Chebyshev spectral collocation method approximations of the Stokes eigenvalue problem based on penalty techniques

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Abstract

Numerical solution strategies for the Stokes eigenvalue problem based on the use of penalty formulations are investigated in this study. It is shown that the penalty method approach can successfully be adapted for the eigenproblem to rectify the associated problems such as the existence of zero diagonal entries in the resulting algebraic system. Two different schemes, namely, the standard penalisation with a small penalty parameter, and the iterative penalisation that enables relatively large parameters, are implemented. The employment of the latter leads to a so-called inhomogeneous generalised eigenvalue problem which requires a special attention. A feasible solution strategy is presented which is adapted from a procedure based on Newton's method proposed for the corresponding standard (inhomogeneous) eigenvalue problems. Concerning the spatial discretisation, among other possible options, the Chebyshev spectral collocation method based on expanding the unknown fields in tensor product of Chebyshev polynomials is employed. It is shown that the method constitutes a novel way of efficiently examining the approximate eigensolutions of the Stokes operator with the use of Chebyshev spectral collocation method directly, without a decoupling of velocity and pressure.

Keywords: Stokes eigenvalue problem, Iterative penalisation, Chebyshev spectral collocation

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1. Introduction

The Stokes eigenvalue problem is a subject of an extensive research due to its significance in both fundamental and practical areas. As such, the Stokes operator is a fundamental ingredient in the theoretical study of the Navier-Stokes equations, including turbulence regimes, its eigenfunctions being a basis of the function spaces where the solutions to Navier-Stokes equations reside [9] (see also [21]). Moreover, the Stokes eigenvalue problem is used as benchmark for analysing convergence and accuracy of the numerical algorithms designed in fluid dynamics. A concise collection of these and similar applications where the Stokes eigenvalue problem is of particular interest can be found in [13] and the references therein. On the other hand, the Stokes eigensolutions on confined domains are known analytically only in the case of periodic boundary conditions [9], hence, the theoretical knowledge in general is limited to asymptotic behaviours (more details on such asymptotics can be found, for instance, in [29]). As a consequence of the lack of the analytical expressions of the corresponding spectrum, one has to rely on its numerical determination. There are numerous works devoted to approximating the Stokes eigenproblem based on different methodologies. Among them are finite element methods [1, 12, 16, 27], mesh free methods based on radial basis functions [10], spectral Chebyshev methods based on decoupling the velocity and pressure operators [14, 15], and spectral Lagrange method using a staggered grid system [6].

There are several difficulties in accurately approximating the eigenmodes of the Stokes problem both in two- and three-dimensional domains. These are largely due to the nonlinearity inherited in the governing equations, and algorithmic limitations arising from the sensitivity of the resulting algebraic system to solution methodologies. These complications are emphasised when the problem is formulated in primitive variables, velocity and pressure, with the solenoidal condition, as the discretisation of the operator leads to a generalised eigenvalue problem with zero diagonal entries in the resulting algebraic system. It is well known that these and similar issues arising in solving the corresponding source problems, can be resolved with the use of penalty methods in which a perturbed form of the problem is considered. Moreover, using a penalty method allows one to avoid the problem of choosing one degree of freedom to prescribe for the pressure that is needed in the original problem; penalty methods intrinsically introduce a zero mean pressure value and pressure is not undetermined by an additive constant.

In this paper, we investigate numerical solution strategies for the Stokes eigenvalue problem based on the use of penalty formulations. The motivation is to

extend the widely used application of penalisation techniques to the Stokes eigenvalue problem. We show that the penalty method approach can successfully be adapted for the eigenproblem to deal with associated issues. Two different schemes, namely, the standard penalisation with a small penalty parameter, and the iterative penalisation that permits the use of relatively large parameters, are implemented. The employment of the iterative method leads to a so-called inhomogeneous generalised eigenvalue problem which cannot be treated in the classical framework of eigenproblems. We adapt an efficient solution strategy from a procedure based on Newton's method proposed for the corresponding standard (inhomogeneous) eigenvalue problems in [17]. We employ the Chebyshev spectral collocation method (CSCM) based on expanding the unknown fields in tensor product of Chebyshev polynomials for the spatial discretisation.

Despite its simplicity, the CSCM is traditionally accepted to be one of the most convenient methods for solving nonlinear partial differential equations as the solutions are sought in the physical space, and direct approximation and evaluation of the derivatives of the unknowns is possible. On the other hand, it results in dense matrices with high condition numbers when the number of collocation points is large, even though it is being extensively used in models of moderate sizes. As the system gets ill conditioned when problems with large scales are involved, one has to use specifically designed preconditioners, see for example [18] and [26]. In addition, there are recent developments of the so-called ultraspherical spectral methods, where the derivatives are represented in terms of ultraspherical polynomials employing a sparse recurrence relation to construct the differential operators [20, 23]. In particular, these methods lead to almost banded and well conditioned matrices, and therefore they have gained interest of a number of researchers in recent years.

As a matter of fact, the idea introduced in the present work can be applied with any type of discretisation; however, the simplicity and efficiency of the CSCM for moderate size problems make it a viable option for the demonstration of the theory. More importantly, we show that the proposed idea constitutes an efficient way of numerically examining the eigensolutions of the Stokes operator with the use of Chebyshev collocation approximation directly, that is, without a decoupling of velocity and pressure.

