, C. J. Tsenoglou, and A. G. Boudouvis, "On hadamard stability and dissipative stability of the molecular stress function model of non-linear viscoelasticity," *International Journal of Non-Linear Mechanics*, vol. 44, no. 7, pp. 727 – 734, 2009.
36. Y. Kwon and A. Leonov, "Stability constraints in the formulation of viscoelastic constitutive equations," *Journal of Non-Newtonian Fluid Mechanics*, vol. 58, no. 1, pp. 25 – 46, 1995.
37. D. A. Siginer, *Stability of Non-Linear Constitutive Formulations for Viscoelastic Fluids*. SpringerBriefs in Thermal Engineering and Applied Science, Springer-Verlag New York, 1 ed., 2014.
38. R. Fattal and R. Kupferman, "Constitutive laws for the matrix-logarithm of the conformation tensor," *Journal of Non-Newtonian Fluid Mechanics*, vol. 123, no. 2-3, pp. 281–285, 2004.
39. M. Hulsen, A. van Heel, and B. van den Brule, "Simulation of viscoelastic flows using brownian configuration fields," *Journal of Non-Newtonian Fluid Mechanics*, vol. 70, no. 1, pp. 79 – 101, 1997.
40. T. Vaithianathan and L. R. Collins, "Numerical approach to simulating turbulent flow of a viscoelastic polymer solution," *Journal of Computational Physics*, vol. 187, no. 1, pp. 1 – 21, 2003.
41. N. Balci, B. Thomases, M. Renardy, and C. R. Doering, "Symmetric factorization of the conformation tensor in viscoelastic fluid models," *Journal of Non-Newtonian Fluid Mechanics*, vol. 166, no. 11, pp. 546 – 553, 2011. XVIth International Workshop on Numerical Methods for Non-Newtonian Flows.
42. A. Afonso, F. Pinho, and M. Alves, "The kernel-conformation constitutive laws," *Journal of Non-Newtonian Fluid Mechanics*, vol. 167-168, pp. 30 – 37, 2012.
43. H.-C. Lee, "A nonlinear weighted least-squares finite element method for the Oldroyd-B viscoelastic flow," *Applied Mathematics and Computation*, vol. 219, no. 1, pp. 421–434, 2012.
44. J. Kwack and A. Masud, "A three-field formulation for incompressible viscoelastic fluids," *International Journal of Engineering Science*, vol. 48, no. 11, pp. 1413–1432, 2010. Special Issue in Honor of K.R. Rajagopal.
45. D. Sandri, "Numerical study of a new finite element method for the approximation of viscoelastic fluid flow problems," *Journal of Non-Newtonian Fluid Mechanics*, vol. 118, no. 2-3, pp. 103–120, 2004.
46. M. Alves, F. Pinho, and P. Oliveira, "Effect of a high-resolution differencing scheme on finite-volume predictions of viscoelastic flows," *Journal of Non-Newtonian Fluid Mechanics*, vol. 93, no. 2-3, pp. 287–314, 2000.
47. M. Alves, F. Pinho, and P. Oliveira, "The flow of viscoelastic fluids past a cylinder: finite-volume high-resolution methods," *Journal of Non-Newtonian Fluid Mechanics*, vol. 97, no. 2-3, pp. 207–232, 2001.
48. M. A. Alves, P. J. Oliveira, and F. T. Pinho, "A convergent and universally bounded interpolation scheme for the treatment of advection," *International Journal for Numerical Methods in Fluids*, vol. 41, no. 1, pp. 47–75, 2003.
49. Saramito, P., "A new θ-scheme algorithm and incompressible FEM for viscoelastic fluid flows," *ESAIM: M2AN*, vol. 28, no. 1, pp. 1–35, 1994.
50. T. He, "The cell-based smoothed finite element method for viscoelastic fluid flows using fractional-step schemes," *Computers & Structures*, vol. 222, pp. 133 – 147, 2019.
51. V. Girault and P.-A. Raviart, *Finite Elements for the Navier Stokes Equations*. Springer-Verlag Berlin, 1986.
52. J. Baranger and D. Sandri, "Finite element approximation of viscoelastic fluid flow: Existence of approximate solutions and error bounds," *Numerische Mathematik*, vol. 63, no. 1, pp. 13–27, 1992.
