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STABILITY, CONVERGENCE, AND ACCURACY OF STABILIZED FINITE ELEMENT METHODS FOR THE WAVE EQUATION IN MIXED FORM*

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Abstract. In this paper we propose two stabilized finite element methods for different functional frameworks of the wave equation in mixed form. These stabilized finite element methods are stable for any pair of interpolation spaces of the unknowns. The variational forms corresponding to different functional settings are treated in a unified manner through the introduction of length scales related to the unknowns. Stability and convergence analysis is performed together with numerical experiments. It is shown that modifying the length scales allows one to mimic at the discrete level the different functional settings of the continuous problem and influence the stability and accuracy of the resulting methods.

Key words. wave equation, stabilized finite element methods, variational multiscale method, orthogonal subgrid scales, convergence, accuracy, stability

AMS subject classifications. 65M60, 65M12, 65M15, 35M13, 35L04, 35F16

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1. Introduction.

When applied to approximate differential equations with several unknowns, and particularly saddle point problems, standard Galerkin mixed finite element (FE) formulations often require the use of inf-sup stable interpolations for the unknowns in order to be stable [10]. Inf-sup stable FE formulations have been formulated for several mixed problems, e.g., for the Stokes problem [2], for the Darcy problem [11], for the Maxwell problem [26], for the Stokes–Darcy problem [1, 25], for the wave equation [8], and for elastodynamics [9, 24].

On the contrary, stabilized FE methods [19] allow one to avoid inf-sup compatibility constraints. As a result, we can deal with different saddle-point problems by using the same equal interpolation for all the unknowns; see, e.g., the unified framework for Stokes, Darcy, and Maxwell problems in [7]. This way, we can certainly ease implementation issues, especially for multiphysics simulations. Stabilized FE methods can be nicely motivated in the variational multiscale (VMS) framework, as shown in [20].

This work is a follow-up of [15] for the wave equation in mixed form. The mixed wave equation is approximated in [15] using the orthogonal subscale stabilization (OSS) method. In the present work, the OSS method is extended, and the algebraic subgrid scale (ASGS) method is also considered. Additionally, length scales associated to the unknowns are introduced, allowing one to treat different functional settings in

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a unified manner. A similar approach for the stationary Stokes–Darcy problem can be found in [5].

We focus on three variational forms of the mixed wave equation and the functional setting for each case. We obtain the different functional settings by *transferring* regularity from the scalar to the vector unknowns or vice-versa. More about functional settings for wave propagation problems of first and second order can be found in [22].

A priori error estimates for the mixed wave equation can be found in the literature. Some only bound L^2-norms of the error of the unknowns [18, 21], whereas others, such as [8], take into account the divergence of the vector unknown too. In this work we bound both the gradient of the scalar unknown and the divergence of the vector unknown using stabilized FEs.

Several stability and convergence analyses have been done so far for the irreducible form of the wave equation (second order space and time derivatives) [17, 23], but not much attention has been paid to the first order in time and space wave equation. As far as the authors are aware, the present work is the first to analyze the convergence properties of stabilized FEs applied to the mixed form of the wave equation.

The organization of this paper is as follows. In section 2 we describe the continuous problem and its variational forms. In section 3 we describe the stabilized discrete problem using the ASGS and OSS methods. In section 4 we state and prove the stability of the discrete formulations. In section 5 we state and prove the convergence of the discrete formulations, including numerical tests. Finally, in section 6 the conclusions of this work are presented.

2. Problem statement.

2.1. Initial and boundary value problem. The problem we consider is an initial and boundary value problem posed in a time interval $(0, T)$ and in a spatial domain $\Omega \subset \mathbb{R}^d$ ($d = 1,\ 2,$ or 3). The long term behavior $T \to \infty$ will not be considered in this work.

Let $\partial\Omega$ be the boundary of the domain Ω. We split this boundary into two disjoint sets denoted as Γ_η and Γ_u, where the boundary conditions corresponding to the unknowns η and $\boldsymbol{u}$ will, respectively, be enforced.

The problem consists of finding $\eta : \Omega \times (0, T) \longrightarrow \mathbb{R}$ and $\boldsymbol{u} : \Omega \times (0, T) \longrightarrow \mathbb{R}^d$ such that

$$\mu_\eta \partial_t \eta + \nabla \cdot \boldsymbol{u} = f_\eta\,, \tag{2.1}$$

$$\mu_u \partial_t \boldsymbol{u} + \nabla \eta = \boldsymbol{f}_u\,, \tag{2.2}$$

with the initial conditions

$$\eta(\boldsymbol{x}, 0) = 0, \quad \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{0}, \quad \boldsymbol{x} \in \Omega, \tag{2.3}$$

and with the boundary conditions

$$\eta = 0 \ \text{ on } \Gamma_\eta, \quad \boldsymbol{n} \cdot \boldsymbol{u} = 0 \ \text{ on } \Gamma_u, \quad t \in (0, T), \tag{2.4}$$

where $\mu_\eta > 0$ and $\mu_u > 0$ are coefficients such that $c^2 = (\mu_\eta \mu_u)^{-1}$, c is the wave speed, f_η and $\boldsymbol{f}_u$ are forcing terms, and $\boldsymbol{n}$ is the unit outward normal to the boundary of the domain. Notice that any general problem with nonhomogeneous initial and boundary conditions can be cast in the form (2.1)–(2.4) properly modifying the forcing terms.

In the previous equations and in what follows, we use the following convention: boldface italic letters represent vectors in $\mathbb{R}^d$ ($d = 1,\ 2,$ or 3), boldface regular uppercase letters represent matrices, and nonbold letters represent scalars.

Let $L^2(\Omega)$ be the space of square integrable functions defined on the domain Ω, let $L^2(\Omega)^d$ be the space of vector valued functions with components in $L^2(\Omega)$, let $H^1(\Omega)$ be the space of functions in $L^2(\Omega)$ with derivatives in $L^2(\Omega)$, let $H^1(\Omega)^d$ be the space of vector valued functions with components in $H^1(\Omega)$, and let $H(\mathrm{div}, \Omega)$ be the space of vector functions with components and divergence in $L^2(\Omega)$. Any of the spaces defined previously will be denoted generically as X. Additionally, given X, a space of (scalar or vector) functions defined over Ω, its spatial norm will be denoted as $\|\cdot\|_X$, and the space of functions whose X-norm is C^k continuous in the time interval $[0,T]$ will be denoted by $C^k([0,T];X)$. We will be interested only in the cases $k=0$, $k=1$, and $k=2$. In the case of $L^2(\Omega)$ or $L^2(\Omega)^d$ the L^2-norm will be simply denoted as $\|\cdot\|$. Functions whose X-norm is L^p in $[0,T]$ will be denoted by $L^p(0,T;X)$; when $X = L^2(\Omega)$ or $X = L^2(\Omega)^d$, the simplification $L^p(L^2)$ will sometimes be used.

Furthermore, let V_η, V_u be spaces associated with η and $\boldsymbol{u}$, respectively. These spaces will be defined afterwards because they depend on the functional setting. Additionally, let us define $V := V_\eta \times V_u$ and $L := L^2(\Omega) \times L^2(\Omega)^d$.

Problem (2.1)–(2.2) will be well posed for

$$\eta \in C^1\big([0,T]\,; L^2(\Omega)\big) \cap C^0\big([0,T]\,; V_\eta\big), \tag{2.5}$$

$$\boldsymbol{u} \in C^1\big([0,T]\,; L^2(\Omega)^d\big) \cap C^0\big([0,T]\,; V_u\big), \tag{2.6}$$

with f_η and $\boldsymbol{f}_u$ in regular enough spaces.