2. Problem statement

The Stokes eigenvalue problem consists of finding $[\boldsymbol{u}, p, \lambda]$ on a bounded and polyhedral domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, where $\boldsymbol{u} : \Omega \to \mathbb{R}^d$, with $\boldsymbol{u} \neq \mathbf{0}$, is the

displacement or velocity field, $p : \Omega \to \mathbb{R}$ is the pressure, and $\lambda \in \mathbb{R}$, such that

$$\begin{cases} -\nu\Delta\boldsymbol{u} + \nabla p = \lambda\boldsymbol{u} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\nu > 0$ is a physical parameter and the system is appropriately nondimensionalised. The eigenproblem that seeks the pressure eigenfunctions (a mixed eigenvalue problem of second type, see [4]) will not be considered in this work. It is well known that the eigenvalues satisfy

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \lambda_k \ldots \leq \lim_{k\to\infty} \lambda_k = \infty,$$

with the associated eigenfunctions

$$[\boldsymbol{u}_1, p_1], [\boldsymbol{u}_2, p_2], \ldots, [\boldsymbol{u}_k, p_k], \ldots$$

which are assumed to satisfy

$$(\boldsymbol{u}_i, \boldsymbol{u}_j) = \delta_{ij}, \quad i, j = 1, 2, \ldots, \tag{2}$$

where $(\cdot, \cdot)$ denotes the standard L^2 inner product in Ω.

As we aim to focus on the demonstration of the idea that the Stokes eigensolutions can be approximated by the use of different penalisation techniques, we consider (1) on both two- and three-dimensional configurations, namely, a square domain and a cube domain. For the former case, the velocity field components are denoted by u and v, whereas, the third dimensional component is denoted by w in the latter case. We note that the accurate determination of the Stokes eigensolutions even on simple domains is a challenging problem. On the other hand, an extension to more general domains can be made by using a domain decomposition technique or a coordinate transformation. Nevertheless, the numerical strategies devised in this study are independent of the choice of domains.

3. Numerical approximation

We explain the spatial discretisation of the Stokes eigenproblem (1), which is carried out by a collocation approach in this section. First, an outline of the CSCM with some relevant details is given in Section 3.1 which is followed by the application of the method to the Stokes eigenproblem in Section 3.2. The

discretisation will lead to an algebraic eigenvalue problem that, in general, can be solved in numerous ways. We will present a procedure based on Newton's method for approximating eigenvalues in Section 3.3. The spatial discretisations as well as the eigenvalue approximations are established in the same fashion for both two- and three-dimensional problems considered in this study; for conciseness, our presentation will mainly be focused on the latter case.

3.1. The essentials of the CSCM

The method we consider is based on requiring the numerical approximation of each unknown to be exactly satisfied on the abscissae of the extreme points of the Chebyshev polynomials. In this approach, each function spans the whole domain under consideration and thus, the derivatives of the function depend on the entire discretisation. For completeness, we first review this method in some detail, and refer to [5, 11, 22, 24] for more details. An approximation in the physical space to a function $\Phi(x)$, $x \in [-1, 1]$, is given in the form

$$P_N(x) = \sum_{j=0}^{N} C_j(x)\Phi(x_j),$$

where $C_j(x)$ is a Cardinal function (or Lagrange basis) of degree N. Thus, $P_N(x)$ is a polynomial of degree at most N, and it is evaluated and also differentiated at the collocation points which are given as $x_i = \cos\dfrac{i\pi}{N}$, for $i = 0, 1, \ldots, N$, the abscissae of the extreme points of the Chebyshev polynomials (in [-1,1]). These points are often referred as Chebyshev-Gauss-Lobatto (CGL) points or sometimes as Chebyshev points of the second kind [28]. They possess the desired property of being clustered at the end points of the interval, consequently in a multi-dimensional domain, having a concentration of grid lines near the boundaries (a distribution of these points in a square domain, where $N = 16$, is illustrated in Figure 1).

Let Φ_N and $\Phi_N^{(r)}$, denote the arrays of size $(N + 1)$ whose components respectively are $\Phi(x_j)$ and $\Phi^{(r)}(x_j)$, $r = 1, 2$ being the order of derivative, and $j = 0, 1, \ldots, N$. Then, the r-th derivative of $\Phi(x)$ is approximated by $D^{(r)}\Phi_N$, where $D^{(r)}$ is the so-called Chebyshev differentiation matrix of order r whose entries are $D_{ij}^{(r)} = C_j^{(r)}(x_i),\ i, j = 0, 1, \ldots, N$.

It is well known that the direct computation of $D^{(r)}$ leads to significant errors. In order to minimise the rounding errors for the calculation, the diagonal elements

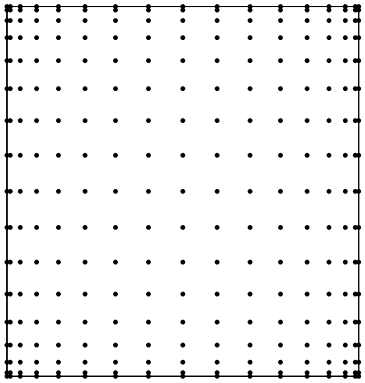

Figure 1: A sample node distribution on a square domain for $N = 16$.

are calculated as

$$D_{ii}^{(r)} = -\sum_{\substack{j=0 \\ j\neq i}}^{N} D_{ij}^{(r)}.$$

This modification is referred as the negative sum trick, and is known to produce distinguishable approximation results, in spite of its ease of implementation, as it is analysed in [3] (see also [2]).

Remark 3.1. *As we have already mentioned, our primary interest in this study is to show that different penalisation techniques can be exploited in the numerical approximation of the Stokes eigensolutions. We have experienced in the numerical simulations that the use of a relatively small number of collocation points successfully served for our illustration purposes (see Section 5). Consequently, the above described aspects of implementation reinforce the choice of the CSCM for spatial discretisation in the present study. Nevertheless, we stress that the ideas introduced here can naturally be applied to any type of discretisation technique.*

The utilisation of the Chebyshev spectral differentiation matrices to construct the approximate discrete operators for solving eigenvalue problems in several space dimensions is described in the subsequent sections.