53. A. C. Bogaerds, W. M. Verbeeten, G. W. Peters, and F. P. Baaijens, "3d viscoelastic analysis of a polymer solution in a complex flow," *Computer Methods in Applied Mechanics and Engineering*, vol. 180, no. 3-4, pp. 413–430, 1999.

54. M. A. Behr, L. P. Franca, and T. E. Tezduyar, "Stabilized finite element methods for the velocity-pressure-stress formulation of incompressible flows," *Computer Methods in Applied Mechanics and Engineering*, vol. 104, no. 1, pp. 31 – 48, 1993.
55. R. Codina, "Finite element approximation of the three-field formulation of the Stokes problem using arbitrary interpolations," *SIAM Journal on Numerical Analysis*, vol. 47, no. 1, pp. 699–718, 2009.
56. J. Bonvin, M. Picasso, and R. Stenberg, "GLS and EVSS methods for a three-field Stokes problem arising from viscoelastic flows," *Computer Methods in Applied Mechanics and Engineering*, vol. 190, no. 29-30, pp. 3893–3914, 2001.
57. D. Sandri, "On a FEM method for a linearized version of the Oldroyd's problem," *Computer Methods in Applied Mechanics and Engineering*, vol. 191, no. 44, pp. 5045–5065, 2002.
58. F. Brezzi and M. Fortin, "Variational formulations and finite element methods," in *Mixed and Hybrid Finite Element Methods* (F. Brezzi and M. Fortin, eds.), vol. 15 of *Springer Series in Computational Mathematics*, pp. 1–35, Springer New York, 1991.
59. J. Marchal and M. Crochet, "A new mixed finite element for calculating viscoelastic flow," *Journal of Non-Newtonian Fluid Mechanics*, vol. 26, no. 1, pp. 77–114, 1987.
60. M. Fortin and A. Fortin, "A new approach for the FEM simulation of viscoelastic flows," *Journal of Non-Newtonian Fluid Mechanics*, vol. 32, no. 3, pp. 295–310, 1989.
61. F. P. Baaijens, "Mixed finite element methods for viscoelastic flow analysis: a review," *Journal of Non-Newtonian Fluid Mechanics*, vol. 79, no. 2-3, pp. 361–385, 1998.
62. A. N. Brooks and T. J. Hughes, "Streamline Upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations," *Computer Methods in Applied Mechanics and Engineering*, vol. 32, no. 1-3, pp. 199–259, 1982.
63. Y. Fan, R. Tanner, and N. Phan-Thien, "Galerkin/least-square finite-element methods for steady viscoelastic flows," *Journal of Non-Newtonian Fluid Mechanics*, vol. 84, no. 2-3, pp. 233–256, 1999.
64. O. M. Coronado, D. Arora, M. Behr, and M. Pasquali, "Four-field Galerkin/least-squares formulation for viscoelastic fluids," *Journal of Non-Newtonian Fluid Mechanics*, vol. 140, no. 1-3, pp. 132–144, 2006.
65. X. Li, X. Han, and X. Wang, "Numerical modeling of viscoelastic flows using equal low-order finite elements," *Computer Methods in Applied Mechanics and Engineering*, vol. 199, no. 9-12, pp. 570–581, 2010.
66. E. Oñate, "A stabilized finite element method for incompressible viscous flows using a finite increment calculus formulation," *Computer methods in applied mechanics and engineering*, vol. 182, pp. 355–370, 2000.
67. H. Damanik, J. Hron, A. Ouazzi, and S. Turek, "A monolithic FEM approach for the log-conformation reformulation LCR of viscoelastic flow problems," *Journal of Non-Newtonian Fluid Mechanics*, vol. 165, pp. 1105–1113, 2010.
68. O. C. Zienkiewicz and R. Codina, "A general algorithm for compressible and incompressible flow-part i. the split, characteristic-based scheme," *International Journal for Numerical Methods in Fluids*, vol. 20, no. 8-9, pp. 869–885, 1995.