2.2. Variational problem. The variational form of problem (2.1)–(2.4) can be expressed in three different ways. Each one requires a certain regularity on the unknowns η and $\boldsymbol{u}$, which is equivalent to saying that η and $\boldsymbol{u}$ should belong to a particular space of functions.

The problem reads as follows: find $[\eta, \boldsymbol{u}] \in C^1([0,T];L) \cap C^0([0,T];V)$ such that

$$\mathcal{B}([\eta, \boldsymbol{u}], [\xi, \boldsymbol{v}]) = \mathcal{L}([\xi, \boldsymbol{v}]) \tag{2.7}$$

for all test functions $[\xi, \boldsymbol{v}] \in C^0([0,T];V)$ and the respective initial conditions. Here, we require that $\eta(\boldsymbol{x},0) = \xi(\boldsymbol{x},0) = 0$ and $\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{v}(\boldsymbol{x},0) = \boldsymbol{0}$. The bilinear form $\mathcal{B}$, the linear form $\mathcal{L}$, and the space V are defined in three different ways depending on the variational form into consideration.

Let us denote as $(\cdot,\cdot)$ the $L^2(\Omega)$ inner product. For simplicity, we will assume that the forcing terms f_η and $\boldsymbol{f}_u$ are square integrable, although we could relax the this regularity requirement and assume they belong to the dual space of V_η and V_u, respectively.

The variational formulation of problem (2.1)–(2.4) can be posed in three different forms, essentially differing in the way integration by parts from the strong form of the problem is performed and in the regularity required for the unknowns. In the problem statement given below, variational form I (2.8)–(2.11) is obtained by not integrating by parts any term. Variational form II (2.12)–(2.15) is obtained integrating by parts the term $(\nabla\eta, \boldsymbol{v})$. Finally, variational form III (2.16)–(2.19) is obtained integrating by parts the term $(\nabla\cdot\boldsymbol{u}, \xi)$. Notice that integration by parts leads to a boundary term. The treatment of the boundary term is explained in each case.

Variational form I.

$$V_\eta = \left\{\xi \in H^1(\Omega) \mid \xi = 0 \text{ on } \Gamma_\eta\right\}, \quad V_u = \{\boldsymbol{v} \in H(\mathrm{div},\Omega) \mid \boldsymbol{n}\cdot\boldsymbol{v} = 0 \text{ on } \Gamma_u\},$$

$$\mathcal{B}\left([\eta, \boldsymbol{u}], [\xi, \boldsymbol{v}]\right) = \mu_\eta \left(\partial_t \eta, \xi\right) + \left(\nabla \cdot \boldsymbol{u}, \xi\right) + \mu_u \left(\partial_t \boldsymbol{u}, \boldsymbol{v}\right) + \left(\nabla \eta, \boldsymbol{v}\right), \tag{2.8}$$
$$\mathcal{L}\left([\xi, \boldsymbol{v}]\right) = \left(f_\eta, \xi\right) + \left(\boldsymbol{f}_u, \boldsymbol{v}\right), \tag{2.9}$$

$$\eta = 0 \text{ on } \Gamma_\eta \qquad \text{strongly imposed}, \tag{2.10}$$
$$\boldsymbol{n} \cdot \boldsymbol{u} = 0 \text{ on } \Gamma_u \qquad \text{strongly imposed}. \tag{2.11}$$

Variational form II.

$$V_\eta = L^2(\Omega), \quad V_u = \{\boldsymbol{v} \in H(\text{div}, \Omega) \mid \boldsymbol{n} \cdot \boldsymbol{v} = 0 \text{ on } \Gamma_u\},$$

$$\mathcal{B}\left([\eta, \boldsymbol{u}], [\xi, \boldsymbol{v}]\right) = \mu_\eta \left(\partial_t \eta, \xi\right) + \left(\nabla \cdot \boldsymbol{u}, \xi\right) + \mu_u \left(\partial_t \boldsymbol{u}, \boldsymbol{v}\right) - \left(\eta, \nabla \cdot \boldsymbol{v}\right), \tag{2.12}$$
$$\mathcal{L}\left([\xi, \boldsymbol{v}]\right) = \left(f_\eta, \xi\right) + \left(\boldsymbol{f}_u, \boldsymbol{v}\right), \tag{2.13}$$

$$\eta = 0 \text{ on } \Gamma_\eta \qquad \text{weakly imposed}, \tag{2.14}$$
$$\boldsymbol{n} \cdot \boldsymbol{u} = 0 \text{ on } \Gamma_u \qquad \text{strongly imposed}. \tag{2.15}$$

Notice that the boundary integral that appears after integration by parts of $(\nabla\eta, \boldsymbol{v})$ vanishes due to (2.14)–(2.15).

Variational form III.

$$V_\eta = \left\{\xi \in H^1(\Omega) \mid \xi = 0 \text{ on } \Gamma_\eta\right\}, \quad V_u = L^2(\Omega)^d,$$

$$\mathcal{B}\left([\eta, \boldsymbol{u}], [\xi, \boldsymbol{v}]\right) = \mu_\eta \left(\partial_t \eta, \xi\right) - \left(\boldsymbol{u}, \nabla \xi\right) + \mu_u \left(\partial_t \boldsymbol{u}, \boldsymbol{v}\right) + \left(\nabla \eta, \boldsymbol{v}\right), \tag{2.16}$$
$$\mathcal{L}\left([\xi, \boldsymbol{v}]\right) = \left(f_\eta, \xi\right) + \left(\boldsymbol{f}_u, \boldsymbol{v}\right), \tag{2.17}$$

$$\eta = 0 \text{ on } \Gamma_\eta \qquad \text{strongly imposed}, \tag{2.18}$$
$$\boldsymbol{n} \cdot \boldsymbol{u} = 0 \text{ on } \Gamma_u \qquad \text{weakly imposed}. \tag{2.19}$$

Notice that the boundary integral that appears after integration by parts of $(\nabla \cdot \boldsymbol{u}, \xi)$ vanishes due to (2.18)–(2.19).

3. Stabilized FE methods for the wave equation in mixed form. In this section, we present two stabilized FE methods, which we will denote by the acronyms ASGS and OSS, aiming to overcome the instability problems of the standard Galerkin method. In general, stabilized FE methods can be used with any type of interpolation for η and $\boldsymbol{u}$. In particular, we focus on equal and continuous interpolations for η and $\boldsymbol{u}$ and, therefore, conforming FE spaces. For conciseness we will consider quasi-uniform FE partitions of size h. For stabilized formulations in general nonuniform nondegenerate cases, see [14].

Let $V_{\eta,h}$ and $V_{u,h}$ be the FE spaces to approximate η and $\boldsymbol{u}$, respectively, with $V_{\eta,h} \subset V_\eta$ and $V_{u,h} \subset V_u$. Additionally, let us define $V_h = V_{\eta,h} \times V_{u,h}$. For any of these spaces we will make frequent use of the classical inverse inequality $\|\nabla v_h\| \leq C_{\text{inv}} h^{-1} \|v_h\|$, with C_{inv} a constant independent of the FE function v_h and the mesh size h.

3.1. The variational multiscale framework. It is not our purpose here to describe in detail the heuristic design of the stabilized FE methods we will consider, for which [15] can be consulted. We shall only sketch the idea of how the method can be motivated.