3.2. The three-dimensional CSCM formulation

In a three-dimensional configuration, we first set a tensor product grid based on CGL points assuming that the domain is a cube. We assume without loss of generality that the same polynomial of degree N is used in all directions; thus, the differentiation matrices $D_N^{(1)}$ and $D_N^{(2)}$, are computed and employed to approximate the derivatives in all coordinates. If the approximations u_N, v_N, w_N and p_N to u, v, w and p, respectively, are substituted together with the approximate differential operators defining this problem into the system of equations describing the Stokes eigenproblem (1), then the discretised equations are written in matrix form as

$$\begin{aligned} -\nu K u_N + G_x p_N &= \lambda u_N, \\ -\nu K v_N + G_y p_N &= \lambda v_N, \\ -\nu K w_N + G_z p_N &= \lambda w_N, \\ G_x u_N + G_y v_N + G_z w_N &= 0. \end{aligned}$$

Here, the matrix associated to the Laplace operator in three dimensions is of order $(N+1)^3$, and it is defined with the use of the Kronecker product as

$$K = I_N \otimes D_N^{(2)} \otimes I_N + I_N \otimes I_N \otimes D_N^{(2)} + D_N^{(2)} \otimes I_N \otimes I_N.$$

Matrices G_x, G_y, and G_z are defined as

$$\begin{aligned} G_x &= I_N \otimes D_N^{(1)} \otimes I_N, \\ G_y &= I_N \otimes I_N \otimes D_N^{(1)}, \\ G_z &= D_N^{(1)} \otimes I_N \otimes I_N. \end{aligned}$$

Next, we can write the generalised matrix eigenvalue problem in the three-dimensional setting as

$$\widetilde{L}\widetilde{\phi} = \lambda \widetilde{R}\widetilde{\phi}, \tag{3}$$

where

$$\widetilde{L} = \begin{bmatrix} -\nu K & 0 & 0 & G_x \\ 0 & -\nu K & 0 & G_y \\ 0 & 0 & -\nu K & G_z \\ G_x & G_y & G_z & 0 \end{bmatrix}, \quad \widetilde{R} = \begin{bmatrix} I_N & 0 & 0 & 0 \\ 0 & I_N & 0 & 0 \\ 0 & 0 & I_N & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \widetilde{\phi} = \begin{bmatrix} u_N \\ v_N \\ w_N \\ p_N \end{bmatrix}.$$

Here we note that the matrix K associated to the Laplace operator is not symmetric, and thus the block matrix $\widetilde{L}$ is not symmetric.

Let us now write the reduced system corresponding to (3) with the boundary conditions imposed as

$$L\phi = \lambda R\phi. \tag{4}$$

3.3. Eigenvalue approximation

Now consider the reduced algebraic eigenproblem (4). This problem is a system of nonlinear equations written as

$$\begin{aligned}(L - \lambda R)\phi &= 0,\\ \phi^{\mathrm{T}} R\phi &= 1,\end{aligned}$$

for the unknowns λ and ϕ. Here, the matrix R, in the term $\phi^{\mathrm{T}} R\phi$, has been introduced for the normalisation of the eigenvectors corresponding to the velocity field associated to (2).

For convenience, we can reformulate this system as

$$\begin{aligned}(L - \lambda R)\phi &= 0,\\ \frac{1 - \phi^{\mathrm{T}} R\phi}{2} &= 0.\end{aligned} \tag{5}$$

If we introduce the partitioned vector Λ defined by

$$\Lambda = \begin{bmatrix}\phi\\ \lambda\end{bmatrix},$$

then the problem can be viewed as a nonlinear equation in the form $F(\Lambda) = 0$. The Jacobian of this system is

$$J_F(\Lambda) = \begin{bmatrix}(L - \lambda R) & -R\phi\\ -\phi^{\mathrm{T}} R & 0\end{bmatrix}.$$

We can then use Newton's method for the solution of the nonlinear problem, with a suitable initial estimate Λ^0, written in the following way:

$$\Lambda^{j+1} = \Lambda^j - [J_F(\Lambda^j)]^{-1} F(\Lambda^j), \quad j = 0, 1, \ldots. \tag{6}$$

Obviously, the Jacobian matrix is not stored and inverted, instead, the linear system $J_F(\Lambda^j)(\Lambda^{j+1} - \Lambda^j) = -F(\Lambda^j)$ is solved for Λ^{j+1}, $j = 0, \ldots$, initialised by a given Λ^0.

4. Penalty methods for the Stokes eigenproblem

The penalty method is a widely used approach in incompressible Stokes and Navier-Stokes models for relaxing the solenoidal condition. It has some advantages, such as the possibility of condensing discontinuous pressures and writing the problem in terms of the velocity only (see, e.g., [7, 19]). Below, we present the extension of the application of the penalisation idea to approximate the spectrum of the Stokes operator.

4.1. The classical penalty problem

The idea is to approximate the solution of the Stokes eigenproblem by the solution to the penalised problem

$$\begin{cases} -\nu\Delta\mathbf{u}_\varepsilon + \nabla p_\varepsilon = \lambda_\varepsilon \mathbf{u}_\varepsilon & \text{in } \Omega, \\ \frac{\varepsilon}{\nu} p_\varepsilon + \nabla \cdot \mathbf{u}_\varepsilon = 0 & \text{in } \Omega, \\ \mathbf{u}_\varepsilon = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where ε is a penalty parameter chosen such that $0 < \varepsilon/\nu < 1$.

The CSCM discretisation steps can be followed as before, yielding

$$\widetilde{L_\varepsilon}\widetilde{\phi_\varepsilon} = \lambda_\varepsilon \widetilde{R}\widetilde{\phi_\varepsilon},$$

where the modified matrix $\widetilde{L_\varepsilon}$ is now given as

$$\widetilde{L_\varepsilon} = \begin{bmatrix} -\nu K & 0 & G_x \\ 0 & -\nu K & G_y \\ G_x & G_y & \frac{\varepsilon}{\nu} I_N \end{bmatrix}.$$

The reduced system after the imposition of the boundary conditions is written in the form

$$L_\varepsilon \phi_\varepsilon = \lambda_\varepsilon R \phi_\varepsilon.$$

The effect of removing zero diagonal entries of the penalisation can readily be seen in Figure 2, where the structures of L and L_ε which are calculated on a square domain with $N = 16$, are visualised using the MATLAB function *spy*.