69. P. Nithiarasu, "A fully explicit characteristic based split CBS scheme for viscoelastic flow calculations," *International Journal for Numerical Methods in Engineering*, vol. 60, no. 5, pp. 949–978, 2004.
70. J. Venkatesan and S. Ganesan, "A three-field local projection stabilized formulation for computations of Oldroyd-B viscoelastic fluid flows," *Journal of Non-Newtonian Fluid Mechanics*, vol. 247, pp. 90 – 106, 2017.
71. S. Varchanis, A. Syrakos, Y. Dimakopoulos, and J. Tsamopoulos, "A new finite element formulation for viscoelastic flows: Circumventing simultaneously the lbb condition and the high-weissenberg number problem," *Journal of Non-Newtonian Fluid Mechanics*, vol. 267, pp. 78 – 97, 2019.
72. R. Codina, J. Principe, O. Guasch, and S. Badia, "Time dependent subscales in the stabilized finite element approximation of incompressible flow problems," *Computer Methods in Applied Mechanics and Engineering*, vol. 196, no. 21, pp. 2413 – 2430, 2007.

73. S. Badia and R. Codina, "On a multiscale approach to the transient Stokes problem: Dynamic subscales and anisotropic space-time discretization," *Applied Mathematics and Computation*, vol. 207, no. 2, pp. 415 – 433, 2009.
74. E. Castillo and R. Codina, "Dynamic term-by-term stabilized finite element formulation using orthogonal subgrid-scales for the incompressible Navier-Stokes problem," *Computer Methods in Applied Mechanics and Engineering*, vol. 349, pp. 701 – 721, 2019.
75. A. González, E. Castillo, and M. Cruchaga, "Numerical verification of a non-residual orthogonal term-by-term stabilized finite element formulation for incompressible convective flow problems," *Computers & Mathematics with Applications*, vol. 80, no. 5, pp. 1009 – 1028, 2020.
76. A. Bonito, P. Clément, and M. Picasso, "Mathematical and numerical analysis of a simplified time-dependent viscoelastic flow," *Numerische Mathematik*, vol. 107, no. 2, pp. 213–255, 2007.
77. E. Carew, P. Townsend, and M. Webster, "On a discontinuity capturing technique for Oldroyd-B fluids," *Journal of Non-Newtonian Fluid Mechanics*, vol. 51, no. 2, pp. 231–238, 1994.
78. R. Codina, "A discontinuity-capturing crosswind-dissipation for the finite element solution of the convection-diffusion equation," *Computer Methods in Applied Mechanics and Engineering*, vol. 110, no. 3-4, pp. 325–342, 1993.
79. P. Nithiarasu, O. C. Zienkiewicz, B. V. K. S. Sai, K. Morgan, R. Codina, and M. Vázquez, "Shock capturing viscosities for the general fluid mechanics algorithm," *International Journal for Numerical Methods in Fluids*, vol. 28, no. 9, pp. 1325–1353, 1998.
80. M. A. Hulsen, R. Fattal, and R. Kupferman, "Flow of viscoelastic fluids past a cylinder at high Weissenberg number: Stabilized simulations using matrix logarithms," *Journal of Non-Newtonian Fluid Mechanics*, vol. 127, no. 1, pp. 27–39, 2005.
81. O. M. Coronado, D. Arora, M. Behr, and M. Pasquali, "A simple method for simulating general viscoelastic fluid flows with an alternate log-conformation formulation," *Journal of Non-Newtonian Fluid Mechanics*, vol. 147, no. 3, pp. 189 – 199, 2007.
82. A. Kane, R. Guénette, and A. Fortin, "A comparison of four implementations of the log-conformation formulation for viscoelastic fluid flows," *Journal of Non-Newtonian Fluid Mechanics*, vol. 164, no. 1, pp. 45 – 50, 2009.
83. P. Saramito, "On a modified non-singular log-conformation formulation for Johnson-Segalman viscoelastic fluids," *Journal of Non-Newtonian Fluid Mechanics*, vol. 211, pp. 16 – 30, 2014.