Let us write the wave equation as

$$M\partial_t U + AU = F, \tag{3.1}$$

where M is a diagonal matrix with entries μ_η, μ_u, $U = [\eta, \boldsymbol{u}]$ is the unknown, A is the differential operator of the problem, and F groups the forcing terms. The weak form of the problem can now be written as

$$\langle AU, W\rangle = \mathcal{B}(U, W) = (F, W) \tag{3.2}$$

for all test functions W, where $\langle\cdot,\cdot\rangle$ is an appropriate duality depending on the functional setting being chosen.

The idea of the VMS framework to approximate problem (3.2) is as follows [20]. Let X be the space where U belongs, and consider $X = X_h \oplus \tilde{X}$, where X_h is an FE approximation space and $\tilde{X}$ is a complement to be specified. It can be thought of as the space where the components of U which cannot be reproduced by the FE mesh live. Functions $\tilde{U} \in \tilde{X}$ will be called *subgrid scales* or *subscales*. As a first approximation, we assume that all terms involving $\tilde{U}$ evaluated on interelement boundaries vanish.

Using the splitting $U = U_h + \tilde{U}$ in (3.2), taking first the test function as $W_h \in X_h$ and then considering the test function in $\tilde{X}$ yields

$$\mathcal{B}(U_h, W_h) + \langle\tilde{U}, A^* W_h\rangle = (F, W_h), \tag{3.3}$$

$$\tilde{P}(M\partial_t\tilde{U} + A\tilde{U}) = \tilde{P}(F - M\partial_t U_h - AU_h), \tag{3.4}$$

where A^* is the adjoint operator of A and $\tilde{P}$ stands for the L^2 projection onto $\tilde{X}$. We shall now introduce two approximations. The first is that $\tilde{P}(M\partial_t\tilde{U}) \approx 0$. This is not crucial, and in fact it can be relaxed, yielding what we call *dynamic* subscales [13]. The second approximation is to replace $\tilde{P}A$ by a diagonal algebraic operator τ^{-1}, with entries τ_η^{-1}, τ_u^{-1}. Using these approximations in (3.4) yields

$$\tilde{U} = \tau\tilde{P}(F - M\partial_t U_h - AU_h).$$

When introduced in (3.3) we obtain the stabilized FE method we were seeking. When applied to our problem, this method will be of the following form: Find a pair $[\eta_h, \boldsymbol{u}_h] \in \mathcal{C}^1([0,T]; V_h)$ satisfying the initial conditions $\eta_h(\boldsymbol{x}, 0) = 0$, $\boldsymbol{u}_h(\boldsymbol{x}, 0) = \boldsymbol{0}$ and such that

$$\mathcal{B}_s\left([\eta_h, \boldsymbol{u}_h], [\xi_h, \boldsymbol{v}_h]\right) = \mathcal{L}_s\left([\xi_h, \boldsymbol{v}_h]\right) \tag{3.5}$$

for all test functions $[\xi_h, \boldsymbol{v}_h] \in C^0\left([0,T]; V_h\right)$ such that $\xi_h(\boldsymbol{x}, 0) = 0$, $\boldsymbol{v}_h(\boldsymbol{x}, 0) = \boldsymbol{0}$, where it can be readily checked that the bilinear form $\mathcal{B}_s$ and the linear form $\mathcal{L}_s$ are given by

$$\begin{aligned}\mathcal{B}_s\left([\eta_h, \boldsymbol{u}_h], [\xi_h, \boldsymbol{v}_h]\right) &= \mathcal{B}\left([\eta_h, \boldsymbol{u}_h], [\xi_h, \boldsymbol{v}_h]\right) + (\tilde{P}(\mu_\eta\partial_t\eta_h + \nabla\cdot\boldsymbol{u}_h), \tau_\eta\nabla\cdot\boldsymbol{v}_h)\\ &\quad + (\tilde{P}(\mu_u\partial_t\boldsymbol{u}_h + \nabla\eta_h), \tau_u\nabla\xi_h),\end{aligned}$$

$$\mathcal{L}_s\left([\xi_h, \boldsymbol{v}_h]\right) = \mathcal{L}\left([\xi_h, \boldsymbol{v}_h]\right) + (\tilde{P}(f_\eta), \tau_\eta\nabla\cdot\boldsymbol{v}_h) + (\tilde{P}(\boldsymbol{f}_u), \tau_u\nabla\xi_h). \tag{3.6}$$

Depending on the choice of $\tilde{X}$ or, equivalently, on the projection $\tilde{P}$, different stabilized methods arise. The two analyzed in this paper are described in the following subsection.

3.2. ASGS method. In this case we take $\tilde{X}$ as the space of FE residuals, so that $\tilde{P}$ is the identity when acting on those residuals. Thus, the problem consists of solving (3.5) with

$$\begin{aligned}\mathcal{B}_s\left([\eta_h,\boldsymbol{u}_h],[\xi_h,\boldsymbol{v}_h]\right)&=\mathcal{B}\left([\eta_h,\boldsymbol{u}_h],[\xi_h,\boldsymbol{v}_h]\right)+\left(\mu_\eta\partial_t\eta_h+\nabla\cdot\boldsymbol{u}_h,\tau_\eta\nabla\cdot\boldsymbol{v}_h\right)\\&\quad+\left(\mu_u\partial_t\boldsymbol{u}_h+\nabla\eta_h,\tau_u\nabla\xi_h\right),\end{aligned}\tag{3.7}$$

$$\mathcal{L}_s\left([\xi_h,\boldsymbol{v}_h]\right)=\mathcal{L}\left([\xi_h,\boldsymbol{v}_h]\right)+\left(f_\eta,\tau_\eta\nabla\cdot\boldsymbol{v}_h\right)+\left(\boldsymbol{f}_u,\tau_u\nabla\xi_h\right).\tag{3.8}$$

3.3. OSS method. This method is an extension of the wave equation in mixed form of the method proposed in [12, 13]. Space $\tilde{X}$ is taken as the orthogonal in the L^2 sense to X_h. Thus, the problem consists of solving problem (3.5) and taking the bilinear form $\mathcal{B}_s$ and the linear form $\mathcal{L}_s$ as

$$\begin{aligned}\mathcal{B}_s\left([\eta_h,\boldsymbol{u}_h],[\xi_h,\boldsymbol{v}_h]\right)&=\mathcal{B}\left([\eta_h,\boldsymbol{u}_h],[\xi_h,\boldsymbol{v}_h]\right)+\left(P_{\eta,h}^{\perp}\left(\nabla\cdot\boldsymbol{u}_h\right),\tau_\eta\nabla\cdot\boldsymbol{v}_h\right)\\&\quad+\left(P_{u,h}^{\perp}\left(\nabla\eta_h\right),\tau_u\nabla\xi_h\right),\end{aligned}\tag{3.9}$$

$$\mathcal{L}_s\left([\xi_h,\boldsymbol{v}_h]\right)=\mathcal{L}\left([\xi_h,\boldsymbol{v}_h]\right)+\left(P_{\eta,h}^{\perp}\left(f_\eta\right),\tau_\eta\nabla\cdot\boldsymbol{v}_h\right)+\left(P_{u,h}^{\perp}\left(\boldsymbol{f}_u\right),\tau_u\nabla\xi_h\right),\tag{3.10}$$

where $P_{\eta,h}^{\perp}(\cdot)=I(\cdot)-P_{\eta,h}(\cdot)$ and $P_{u,h}^{\perp}(\cdot)=I(\cdot)-P_{u,h}(\cdot)$, $P_{\eta,h}(\cdot)$ being the $L^2(\Omega)$ projection on $V_{\eta,h}$ and $P_{u,h}(\cdot)$ the $L^2(\Omega)$ projection on $V_{u,h}$. This, in particular, implies that $P_{\eta,h}(\cdot)=0$ on Γ_η for variational forms I and III and that $\boldsymbol{n}\cdot P_{u,h}(\cdot)=0$ on Γ_u for variational forms I and II.