It is well known that the solutions $[\boldsymbol{u}, p]$ and $[\boldsymbol{u}_\varepsilon, p_\varepsilon]$ to the source problems that correspond to (1) and (7), respectively, satisfy (see [8], for example)

$$\nu\|\nabla\mathbf{u} - \nabla\mathbf{u}_\varepsilon\|^2 + \nu^{-1}\|p - p_\varepsilon\|^2 \leq C_1\varepsilon^2 \left(\nu\|\nabla\mathbf{u}\|^2 + \nu^{-1}\|p\|^2\right),$$

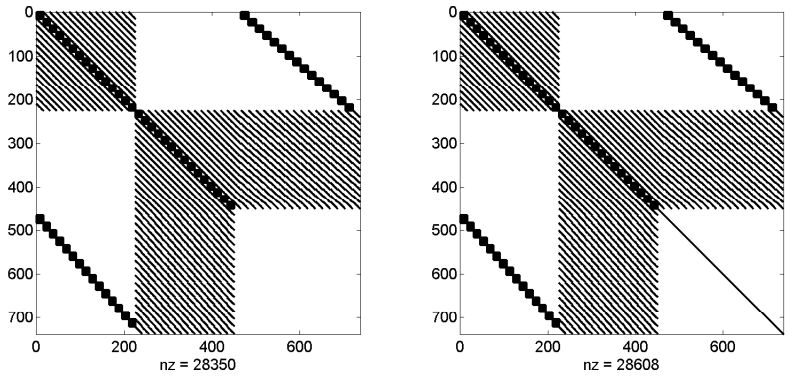

Figure 2: Structures of the eigensystem matrices L (Left) and L_ε (Right), where $N = 16$.

where C_1 is a positive constant independent of ε and $\|\cdot\|$ is the L^2 norm over Ω. That is, the convergence of both $\mathbf{u}_\varepsilon$ and p_ε (for the source problem) is $\mathcal{O}(\varepsilon)$. This can be proved using the variational formulation of the problem, but it is obviously also true for any continuous solution.

Our calculations have revealed the fact that the convergence order is the same for the eigenvalue problem approximated with the CSCM. Furthermore, we have observed that the approximations obtained from the classical penalty method satisfy

$$|\lambda - \lambda_\varepsilon| \leq C_2 \varepsilon |\lambda|,$$

for any eigenvalue λ, where C_2 is a positive constant. We do not provide a proof for this estimate, which can be obtained following the perturbation analysis given in [17].

4.2. The iterative penalty method

The iterative penalisation idea that we consider here was proposed in [7] for the Stokes source problem. In this approach, the penalised equations are solved in each iteration with the addition of the residual of the incompressibility equation of the previous iteration. This method allows the use of large penalty parameters, leading to a system with better conditioning.

The resulting problem reads as: given initially p_ε^0; find ${\mathbf{u}_\varepsilon}^i$, p_ε^i, and λ_ε^i such

that

$$\begin{cases} -\nu\Delta\mathbf{u}_\varepsilon^i + \nabla p_\varepsilon^i = \lambda_\varepsilon \mathbf{u}_\varepsilon^i & \text{in } \Omega, \\ \frac{\varepsilon}{\nu} p_\varepsilon^i + \nabla \cdot \mathbf{u}_\varepsilon^i = \frac{\varepsilon}{\nu} p_\varepsilon^{i-1} & \text{in } \Omega, \\ \mathbf{u}_\varepsilon{}^i = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{8}$$

for $i = 1, 2, \ldots$.

The discretised system of (8) can be written as

$$L_\varepsilon^i \phi_\varepsilon^i = \lambda_\varepsilon^i R \phi_\varepsilon^i + \varphi_\varepsilon^{i-1}, \tag{9}$$

at each iteration i, where $\varphi_\varepsilon^{i-1}$ is the inhomogeneity vector partitioned as

$$\varphi_\varepsilon^{i-1} = \begin{bmatrix} 0 \\ \frac{\varepsilon}{\nu} p_\varepsilon^{i-1} \end{bmatrix}.$$

Clearly, the first iterate, that is, the case $i = 1$, with $p_\varepsilon^0 = 0$, and thus $\varphi_\varepsilon^0 = 0$, corresponds to the classical penalty problem and can be solved as described in the previous section. On the other hand, the next iterations where $\varphi_\varepsilon^{i-1} \neq 0$ are in the form of an inhomogeneous generalised eigenvalue problem which cannot be solved as a standard eigenvalue problem. A natural contender for solving such a problem is a procedure based on Newton's method proposed for the corresponding inhomogeneous but standard eigenvalue problems in [17].

To begin with, we assume that a normalisation condition in the form

$$(\phi_\varepsilon^i)^T R \phi_\varepsilon^i = 1,$$

accompanies Equation (9), since the eigenproblem we consider corresponds to finding velocity eigenfunctions. Consequently, we can write the iterative problem in the form

$$\begin{aligned} L_\varepsilon^i \phi_\varepsilon^i - \lambda_\varepsilon^i R \phi_\varepsilon^i - \varphi_\varepsilon^{i-1} &= 0, \\ \frac{1 - (\phi_\varepsilon^i)^T R \phi_\varepsilon^i}{2} &= 0. \end{aligned}$$

This problem can be viewed as a nonlinear equation in the form $H(\Lambda_\varepsilon^i) = 0$, where the partitioned vector Λ_ε^i is defined by

$$\Lambda_\varepsilon^i = \begin{bmatrix} \phi_\varepsilon^i \\ \lambda_\varepsilon^i \end{bmatrix}.$$

We can then introduce Newton's method for the solution of the nonlinear problem at each iteration $i = 1, 2, \ldots$, yielding the nested algorithm in the following way:

$$\Lambda_\varepsilon^{i,j+1} = \Lambda_\varepsilon^{i,j} - [J_H(\Lambda_\varepsilon^{i,j})]^{-1} H(\Lambda_\varepsilon^{i,j}), \quad j = 0, 1, \ldots, \tag{10}$$

where J_H is the Jacobian of the system $H(\Lambda_\varepsilon^i) = 0$. A convergence criteria is set at each inner iteration (with respect to j), i.e. Newton's method, and then the outer cycle (with respect to i), i.e. the iterative penalisation, is carried out until a corresponding condition for the convergence is met.