84. P. Knechtges, M. Behr, and S. Elgeti, "Fully-implicit log-conformation formulation of constitutive laws," *Journal of Non-Newtonian Fluid Mechanics*, vol. 214, pp. 78 – 87, 2014.
85. E. Fernández-Cara, F. Guillén, and R. Ortega, *Mathematical modeling and analysis of viscoelastic fluids of the Oldroyd kind,* in Handbook of Numerical Analysis, VIII. North-Holland, 2002.
86. M. Renardy, "Existence of slow steady flows of viscoelastic fluids with differential constitutive equations," *ZAMM Journal of Applied Mathematics and Mechanics*, vol. 65, no. 9, pp. 449–451, 1985.
87. C. Guillopé and J. Saut, "Existence results for the flow of viscoelastic fluids with a differential constitutive law," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 15, no. 9, pp. 849–869, 1990.
88. P. L. Lions and N. Masmoudi, "Global solutions for some Oldroyd models of non-Newtonian flows," *Chinese Annals of Mathematics*, vol. 21, no. 3, pp. 131–146, 2000.
89. J. Barrett and S. Boyaval, "Existence and approximation of a (regularized) oldroyd-b model," *Mathematical Models and Methods in Applied Sciences*, vol. 21, pp. 1783–1837, 2011.
90. R. Sureshkumar and A. Beris, "Effect of artificial stress diffusivity on the stability of numerical calculations and the flow dynamics of time-dependent viscoelastic flows," *Journal of Non-Newtonian Fluid Mechanics*, vol. 60, pp. 53–80, 1995.
91. D. Sandri, "Finite element approximation of viscoelastic fluid flow: existence of approximate solutions and error bounds. continuous approximation of the stress," *SIAM Journal on Numerical Analysis*, vol. 31, no. 2, pp. 362–377, 1994.

92. M. Picasso and J. Rappaz, "Existence, a priori and a posteriori error estimates for a nonlinear three-field problem arising from Oldroyd-B viscoelastic flows," *ESAIM: Mathematical Modelling and Numerical Analysis*, vol. 35, no. 5, pp. 879–897, 2001.
93. M. Lukáčová-Medvidová, B. Mizerová, H.and She, and J. Stebel, "Error analysis of finite element and finite volume methods for some viscoelastic fluids," *Journal of Numerical Mathematics*, vol. 24, no. 2, pp. 105–123, 2016.
94. A. Bonito and E. Burman, "A continuous interior penalty method for viscoelastic flows," *SIAM Journal on Scientific Computing*, vol. 30, no. 3, pp. 1156–1177, 2007.
95. V. J. Ervin, H. K. Lee, and L. N. Ntasin, "Analysis of the Oseen-viscoelastic fluid flow problem," *Journal of Non-Newtonian Fluid Mechanics*, vol. 127, no. 2-3, pp. 157–168, 2005.
96. V. J. Ervin and W. W. Miles, "Approximation of time-dependent viscoelastic fluid flow: SUPG approximation," *SIAM Journal on Numerical Analysis*, vol. 41, no. 2, pp. 457–486, 2003.
97. A. K. Pani and J. Y. Yuan, "Semidiscrete finite element Galerkin approximations to the equations of motion arising in the Oldroyd model," *IMA Journal of Numerical Analysis*, vol. 25, pp. 750–782, 2005.
98. R. Codina and L. Moreno, "Analysis of stabilized finite element approximation for a linearized logarithmic reformulation of the viscoelastic flow problem," *Mathematical Modelling and Nuerical Analysis*.
99. J. S. Howell, "Dual-mixed finite element approximation of Stokes and nonlinear Stokes problems using trace-free velocity gradients," *Journal of Computational and Applied Mathematics*, vol. 231, no. 2, pp. 780–792, 2009.
100. E. Castillo and R. Codina, "Stabilized stress-velocity-pressure finite element formulations of the Navier-Stokes problem for fluids with non-linear viscosity," *Computer Methods in Applied Mechanics and Engineering*, vol. 279, pp. 554–578, 2014.