From the implementation point of view, the terms $P_{\eta,h}^{\perp}\left(\nabla\cdot\boldsymbol{u}_h\right)$ and $P_{u,h}^{\perp}\left(\nabla\eta_h\right)$ obviously imply an augmented stencil in the matrix of the final algebraic system of equations to be solved. However, using iterative methods it is possible to deal with this without increasing the memory storage and at a very low computation cost, as shown in [14]. In the time discrete problem, it is also possible to approximate $P_{\eta,h}^{\perp}\left(\nabla\cdot\boldsymbol{u}_h^n\right)\approx\nabla\cdot\boldsymbol{u}_h^n-P_{\eta,h}\left(\nabla\cdot\boldsymbol{u}_h^{n-1}\right)$ (and likewise for $P_{u,h}^{\perp}\left(\nabla\eta_h^n\right)$), where the superscript denotes the time step counter. Other possibilities are discussed in [16], and a modified projection with lower sparsity is proposed in [3]. In any case, the increase in computational effort because of the projections can be made very low.

3.4. The stabilization parameters. Important components of stabilized formulations are the stabilization parameters. Following the motivation presented in subsection 3.1, they appear when trying to approximate $\tilde{P}A\tilde{U}\approx\tau^{-1}\tilde{U}$ in a certain sense, which in our case is $[\nabla\cdot\tilde{\boldsymbol{u}},\nabla\tilde{\eta}]\approx[\tau_\eta^{-1}\tilde{\eta},\tau_u^{-1}\tilde{\boldsymbol{u}}]$. Again, the details of the arguments for this approximation can be found in [15, 6], and here we will only describe the essential ideas.

Consider (3.1). In general, the dimensions of the components of F are heterogeneous. Let S be a positive-definite scaling matrix such that the product $F^{\mathrm{t}}SF=:|F|_S^2$ is dimensionally well defined. Within each element K of the FE partition, we assume that the Fourier transform of $\tilde{U}$ is dominated by wave numbers of the form $h^{-1}\tilde{\boldsymbol{k}}$, with h the diameter of K and $\tilde{\boldsymbol{k}}$ dimensionless and of order one. We may then take the Fourier transform $\mathcal{F}$ of $\tilde{P}A\tilde{U}$ and get $\mathcal{F}(\tilde{P}A\tilde{U})=h^{-1}\mathcal{S}(\tilde{\boldsymbol{k}})\mathcal{F}(\tilde{U})$, where $\mathcal{S}(\tilde{\boldsymbol{k}})$ is a complex matrix.

Let $\|F\|_{K,S}$ be the $L^2(K)$-norm of $|F|_S$. The way we propose to approximate $\tilde{P}A\tilde{U}\approx\tau^{-1}\tilde{U}$ is to choose τ^{-1} such that $\|h^{-1}\mathcal{S}(\tilde{\boldsymbol{k}})\mathcal{F}(\tilde{U})\|_{K,S}=\|\tau^{-1}\tilde{U}\|_{K,S}$. This, in particular, can be accomplished by imposing that the spectrum of $h^{-1}\mathcal{S}(\tilde{\boldsymbol{k}})\mathcal{F}(\tilde{U})$ and of $\tau^{-1}\tilde{U}$ with respect to the scaling matrix S coincide. Choosing S as diagonal, with entries

$$S_\eta = \mu_u\sqrt{\frac{\ell_\eta}{\mu_\eta}}, \qquad S_u = \mu_\eta\sqrt{\frac{\ell_u}{\mu_u}},$$

with ℓ_η, ℓ_u length scales corresponding to η and $\boldsymbol{u}$, respectively, it can be shown that

$$\tau_\eta = C_\tau\sqrt{\frac{\mu_u}{\mu_\eta}}h\sqrt{\frac{\ell_\eta}{\ell_u}}, \qquad \tau_u = C_\tau\sqrt{\frac{\mu_\eta}{\mu_u}}h\sqrt{\frac{\ell_u}{\ell_\eta}}, \tag{3.11}$$

where C_τ is a dimensionless algorithmic constant that corresponds to the norm of $\mathcal{S}(\tilde{\boldsymbol{k}})$ evaluated at a certain wave number $\tilde{\boldsymbol{k}}$. See [15, 6] for the details of the derivation.

As will be shown in the analysis to be presented, in order to mimic at the discrete level the proper functional setting of the continuous problem the length scales ℓ_η and ℓ_u should be taken as shown in Table 1, where L_0 is a fixed length scale of the problem that can be fixed a priori. The motivation for designing the stabilization parameters can be found in [5, 15].

Table 1
Stabilization parameters order and length scales definition.

Variational form	I	II	III
τ_η	$\mathcal{O}(h)$	$\mathcal{O}(1)$	$\mathcal{O}(h^2)$
τ_u	$\mathcal{O}(h)$	$\mathcal{O}(h^2)$	$\mathcal{O}(1)$
ℓ_η	$\ell_\eta = \ell_u$	L_0^2/h	h
ℓ_u	$\ell_\eta = \ell_u$	h	L_0^2/h

4. Stability analysis.

In this section, we state and prove stability for the ASGS and OSS methods. First, we use the concept of Λ-coercivity, which will aid us in the proof of stability and later in the convergence analysis.

4.1. Λ-coercivity.

In this section, we state and prove Λ-coercivity in the same sense as in [4] for the ASGS method and with some modifications for the OSS method. The results obtained apply to any of the variational forms defined in (2.8)–(2.19). We only prove Λ-coercivity for the variational form I (2.8)–(2.11) because variational forms II and III only differ in two of the Galerkin terms, namely $(\nabla\eta_h, \boldsymbol{v}_h)$ and $(\nabla\cdot\boldsymbol{u}_h, \xi_h)$. That difference makes the proof just slightly different among the three variational forms. Therefore, we only include the proof for the variational form I.

In what follows, C denotes a positive constant, independent of μ_η, μ_u, ℓ_η, and ℓ_u, but which might depend on the computational domain Ω. In the discrete formulation C will be independent of the mesh size h. The value of C may be different at different occurrences. Additionally, we will use the notation $A \gtrsim B$ and $A \lesssim B$ to indicate that $A \geq CB$ and $A \leq CB$, respectively, where A and B are two quantities that might depend on the solution or mesh size.

The previous methods are not coercive in the norms of interest. The well-posedness is proved via an inf-sup condition. Let $\mathcal{V}$ be a normed space with norm $|\cdot|_\mathcal{V}$ and let $\zeta : \mathcal{V}\times\mathcal{V} \longrightarrow \mathbb{R}$ be a bilinear form. The inf-sup condition implies that $\forall u \in \mathcal{V}\ \exists v \in \mathcal{V}$ such that $\zeta(u,v) \gtrsim |u|_\mathcal{V}|v|_\mathcal{V}$, with $|v|_\mathcal{V} \lesssim |u|_\mathcal{V}$.