The coupled iterations

Instead of following the nested iterative scheme (10), we may implement a combined iterative scheme defined as

$$\Lambda_\varepsilon^{k+1} = \Lambda_\varepsilon^{k} - [J_H(\Lambda_\varepsilon^{k})]^{-1} H(\Lambda_\varepsilon^{k}), \quad k = 0, 1, \ldots, \tag{11}$$

starting from an initial φ_ε^0 formed by a given p^0. In this way, a computationally cheaper scheme is obtained that is also capable of approximating the eigenspectrum of the Stokes operator.

It is shown in [7] that for the corresponding source problem, the error of the iterative penalty method $\nu^{1/2}\|\nabla \mathbf{u} - \nabla \mathbf{u}_\varepsilon^i\| + \nu^{-1/2}\|p - p_\varepsilon^i\|$ is of order $\mathcal{O}(\varepsilon^i)$, so it tends to zero both if $\varepsilon \to 0$ or if $i \to \infty$. For the Navier-Stokes source problem, it is possible to couple the iterations due to penalisation and to nonlinearity as explained for the eigenvalue problem. It is shown in [7] that convergence is driven by the slower of the two errors. We have experimentally observed the same behaviour for the eigenvalue problem.

5. Numerical examples

In this section we present the numerical results for approximating the Stokes eigensolutions on the square domain $[0, 1]^2$ and the cube domain $[-1, 1]^3$. The case $\nu = 1$ is considered, and the results are obtained with the different penalisation techniques described in Section 4. For the iterative penalty method, the convergence criteria has been set to 10^{-10} for both Newton's method and the iterative penalisation method.

The computations are carried out by a computer program created by us, using MATLAB exploiting its linear system as well as matrix eigenvalue solvers based on ARPACK. Even though the orders of complexity of these routines are not generally documented, it is known that they are optimised in the sense that

the structures of the matrices involved are taken into account, and appropriate decompositions are applied. We have taken 17 collocation points in each spatial dimension, that is, the interpolation polynomials are of degree $N = 16$, in all the simulations presented below. Not surprisingly, the naive routines for approximating the solutions to the linear systems involved in our study proved to be significantly efficient, having yielded fairly reasonable results. To be more specific, the computational time is on average 0.311 seconds for a two-dimensional test, and 144.338 seconds for a three-dimensional test which have been performed using an Intel i7 processor 2.00 GHz$\times$4 CPU with 8 GB RAM, running Linux. In general, special attention must be paid in the handling of large scale systems as we have noted in Section 3.1.

The eigensolutions to the Stokes problem for all cases we consider are not known analytically, and therefore, we take reference values from the literature to compare our results with. To quantitatively illustrate the convergence behaviours of the iterative schemes, we calculate the estimated order of convergence (EOC) which is defined as

$$\frac{\log(|\lambda_i - \lambda_{i-1}|/|\lambda_{i-1} - \lambda_{i-2}|)}{\log(|\lambda_{i-1} - \lambda_{i-2}|/|\lambda_{i-2} - \lambda_{i-3}|)}, \quad \text{for } i = 3, 4, \ldots,$$

for the case of nested loops; and as

$$\frac{\log(|\lambda_k - \lambda_{\text{ref}}|/|\lambda_{k-1} - \lambda_{\text{ref}}|)}{\log(|\lambda_{k-1} - \lambda_{\text{ref}}|/|\lambda_{k-2} - \lambda_{\text{ref}}|)}, \quad \text{for } k = 3, 4, \ldots,$$

in the case of combined loops, where λ_{ref} is the reference value for the corresponding problem.

Let us note here that the convergence behaviour of Newton's method depends mainly on its initialisation. In the following tests, we have chosen to start the iterative procedures by the implementation of a single step in the corresponding homogeneous case, to acquire a scaled partitioned vector with a conceivable initial direction. Note also that regardless of the choice of the starting vectors, the convergence rate could always be enhanced by incorporating appropriate shifts in the scheme.

5.1. The square domain

5.1.1. The classical penalty method

Firstly, we consider approximating the Stokes eigenproblem on the square using the classical penalty method described in Subsection 4.1. The first 10 eigenvalues are listed in Table 1, where the penalty parameter is taken as $\varepsilon = 10^{-6}$.

The tabulated values show that the approximations obtained from the classical penalisation agree reasonably well with the reference values. Further, the plot of the eigenfunction associated to the minimum eigenvalue is given in Figure 3, revealing a good agreement with the existing ones in the literature (see, [25]).

Table 1: Computed first 10 eigenvalues on the square domain with $\varepsilon = 10^{-6}$.

Ref. [25]	The classical penalty method
52.3447	52.3447
92.1245	92.1243
92.1246	92.1243
128.2100	128.2096
154.1260	154.1254
167.0298	167.0292
189.5729	189.5718
189.5735	189.5718
246.3240	246.3227
246.3243	246.3227

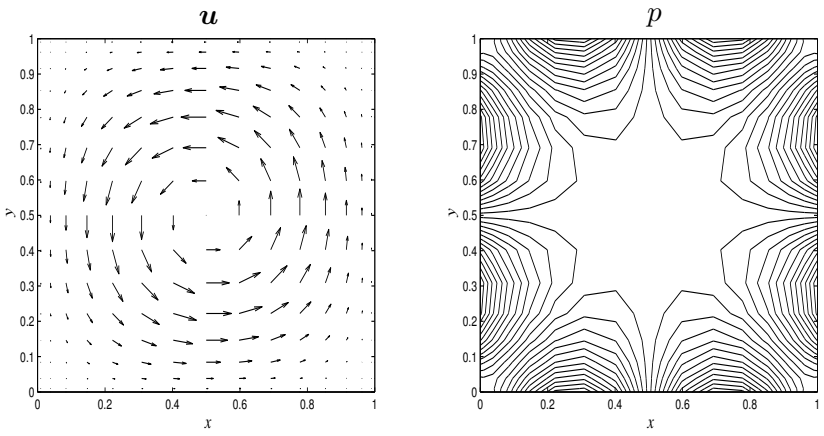

Figure 3: Plot of the first eigenfunction (Left) and the associated pressure contours (Right), obtained with $\varepsilon = 10^{-6}$.