101. R. Codina, S. Badia, J. Baiges, and J. Principe, *Variational Multiscale Methods in Computational Fluid Dynamics*, pp. 1–28. American Cancer Society, 2017.
102. R. Araya, G. Barrenechea, and F. Valentin, "Stabilized finite element methods based on multiscale enrichment for the stokes problem," *SIAM Journal on Numerical Analysis*, vol. 44, no. 1, pp. 322–348, 2006.
103. R. Codina, J. Principe, and J. Baiges, "Subscales on the element boundaries in the variational two-scale finite element method," *Computer Methods in Applied Mechanics and Engineering*, vol. 198, no. 5-8, pp. 838–852, 2009.
104. M. Avila, J. Principe, and R. Codina, "A finite element dynamical nonlinear subscale approximation for the low Mach number flow equations," *Journal of Computational Physics*, vol. 230, no. 22, pp. 7988–8009, 2011.
105. R. Codina, "A stabilized finite element method for generalized stationary incompressible flows," *Computer Methods in Applied Mechanics and Engineering*, vol. 190, no. 20-21, pp. 2681–2706, 2001.
106. R. Codina, "Stabilized finite element approximation of transient incompressible flows using orthogonal subscales," *Computer Methods in Applied Mechanics and Engineering*, vol. 191, no. 39-40, pp. 4295–4321, 2002.
107. T. J. Hughes, "Multiscale phenomena: Green's functions, the Dirichlet-to-Neumann formulation, subgrid scale models, bubbles and the origins of stabilized methods," *Computer Methods in Applied Mechanics and Engineering*, vol. 127, no. 1-4, pp. 387–401, 1995.
108. J. Principe and R. Codina, "On the stabilization parameter in the subgrid scale approximation of scalar convection–diffusion–reaction equations on distorted meshes," *Computer Methods in Applied Mechanics and Engineering*, vol. 199, no. 21, pp. 1386 – 1402, 2010. Multiscale Models and Mathematical Aspects in Solid and Fluid Mechanics.
109. R. Codina, J. Principe, O. Guasch, and S. Badia, "Time dependent subscales in the stabilized finite element approximation of incompressible flow problems," *Computer Methods in Applied Mechanics and Engineering*, vol. 196, no. 21-24, pp. 2413–2430, 2007.
110. F. Shakib and T. J. Hughes, "A new finite element formulation for computational fluid dynamics: IX. Fourier analysis of space-time Galerkin/least-squares algorithms,"

Computer Methods in Applied Mechanics and Engineering, vol. 87, no. 1, pp. 35 – 58, 1991.
111. P. B. Bochev, M. D. Gunzburger, and R. B. Lehoucq, "On stabilized finite element methods for the Stokes problem in the small time step limit," *International Journal for Numerical Methods in Fluids*, vol. 53, no. 4, pp. 573–597, 2007.
112. R. Codina and J. Principe, "Dynamic subscales in the finite element approximation of thermally coupled incompressible flows," *International Journal for Numerical Methods in Fluids*, vol. 54, pp. 707–730, 2007.
113. T. Tezduyar and S. Sathe, "Stabilization parameters in SUPG and PSPG formulations," *Journal of Computational and Applied Mechancis*, vol. 4, no. 1, pp. 71 – 88, 2003.
114. R. Codina, "Stabilization of incompressibility and convection through orthogonal subscales in finite element methods," *Computer Methods in Applied Mechanics and Engineering*, vol. 190, no. 13-14, pp. 1579–1599, 2000.
115. R. Codina, "Analysis of a stabilized finite element approximation of the Oseen equations using orthogonal subscales," *Applied Numerical Mathematics*, vol. 58, no. 3, pp. 264–283, 2008.
116. A. Villota and R. Codina, "Aproximación de la ecuación escalar de convección-difusión-reacción con formulaciones estabilizadas de elementos finitos de alto orden," *Revista Internacional de Métodos Numéricos para Cálculo y Diseño en Ingeniería*, vol. 34, no. 1, p. 28, 2018.
117. R. Codina, "Analysis of a stabilized finite element approximation of the Oseen equations using orthogonal subscales," *Applied Numerical Mathematics*, vol. 58, no. 3, pp. 264–283, 2008.