For the subsequent analysis we want to prove a more descriptive property than the inf-sup condition, which does not give any clue on how to choose v. We will define an operator $\Lambda : \mathcal{V} \longrightarrow \mathcal{V}$ such that $\zeta(u,\Lambda(u)) \gtrsim |u|_\mathcal{V}|\Lambda(u)|_\mathcal{V}$ $\forall u \in \mathcal{V}$, with

$|\Lambda(u)|_{\mathcal{V}} \lesssim |u|_{\mathcal{V}}$. This property has been defined as Λ-coercivity in [4]. It implies the inf-sup condition for a particular definition of norms, but it also provides additional information on how to choose a v such that the inf-sup condition holds.

DEFINITION 4.1. *Let us define the following norm in $\mathcal{C}^1([0,T];V_h)$:*

$$\begin{aligned}\| [\xi_h, \boldsymbol{v}_h] \|^2_{W,h} &:= \mu_\eta \|\xi_h\|^2_{L^\infty(0,T;L^2(\Omega))} + \mu_u \|\boldsymbol{v}_h\|^2_{L^\infty(0,T;L^2(\Omega))} \\ &\quad + \tau_\eta \|\mu_\eta \partial_t \xi_h + \nabla\cdot \boldsymbol{v}_h\|^2_{L^2(0,T;L^2(\Omega))} + \tau_u \|\mu_u \partial_t \boldsymbol{v}_h + \nabla \xi_h\|^2_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{4.1}$$

The way the norm $\|\cdot\|_{W,h}$ is written and the spaces to which $[\eta_h, \boldsymbol{u}_h]$ belong allow us to determine the expression of the length scales ℓ_η and ℓ_u. In particular, the terms that contain $\nabla\eta_h$ and $\nabla\cdot\boldsymbol{u}_h$ are the ones that allow us to define the length scales. For instance, in the case of variational form II, as $\eta_h \in L^2(\Omega)$, the term containing the $\nabla\eta_h$ should include a factor h^2, which means we choose $\tau_u = \mathcal{O}(h^2)$. Following the same reasoning, for $\nabla\cdot\boldsymbol{u}_h \in L^2(\Omega)$ we arrive at $\tau_\eta = \mathcal{O}(1)$. In Table 1 we have summarized this reasoning and extended it to the remaining variational forms.

Note that some of the results to be presented hold even in the case $\tau_\eta = 0$, $\tau_u = 0$. However, in this case the norm in (4.1) only contains $L^\infty(0,T;L^2(\Omega))$-norms, and this is not enough to avoid point-to-point oscillations. This stability would be found in most first-order-in-time problems using the standard Galerkin method, even if the spatial approximation is completely unstable.

4.1.1. ASGS method. Here we state and prove Λ-coercivity for the ASGS method.

LEMMA 4.2 (weak Λ-coercivity, ASGS). *The bilinear form* (3.7) *satisfies*

$$\| [\xi_h, \boldsymbol{v}_h] \|^2_{W,h} \lesssim \int_0^T \mathcal{B}_s\left([\xi_h, \boldsymbol{v}_h], \Lambda\left([\xi_h, \boldsymbol{v}_h]\right)\right) \mathrm{d}t \quad \forall [\xi_h, \boldsymbol{v}_h], \tag{4.2}$$

where the norm $\|\cdot\|_{W,h}$ is defined in (4.1) *and*

$$\Lambda\left([\xi_h, \boldsymbol{v}_h]\right) := [\xi_h + \tau_\eta \mu_\eta \partial_t \xi_h, \boldsymbol{v}_h + \tau_u \mu_u \partial_t \boldsymbol{v}_h]. \tag{4.3}$$

Proof. Let us test (3.7) with (4.3):

$$\begin{aligned}&\mathcal{B}_s\left([\xi_h, \boldsymbol{v}_h], \Lambda\left([\xi_h, \boldsymbol{v}_h]\right)\right) \\ &\quad = (\mu_\eta \partial_t \xi_h + \nabla\cdot\boldsymbol{v}_h, \xi_h) + (\mu_\eta \partial_t \xi_h + \nabla\cdot\boldsymbol{v}_h, \tau_\eta \mu_\eta \partial_t \xi_h + \tau_\eta \nabla\cdot\boldsymbol{v}_h) \\ &\qquad + (\mu_u \partial_t \boldsymbol{v}_h + \nabla\xi_h, \boldsymbol{v}_h) + (\mu_u \partial_t \boldsymbol{v}_h + \nabla\xi_h, \tau_u \mu_u \partial_t \boldsymbol{v}_h + \tau_u \nabla\xi_h) \\ &\qquad + (\mu_\eta \partial_t \xi_h + \nabla\cdot\boldsymbol{v}_h, \tau_\eta \nabla\cdot(\tau_u \mu_u \partial_t \boldsymbol{v}_h)) + (\mu_u \partial_t \boldsymbol{v}_h + \nabla\xi_h, \tau_u \nabla(\tau_\eta \mu_\eta \partial_t \xi_h)). \end{aligned} \tag{4.4}$$

Using the divergence theorem, recalling that $\xi_h(\boldsymbol{n}\cdot\boldsymbol{v}_h) = 0$ and $\partial_t\xi_h \partial_t(\boldsymbol{n}\cdot\boldsymbol{v}_h) = 0$ on $\partial\Omega$ due to (2.4), we get

$$\begin{aligned}\mathcal{B}_s\left([\xi_h, \boldsymbol{v}_h], \Lambda\left([\xi_h, \boldsymbol{v}_h]\right)\right) &= \frac{1}{2}\mu_\eta \frac{\mathrm{d}}{\mathrm{d}t}\|\xi_h\|^2 + \frac{1}{2}\mu_u \frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}_h\|^2 + \frac{1}{2}\mu_\eta \tau_\eta \tau_u \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\xi_h\|^2 \\ &\quad + \frac{1}{2}\mu_u \tau_\eta \tau_u \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\cdot\boldsymbol{v}_h\|^2 + \tau_\eta \|\mu_\eta \partial_t \xi_h + \nabla\cdot\boldsymbol{v}_h\|^2 + \tau_u \|\mu_u \partial_t \boldsymbol{v}_h + \nabla\xi_h\|^2, \end{aligned} \tag{4.5}$$

and integrating (4.5) from $t = 0$ up to any $t = t^* \leq T$ we get

$$\int_0^{t^*} \mathcal{B}_s\left([\xi_h, \boldsymbol{v}_h], \Lambda\left([\xi_h, \boldsymbol{v}_h]\right)\right) \mathrm{d}t$$

$$=\frac{1}{2}\mu_\eta\|\xi_h(t^*)\|^2+\frac{1}{2}\mu_u\|\boldsymbol{v}_h(t^*)\|^2+\frac{1}{2}\tau_\eta\tau_u\mu_\eta\|\nabla\xi_h(t^*)\|^2+\frac{1}{2}\tau_\eta\tau_u\mu_u\|\nabla\cdot\boldsymbol{v}_h(t^*)\|^2$$
$$+\tau_\eta\int_0^{t^*}\|\mu_\eta\partial_t\xi_h(t)+\nabla\cdot\boldsymbol{v}_h(t)\|^2\,\mathrm{d}t+\tau_u\int_0^{t^*}\|\mu_u\partial_t\boldsymbol{v}_h(t)+\nabla\xi_h(t)\|^2\,\mathrm{d}t,$$

from which $L^\infty(0,T;L^2(\Omega))$ stability follows. Choosing $t^*=T$, we complete the proof. ∎

4.1.2. OSS method. Here we state and prove Λ-coercivity for the OSS method. In this case we express Λ-coercivity in two norms: a weak norm defined in (4.1) and a stronger norm defined in (4.12) below. The norm in (4.12) is stronger, since it provides full control over $\nabla\eta_h$ and $\nabla\cdot\boldsymbol{u}_h$.