In order to investigate the convergence behaviour of the approximated eigen-

value with respect to the penalty parameter, the minimum eigenvalue (denoted by λ_1) is calculated for ε values varying from 10^{-1} to 10^{-10}. For this test, we consider the corresponding reference value as $\lambda_{\text{ref}} = 52.34469138411319$. This value is obtained by manually removing the spurious pressure modes from the nonpenalised problem (for the case $N = 16$). The variation of $|\lambda_1 - \lambda_{\text{ref}}|/\lambda_{\text{ref}}$ with respect to ε is presented in Figure 4, from which it is inferred that the convergence is linear.

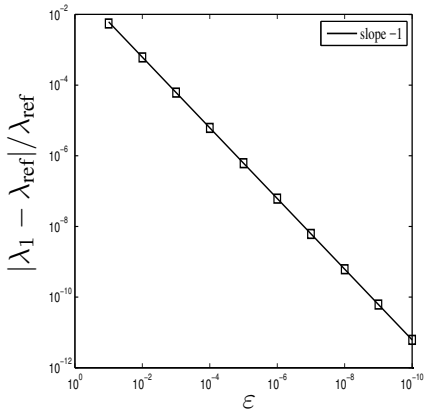

Figure 4: Convergence of λ_1 with respect to ε, on the square domain, where $\lambda_{\text{ref}} = 52.34469138411319$.

5.1.2. The iterative penalty method

Here we present numerical results obtained from the techniques described in Subsection 4.2, for the approximation of the iterative penalty problem (8).

A plot of the relative change in the approximated first eigenvalue, defined by $|\lambda_i - \lambda_{i-1}|/\lambda_i$ against the number of the accumulated iterations i is provided in Figure 5. Two different penalty parameters $\varepsilon = 10^{-1}$ and $\varepsilon = 10^{-3}$ are tested. In both cases, the quadratic dependence on the iteration number at initial steps shows that Newton's method governs the convergence behaviour of the procedure. The jumps in the error correspond to the new outer iterations of the nested loops. These observations are supported qualitatively by the EOCs which are calculated in the first inner loops as 2.2796, 2.1390 for $\varepsilon = 10^{-1}$, and as 2.2906, 2.1284 for $\varepsilon = 10^{-3}$.

The previous experiment for $\varepsilon = 10^{-1}$ and $\varepsilon = 10^{-3}$ is now repeated by implementing the iterative penalisation procedure in a combined loop defined in

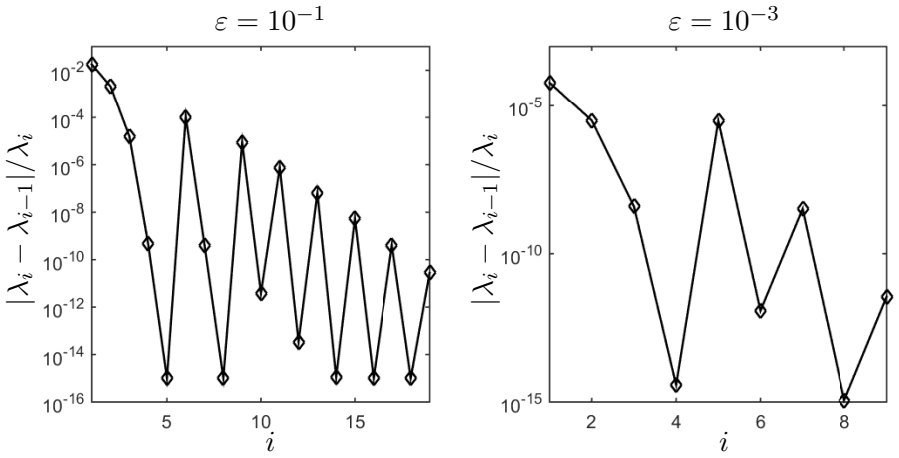

Figure 5: Convergence plot of the relative error with respect to the accumulated iterations of the nested loops (denoted by i), on the square domain, where $\varepsilon = 10^{-1}$ (Left) and $\varepsilon = 10^{-3}$ (Right).

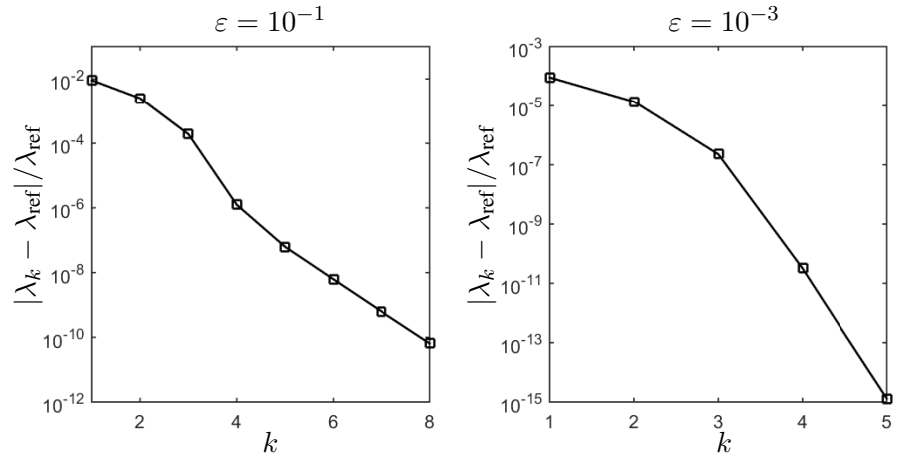

Figure 6: Convergence plot of the relative error, with respect to the number of combined iterations (denoted by k), on the square domain, where $\varepsilon = 10^{-1}$ (Left) and $\varepsilon = 10^{-3}$ (Right).