118. R. Becker and M. Braack, "A finite element pressure gradient stabilization for the Stokes equations based on local projections," *CALCOLO*, vol. 38, no. 4, pp. 173–199, 2001.
119. G. Matthies, P. Skrzypacz, and L. Tobiska, "A unified convergence analysis for local projection stabilisations applied to the Oseen problem," *ESAIM: Mathematical Modelling and Numerical Analysis*, vol. 41, pp. 713–742, 7 2007.
120. Chacón Rebollo, T., "A term by term stabilization algorithm for finite element solution of incompressible flow problems," *Numerische Mathematik*, vol. 79, no. 1, pp. 283–319, 1998.
121. R. Guénette and M. Fortin, "A new mixed finite element method for computing viscoelastic flows," *Journal of Non-Newtonian Fluid Mechanics*, vol. 60, no. 1, pp. 27–52, 1995.
122. T. Knopp, G. Lube, and G. Rapin, "Stabilized finite element methods with shock capturing for advection–diffusion problems," *Computer Methods in Applied Mechanics and Engineering*, vol. 191, no. 27, pp. 2997 – 3013, 2002.
123. V. John and P. Knobloch, "On spurious oscillations at layers diminishing (SOLD) methods for convection–diffusion equations: Part i - a review," *Computer Methods in Applied Mechanics and Engineering*, vol. 196, no. 17, pp. 2197 – 2215, 2007.
124. R. Glowinsky and O. Pironneau, "Finite-element methods for Navier-Stokes equations," *Ann. Rev. Fluid Mech.*, vol. 24, pp. 164–204, 1992.
125. P. Singh and L. Leal, "Finite-element simulation of the start-up problem for a viscoelastic fluid in an eccentric rotating cylinder geometry using a third-order upwind scheme," *Theoretical and Computational Fluid Dynamics*, vol. 5, pp. 107–137, 1993.
126. P. Saramito, "A new θ-scheme algorithm and incompressible fem for viscoelastic fluid flows," *ESAIM: Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique*, vol. 28, pp. 1–35, 1994.
127. J. Chrispell, V. Ervin, and E. Jenkins, "A fractional step θ-method approximation of time-dependent viscoelastic fluid flow," *Journal of Computational and Applied Mathematics*, vol. 232, pp. 159–175, 2009.
128. G. D'Avino and M. Hulsen, "Decoupled second-order transient schemes for the flow of viscoelastic fluids without a viscous solvent contribution," *Journal of Non-Newtonian Fluid Mechanics*, vol. 165, pp. 1602–1612, 2010.
129. G. D'Avino, M. Hulsen, and P. Maffettone, "Decoupled transient schemes for viscoelastic fluid flow with inertia," *Computers & Fluids*, vol. 66, pp. 183–193, 2012.

130. J. M. Kim, C. Kim, J. H. Kim, C. Chung, K. H. Ahn, and S. J. Lee, "High-resolution finite element simulation of 4:1 planar contraction flow of viscoelastic fluid," *Journal of Non-Newtonian Fluid Mechanics*, vol. 129, no. 1, pp. 23–37, 2005.
131. H. Choi, H. Choi, and J. Yoo, "A fractional four-step finite element formulation of the unsteady incompressible Navier-Stokes equations using SUPG and linear equal-order element methods," *Computer Methods in Applied Mechanics and Engineering*, vol. 143, pp. 333–348, 1997.
132. L. Liu and D. E. Keyes, "Field-split preconditioned inexact newton algorithms," *SIAM Journal on Scientific Computing*, vol. 37, no. 3, pp. A1388–A1409, 2015.
133. E. Burman and M. A. Fernández, "Continuous interior penalty finite element method for the time-dependent Navier-Stokes equations: space discretization and convergence," *Numerische Mathematik*, vol. 107, no. 1, pp. 39–77, 2007.
134. S. Badia and R. Codina, "On a multiscale approach to the transient Stokes problem: Dynamic subscales and anisotropic space-time discretization," *Applied Mathematics and Computation*, vol. 207, no. 2, pp. 415–433, 2009.