LEMMA 4.3 (weak Λ-coercivity, OSS). *The bilinear form* (3.9) *satisfies*

$$\|\!|\!|\,[\xi_h,\boldsymbol{v}_h]\,|\!|\!|^2_{W,h}\lesssim\int_0^T\mathcal{B}_s\left([\xi_h,\boldsymbol{v}_h],\Lambda\left([\xi_h,\boldsymbol{v}_h]\right)\right)\mathrm{d}t\quad\forall\,[\xi_h,\boldsymbol{v}_h]\,, \tag{4.6}$$

where the norm $|\!|\!|\cdot|\!|\!|_{W,h}$ *is defined in* (4.1) *and*

$$\Lambda\left([\xi_h,\boldsymbol{v}_h]\right)=[\xi_h,\boldsymbol{v}_h]+\beta\left[\tau_\eta(\mu_\eta\partial_t\xi_h+P_{\eta,h}(\nabla\cdot\boldsymbol{v}_h)),\tau_u(\mu_u\partial_t\boldsymbol{v}_h+P_{u,h}(\nabla\xi_h))\right], \tag{4.7}$$

with a small enough $\beta>0$.

Proof. Let us test (3.9) against $\Lambda_a([\xi_h,\boldsymbol{v}_h])=[\xi_h,\boldsymbol{v}_h]$. Using the divergence theorem and the boundary condition $\xi_h(\boldsymbol{n}\cdot\boldsymbol{v}_h)=0$ and integrating from $t=0$ to $t=T$, we can readily get

$$\int_0^T\mathcal{B}_s\left([\xi_h,\boldsymbol{v}_h],\Lambda_a\left([\xi_h,\boldsymbol{v}_h]\right)\right)\,\mathrm{d}t\geq\frac{1}{2}\mu_\eta\|\xi_h\|^2_{L^\infty(L^2)}+\frac{1}{2}\mu_u\|\boldsymbol{v}_h\|^2_{L^\infty(L^2)}$$
$$+\tau_\eta\|P^\perp_{\eta,h}\left(\nabla\cdot\boldsymbol{v}_h\right)\|^2_{L^2(L^2)}+\tau_u\|P^\perp_{u,h}\left(\nabla\xi_h\right)\|^2_{L^2(L^2)}. \tag{4.8}$$

Now, let $\Lambda_b([\xi_h,\boldsymbol{v}_h])=\left[\tau_\eta(\mu_\eta\partial_t\xi_h+P_{\eta,h}\left(\nabla\cdot\boldsymbol{v}_h\right)),\tau_u(\mu_u\partial_t\boldsymbol{v}_h+P_{u,h}\left(\nabla\xi_h\right))\right]$, and let us test (3.9) against $\Lambda_b([\xi_h,\boldsymbol{v}_h])$:

$$\mathcal{B}_s\left([\xi_h,\boldsymbol{v}_h],\Lambda_b\left([\xi_h,\boldsymbol{v}_h]\right)\right)=(\mu_\eta\partial_t\xi_h+\nabla\cdot\boldsymbol{v}_h,\tau_\eta(\mu_\eta\partial_t\xi_h+P_{\eta,h}\left(\nabla\cdot\boldsymbol{v}_h\right)))$$
$$+(\mu_u\partial_t\boldsymbol{v}_h+\nabla\xi_h,\tau_u(\mu_u\partial_t\boldsymbol{v}_h+P_{u,h}\left(\nabla\xi_h\right)))$$
$$+\left(P^\perp_{\eta,h}(\nabla\cdot\boldsymbol{v}_h),\tau_\eta\nabla\cdot\left[\tau_u(\mu_u\partial_t\boldsymbol{v}_h+P_{u,h}\left(\nabla\xi_h\right))\right]\right)$$
$$+\left(P^\perp_{u,h}(\nabla\xi_h),\tau_u\nabla[\tau_\eta(\mu_\eta\partial_t\xi_h+P_{\eta,h}\left(\nabla\cdot\boldsymbol{v}_h\right))]\right). \tag{4.9}$$

Using the Cauchy–Schwarz inequality, the inverse inequality, the fact that $\tau_\eta\tau_u=C_\tau^2h^2$, and Young's inequality, we get

$$\mathcal{B}_s\left([\xi_h,\boldsymbol{v}_h],\Lambda_b\left([\xi_h,\boldsymbol{v}_h]\right)\right)$$
$$\geq\tau_\eta\|\mu_\eta\partial_t\xi_h+P_{\eta,h}\left(\nabla\cdot\boldsymbol{v}_h\right)\|^2+\tau_u\|\mu_u\partial_t\boldsymbol{v}_h+P_{u,h}\left(\nabla\xi_h\right)\|^2$$
$$-\frac{1}{2\alpha_1}\tau_\eta\|\mu_\eta\partial_t\xi_h+P_{\eta,h}\left(\nabla\cdot\boldsymbol{v}_h\right)\|^2-\frac{\alpha_1}{2}C_\tau^2C_{\mathrm{inv}}^2\tau_u\|P^\perp_{u,h}(\nabla\xi_h)\|^2$$
$$-\frac{1}{2\alpha_2}\tau_u\|\mu_u\partial_t\boldsymbol{v}_h+P_{u,h}\left(\nabla\xi_h\right)\|^2-\frac{\alpha_2}{2}C_\tau^2C_{\mathrm{inv}}^2\tau_\eta\|P^\perp_{\eta,h}(\nabla\cdot\boldsymbol{v}_h)\|^2 \tag{4.10}$$

for any $\alpha_1,\alpha_2>0$. Next, we integrate (4.10) in time from 0 to T and multiply it by a positive constant β. The resulting inequality is added to (4.8). Taking β small enough and α_i large enough, we prove the lemma. ∎

As was explained before, now we introduce a norm stronger than (4.1). Using this norm we establish a new Λ-coercivity result.

LEMMA 4.4 (strong Λ-coercivity, OSS). *The bilinear form* (3.9) *satisfies*

$$\begin{aligned}\|\!|\, [\xi_h, \boldsymbol{v}_h] \,\|\!|^2_{S,h} \lesssim \int_0^T &\mathcal{B}_s\left([\xi_h, \boldsymbol{v}_h], \Lambda_1([\xi_h, \boldsymbol{v}_h])\right) \mathrm{d}t \\ &+ \int_0^T \mathcal{B}_s\left([\partial_t\xi_h, \partial_t\boldsymbol{v}_h], \Lambda_2([\xi_h, \boldsymbol{v}_h])\right) \mathrm{d}t \\ &+ \mathcal{B}_s\left([\xi_h, \boldsymbol{v}_h], \Lambda_2([\xi_h, \boldsymbol{v}_h])\right)\big|_{t=0},\end{aligned} \tag{4.11}$$