(11). The convergence behaviours are illustrated in Figure 6, and the corresponding EOCs are listed in Table 2. In progressive iterations, the convergence is driven by the penalisation scheme where an almost linear profile is observed. This phenomena is more pronounced for the larger value of the penalty parameter where the relative errors are larger compared to the smaller value for all iterates as expected. For the larger ε case, the profile changes to linear after the fourth iteration.

A similar tendency can be observed for the smaller ε case, however, with faster decrease in the residual.

Table 2: EOCs with respect to the reference value in the accumulated iterations for the square domain.

k	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-3}$
3	2.0160	2.1920
4	1.9493	2.1529
5	0.5939	1.1395
6	0.7737	-
7	0.9913	-
8	0.9905	-

5.2. The cube domain

In the previous subsection, we have focused on the square domain to show that the penalty formulations are successfully applied to solve the Stokes eigenvalue problem with the use of CSCM. Now we proceed to present the corresponding results for the cube domain.

5.2.1. The classical penalty method

The classical penalisation approach has been tested to approximate the eigensolutions of the Stokes operator on the cube domain. The approximations to the first 33 eigenvalues are listed in Table 3, together with their multiplicities. We have obtained these results using $\varepsilon = 10^{-6}$. Comparing them with the reference values published in [15], one sees that they are reasonably well approximated, with exactly matching multiplicities.

The minimum eigenvalue (having multiplicity three, and denoted by λ_1), is calculated for ε values varying from 10^{-1} to 10^{-10}. In the same way as in the square domain case, for this test we take the corresponding reference value as $\lambda_{\text{ref}} = 15.543353940134807$, which is obtained by manually removing the spurious pressure modes from the nonpenalised problem. The variation of $|\lambda_1 - \lambda_{\text{ref}}|/\lambda_{\text{ref}}$ with respect to ε is depicted in Figure 7, to examine the convergence behaviour of the approximated eigenvalue with respect to the penalty parameter. The convergence rate to the reference value with respect to the penalty parameter is clearly linear, as can be seen from the figure.

Table 3: Computed first 33 eigenvalues on the cube domain for $\varepsilon = 10^{-6}$.

(Multiplicity) Ref. [15]	(Multiplicity) The classical penalty method
(3) 15.54335376	(3) 15.54335314
(2) 22.90746669	(2) 22.90746812
(3) 24.07918373	(3) 24.07915406
(3) 27.06027940	(3) 27.06027842
(3) 32.31421538	(3) 32.31420328
(2) 33.53829871	(2) 33.53828591
(3) 35.17427505	(3) 35.17426715
(1) 36.68074859	(1) 36.68074764
(3) 41.51396629	(3) 41.51394605
(3) 41.99664874	(3) 41.99651688
(3) 44.20838149	(3) 44.20837963
(1) 45.36635367	(1) 45.36633127
(3) 46.41314479	(3) 46.41313479

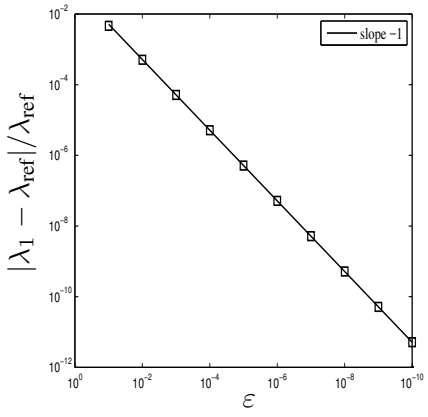

Figure 7: Convergence of λ_1 with respect to ε, on the cube domain, where $\lambda_{\text{ref}} = 15.543353940134807$.

Before passing to the results of the iterative procedure, we remark on the conditioning of the systems for both two- and three-dimensional cases. This is studied by means of the condition numbers of L' (two-dimensional case) and L

(three-dimensional case), denoted by κ' and κ, respectively. Figure 8 illustrates the variation of each condition number (calculated in the 2-norm) with the penalty parameter. As can be inferred from this figure, in each case, the condition number grows linearly with the penalty parameter, and the order of magnitude does not depend on the spatial dimension.

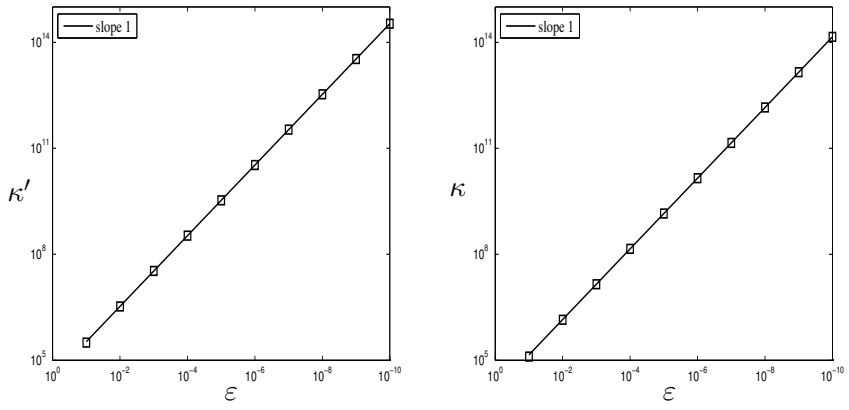

Figure 8: The variation of the condition numbers with the penalty parameter for the square (Left) and the cube (Right).

5.2.2. The iterative penalty method

In analogy with the square domain case, a plot of the relative change in the approximated first eigenvalue defined by $|\lambda_i - \lambda_{i-1}|/\lambda_i$ against the number of the accumulated iterations is given in Figure 9, for $\varepsilon = 10^{-1}$ and $\varepsilon = 10^{-3}$. In both cases corresponding to different penalty parameter, the quadratic dependence on the iteration number at initial steps shows that Newton's method governs the convergence behaviour of the procedure. In progressive iterations, the convergence is driven by the penalisation scheme where an almost linear profile is observed. These feature are also confirmed quantitatively by the EOCs that are computed (in the first inner loops) as 2.2649, 2.2208, 2.1097 and 2.2662, 2.1238, respectively for $\varepsilon = 10^{-1}$ and $\varepsilon = 10^{-3}$. The results are in accordance with the corresponding two-dimensional case (see Figure 5).

Finally, we present the results obtained from the application of the iterative penalisation procedure in the combined loop given in (11). As before, we repeat the experiments for $\varepsilon = 10^{-1}$ and $\varepsilon = 10^{-3}$. The variation of the relative error

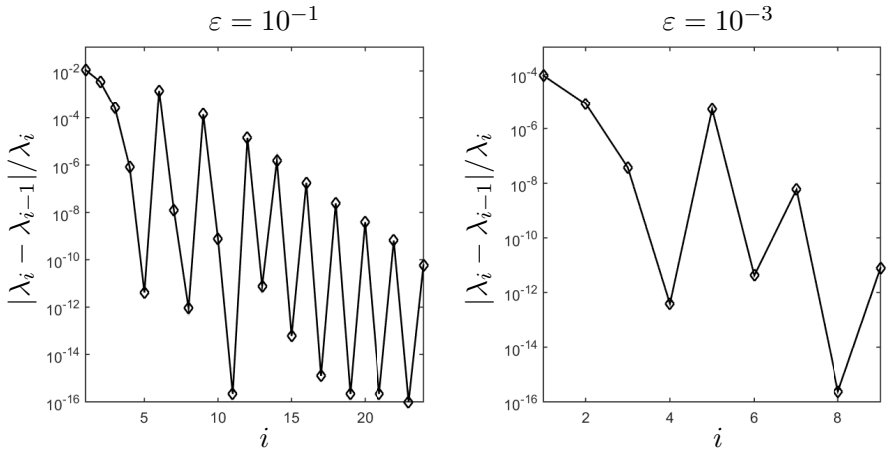

Figure 9: Convergence plot of the relative error with respect to the accumulated iterations of the nested loops (denoted by i), on the cube domain, where $\varepsilon = 10^{-1}$ (Left) and $\varepsilon = 10^{-3}$ (Right).

with respect to the iteration number k is given in Figure 10 for each case, and the corresponding EOCs are listed in Table 4. We see that the convergence properties are similar to those of the two-dimensional counterpart, especially in the case of smaller ϵ. For $\varepsilon = 10^{-1}$ convergence is initially quadratic, driven by Newton's method to deal with the nonlinearity; after the fourth iteration, the convergence curve adapts to that determined by the iterative penalisation. For $\varepsilon = 10^{-3}$ the residual of the iterative penalisation is smaller than that of the nonlinearity and therefore we only observe the latter. In this case, the iterative penalisation allows obtaining a very good approximation to the incompressibility condition with no cost.

6. Conclusions

We have presented two different methods, namely, the classical method and iterative method, based on the penalisation idea applied to the Stokes eigenproblem. We have shown that both procedures circumvent the difficulties related to the eigensystem solution, and further, provide an efficient means of approximating the Stokes eigenproblem directly with the use of CSCM. The numerical calculations suggest that the classical penalty method converges linearly to the reference values, for the two- and three-dimensional examples, namely a square and a cube domain, we considered. The iterative penalty method allows the use of large penalty parameters, leading to a system with better conditioning. On the other

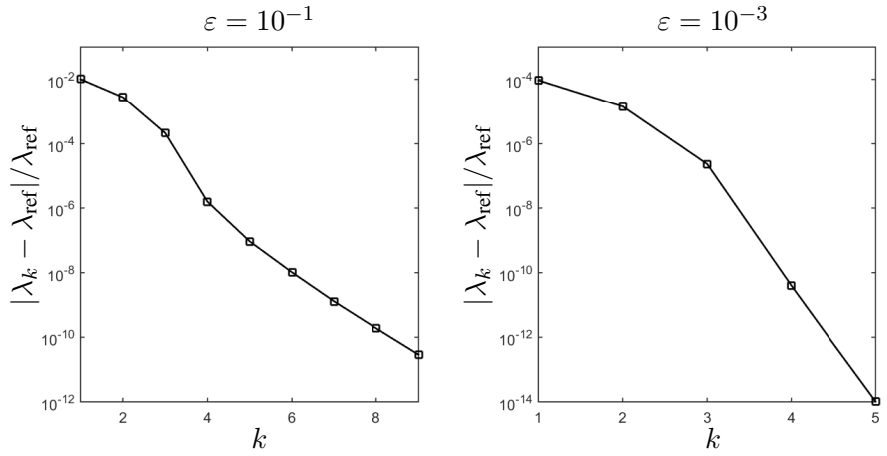

Figure 10: Convergence plot of the relative error, with respect to the number of combined iterations (denoted by k), on the cube domain, where $\varepsilon = 10^{-1}$ (Left) and $\varepsilon = 10^{-3}$ (Right).

Table 4: EOCs with respect to the reference value in the accumulated iterations for the square domain.

k	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-3}$
3	2.0132	2.1911
4	1.9069	2.1070
5	0.5769	0.9553
6	0.7774	-
7	0.9405	-
8	0.9255	-
9	0.9809	-

hand, it necessitates a novel algorithm to deal with the nonlinearity inherited by the inhomogeneous nature of the eigenproblem. We have implemented this novel procedure based on Newton's method for approximating the eigenvalues of the resulting inhomogeneous generalised problem. Our results concerning two different applications, the nested loops and the single loop options, reflect the characteristic behaviours of the iterative penalisation. For the nested loop option, within each step of the iterative penalisation Newton's method yields quadratic convergence, but the outer penalisation loop yields linear convergence only; a jump in the residual is observed when pressure in the continuity equation is updated. For the single

loop option, convergence is driven by the slowest of the two iterative procedures, i.e., Newton's linearisation and the iterative penalisation. If the penalty parameter is relatively small, the former dominates the latter, and incompressible solutions can be found with no added cost with respect to the classical Newton's scheme.

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