$\forall [\xi_h, \boldsymbol{v}_h]$, *where*

$$\begin{aligned}\|\!|\, [\xi_h, \boldsymbol{v}_h] \,\|\!|^2_{S,h} :=& \mu_\eta \|\xi_h\|^2_{L^\infty(0,T;L^2(\Omega))} + \mu_u \|\boldsymbol{v}_h\|^2_{L^\infty(0,T;L^2(\Omega))} \\ &+ \tau_\eta \|\nabla\cdot\boldsymbol{v}_h\|^2_{L^2(0,T;L^2(\Omega))} + \tau_u \|\nabla\xi_h\|^2_{L^2(0,T;L^2(\Omega))}\end{aligned} \tag{4.12}$$

and

$$\Lambda_1([\xi_h, \boldsymbol{v}_h]) = [\xi_h + \beta_1\tau_\eta P_{\eta,h}(\nabla\cdot\boldsymbol{v}_h), \boldsymbol{v}_h + \beta_1\tau_u P_{u,h}(\nabla\xi_h)], \tag{4.13}$$

$$\Lambda_2([\xi_h, \boldsymbol{v}_h]) = \beta_2 [\partial_t\xi_h, \partial_t\boldsymbol{v}_h], \tag{4.14}$$

with $\beta_1 > 0$ *small enough,*

$$\beta_2 = \mu_\eta\gamma_\eta + \mu_u\gamma_u, \quad \gamma_\eta = \frac{\alpha}{2}T\left(\tau_\eta + \tau_u\frac{\mu_u}{\mu_\eta}\right), \quad \gamma_u = \frac{\alpha}{2}T\left(\tau_u + \tau_\eta\frac{\mu_\eta}{\mu_u}\right), \tag{4.15}$$

and $\alpha > 0$ *large enough.*

Proof. Let us test (3.9) with (4.13):

$$\begin{aligned}&\mathcal{B}_s\left([\xi_h, \boldsymbol{v}_h], \Lambda_1([\xi_h, \boldsymbol{v}_h])\right) \\ &= (\mu_\eta\partial_t\xi_h + \nabla\cdot\boldsymbol{v}_h, \xi_h + \beta_1\tau_\eta P_{\eta,h}(\nabla\cdot\boldsymbol{v}_h)) + (\mu_u\partial_t\boldsymbol{v}_h + \nabla\xi_h, \boldsymbol{v}_h + \beta_1\tau_u P_{u,h}(\nabla\xi_h)) \\ &\quad + \left(P^\perp_{\eta,h}(\nabla\cdot\boldsymbol{v}_h), \tau_\eta\nabla\cdot(\boldsymbol{v}_h + \beta_1\tau_u P_{u,h}(\nabla\xi_h))\right) \\ &\quad + \left(P^\perp_{u,h}(\nabla\xi_h), \tau_u\nabla(\xi_h + \beta_1\tau_\eta P_{\eta,h}(\nabla\cdot\boldsymbol{v}_h))\right).\end{aligned} \tag{4.16}$$

Using the divergence theorem, the fact that $\xi_h(\boldsymbol{n}\cdot\boldsymbol{v}_h) = 0$ on the boundary, the Cauchy–Schwarz inequality, Young's inequality, and integrating in time from 0 to T, we readily get

$$\begin{aligned}\int_0^T \mathcal{B}_s\left([\xi_h, \boldsymbol{v}_h], \Lambda_1([\xi_h, \boldsymbol{v}_h])\right) \mathrm{d}t \geq& \frac{1}{2}\mu_\eta\|\xi_h\|^2_{L^\infty(L^2)} + \frac{1}{2}\mu_u\|\boldsymbol{v}_h\|^2_{L^\infty(L^2)} \\ &+ \beta_3\tau_\eta\|\nabla\cdot\boldsymbol{v}_h\|^2_{L^2(L^2)} - \frac{\alpha_1}{2}\tau_\eta\|\mu_\eta\partial_t\xi_h\|^2_{L^2(L^2)} \\ &+ \beta_3\tau_u\|\nabla\xi_h\|^2_{L^2(L^2)} - \frac{\alpha_1}{2}\tau_u\|\mu_u\partial_t\boldsymbol{v}_h\|^2_{L^2(L^2)},\end{aligned} \tag{4.17}$$

with

$$\beta_3 = \min\left\{\beta_1\left(1 - \frac{\beta_1}{2\alpha_1} - \frac{\beta_1 C_\tau^2 C_{\mathrm{inv}}^2}{2}\right), \frac{1}{2}\right\} \tag{4.18}$$

and β_1 small enough, so that β_3 is positive.

Now, let us take Λ_2 as defined in (4.14). Using the fact that $\partial_t \xi_h \partial_t (\boldsymbol{n} \cdot \boldsymbol{v}_h) = 0$ on the boundary and integrating in time, we get

$$\int_0^T \mathcal{B}_s \left([\partial_t \xi_h, \partial_t \boldsymbol{v}_h] , \Lambda_2 \left([\xi_h, \boldsymbol{v}_h] \right) \right) \, \mathrm{d}t \geq \gamma_\eta \left(\|\mu_\eta \partial_t \xi_h\|^2_{L^\infty(L^2)} - \|\mu_\eta \partial_t \xi_h(0)\|^2 \right) + \gamma_u \left(\|\mu_u \partial_t \boldsymbol{v}_h\|^2_{L^\infty(L^2)} - \|\mu_u \partial_t \boldsymbol{v}_h(0)\|^2 \right) . \tag{4.19}$$

Additionally, we take α large enough so that γ_η and γ_u are large enough and the combination of (4.17) and (4.19) results in a positive factor multiplying both $\|\mu_\eta \partial_t \xi_h\|_{L^\infty(L^2)}$ and $\|\mu_u \partial_t \boldsymbol{u}_h\|_{L^\infty(L^2)}$.

Now, let us test (3.9) with (4.14) and evaluate it at $t = 0$:

$$\begin{aligned} &\mathcal{B}_s \left([\xi_h, \boldsymbol{v}_h] , \Lambda_2 \left([\xi_h, \boldsymbol{v}_h] \right) \right) \big|_{t=0} \\ &\quad = \left(\mu_\eta \partial_t \xi_h(0) + \nabla \cdot \boldsymbol{v}_h(0), \beta_2 \partial_t \xi_h(0) \right) + \left(\mu_u \partial_t \boldsymbol{v}_h(0) + \nabla \xi_h(0), \beta_2 \partial_t \boldsymbol{v}_h(0) \right) \\ &\quad + \left(P^\perp_{\eta,h}(\nabla \cdot \boldsymbol{v}_h(0)), \tau_\eta \nabla \cdot (\beta_2 \partial_t \boldsymbol{v}_h(0)) \right) + \left(P^\perp_{u,h}(\nabla \xi_h(0)), \tau_u \nabla (\beta_2 \partial_t \xi_h(0)) \right) . \end{aligned} \tag{4.20}$$

Noticing that $\nabla \xi_h(0) = \mathbf{0}$ and $\nabla \cdot \boldsymbol{v}_h(0) = 0$, we get

$$\mathcal{B}_s \left([\xi_h, \boldsymbol{v}_h] , \Lambda_2 \left([\xi_h, \boldsymbol{v}_h] \right) \right) \big|_{t=0} \geq 2\gamma_\eta \|\mu_\eta \partial_t \xi_h(0)\|^2 + 2\gamma_u \|\mu_u \partial_t \boldsymbol{v}_h(0)\|^2. \tag{4.21}$$

Combining (4.17), (4.19), and (4.21), the proof is complete. ☐

4.2. Stability. Here, using the previous Λ-coercivity lemmata, we state and prove stability for the ASGS and the OSS methods; i.e., we prove that the solution is bounded by the initial conditions and forcing terms. The results obtained apply to any of the variational forms defined in (2.8)–(2.19).

4.2.1. ASGS method. In this section we define the external forces norm and prove stability for the ASGS method.

THEOREM 4.5 (ASGS stability). *The solution* $[\eta_h, \boldsymbol{u}_h]$ *of* (3.5) *obtained with the ASGS method* (3.7)–(3.8) *satisfies*

$$\| [\eta_h, \boldsymbol{u}_h] \|^2_{W,h} \lesssim \| [f_\eta, \boldsymbol{f}_u] \|^2_{W,h}, \tag{4.22}$$

with the norm $\|\cdot\|_{W,h}$ *defined in* (4.1) *and with*

$$\begin{aligned} \| [f_\eta, \boldsymbol{f}_u] \|^2_{W,h} := &\frac{1}{\mu_\eta} \|f_\eta\|^2_{L^1(0,T;L^2(\Omega))} + \frac{1}{\mu_u} \|\boldsymbol{f}_u\|^2_{L^1(0,T;L^2(\Omega))} \\ &+ \tau_\eta \|f_\eta\|^2_{L^2(0,T;L^2(\Omega))} + \tau_u \|\boldsymbol{f}_u\|^2_{L^2(0,T;L^2(\Omega))} \\ &+ \tau_\eta \tau_u \mu_u \|\partial_t f_\eta\|^2_{L^1(0,T;L^2(\Omega))} + \tau_\eta \tau_u \mu_\eta \|\partial_t \boldsymbol{f}_u\|^2_{L^1(0,T;L^2(\Omega))} \\ &+ \tau_\eta \tau_u \mu_u \|f_\eta\|^2_{L^\infty(0,T;L^2(\Omega))} + \tau_\eta \tau_u \mu_\eta \|\boldsymbol{f}_u\|^2_{L^\infty(0,T;L^2(\Omega))}. \end{aligned} \tag{4.23}$$

Proof. Using weak Λ-coercivity for the ASGS method (4.2) and the definition of the operator Λ in (4.3), we arrive at

$$\begin{aligned} \| [\eta_h, \boldsymbol{u}_h] \|^2_{W,h} \lesssim \int_0^T \Big[&\left(f_\eta, \eta_h + \tau_\eta \mu_\eta \partial_t \eta_h \right) + \tau_\eta \left(f_\eta, \nabla \cdot \left(\boldsymbol{u}_h + \tau_u \mu_u \partial_t \boldsymbol{u}_h \right) \right) \\ &+ \left(\boldsymbol{f}_u, \boldsymbol{u}_h + \tau_u \mu_u \partial_t \boldsymbol{u}_h \right) + \tau_u \left(\boldsymbol{f}_u, \nabla \left(\eta_h + \tau_\eta \mu_\eta \partial_t \eta_h \right) \right) \Big] \, \mathrm{d}t. \end{aligned} \tag{4.24}$$

Combining some terms and using the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\| [\eta_h, \boldsymbol{u}_h] \|_{W,h}^2 \lesssim \int_0^T \Big[&\|f_\eta\| \|\eta_h\| + \tau_\eta \|f_\eta\| \|\mu_\eta \partial_t \eta_h + \nabla \cdot \boldsymbol{u}_h\| \\ &+ (f_\eta, \tau_\eta \tau_u \mu_u \nabla \cdot \partial_t \boldsymbol{u}_h) + (\boldsymbol{f}_u, \tau_\eta \tau_u \mu_\eta \nabla \partial_t \eta_h) \\ &+ \|\boldsymbol{f}_u\| \|\boldsymbol{u}_h\| + \tau_u \|\boldsymbol{f}_u\| \|\mu_u \partial_t \boldsymbol{u}_h + \nabla \eta_h\| \Big] \,\mathrm{d}t. \end{aligned} \tag{4.25}$$

Most of the terms of (4.25) are easy to bound; the only ones that require special treatment are the ones containing $\nabla \cdot \partial_t \boldsymbol{u}_h$ and $\nabla \partial_t \eta_h$. Those terms can be bounded as

$$\begin{aligned} &(f_\eta, \tau_\eta \tau_u \mu_u \nabla \cdot \partial_t \boldsymbol{u}_h) + (\boldsymbol{f}_u, \tau_\eta \tau_u \mu_\eta \nabla \partial_t \eta_h) \\ &\quad = - \int_0^T (\partial_t f_\eta, \tau_\eta \tau_u \mu_u \nabla \cdot \boldsymbol{u}_h) \,\mathrm{d}t + (f_\eta(T), \tau_\eta \tau_u \mu_u \nabla \cdot \boldsymbol{u}_h(T)) \\ &\qquad - \int_0^T (\partial_t \boldsymbol{f}_u, \tau_\eta \tau_u \mu_\eta \nabla \eta_h) \,\mathrm{d}t + (\boldsymbol{f}_u(T), \tau_\eta \tau_u \mu_\eta \nabla \eta_h(T)) \\ &\quad \le \frac{\alpha_1}{2} \tau_\eta \tau_u \mu_u \|\partial_t f_\eta\|_{L^1(L^2)}^2 + \frac{1}{2\alpha_1} \tau_\eta \tau_u \mu_u \|\nabla \cdot \boldsymbol{u}_h\|_{L^\infty(L^2)}^2 \\ &\qquad + \frac{\alpha_2}{2} \tau_\eta \tau_u \mu_u \|f_\eta\|_{L^\infty(L^2)}^2 + \frac{1}{2\alpha_2} \tau_\eta \tau_u \mu_u \|\nabla \cdot \boldsymbol{u}_h\|_{L^\infty(L^2)}^2 \\ &\qquad + \frac{\alpha_3}{2} \tau_\eta \tau_u \mu_\eta \|\partial_t \boldsymbol{f}_u\|_{L^1(L^2)}^2 + \frac{1}{2\alpha_3} \tau_\eta \tau_u \mu_\eta \|\nabla \eta_h\|_{L^\infty(L^2)}^2 \\ &\qquad + \frac{\alpha_4}{2} \tau_\eta \tau_u \mu_\eta \|\boldsymbol{f}_u\|_{L^\infty(L^2)}^2 + \frac{1}{2\alpha_4} \tau_\eta \tau_u \mu_\eta \|\nabla \eta_h\|_{L^\infty(L^2)}^2. \end{aligned} \tag{4.26}$$

Finally, taking α_i sufficiently large in (4.26) and replacing it in (4.25), it is easy to arrive at (4.22), which is what we wanted to prove. ∎

4.2.2. OSS method. In this section we define the external forces norm and prove stability for the OSS method.

THEOREM 4.6 (weak OSS stability). *The solution* $[\eta_h, \boldsymbol{u}_h]$ *of* (3.5) *obtained with the OSS method* (3.9)–(3.10) *satisfies*

$$\| [\eta_h, \boldsymbol{u}_h] \|_{W,h}^2 \lesssim \| [f_\eta, \boldsymbol{f}_u] \|_{W,h}^2, \tag{4.27}$$

with the norm $\| \cdot \|_{W,h}$ *defined in* (4.1) *and with*

$$\begin{aligned} \| [f_\eta, \boldsymbol{f}_u] \|_{W,h}^2 := &\frac{1}{\mu_\eta} \|f_\eta\|_{L^1(0,T;L^2(\Omega))}^2 + \frac{1}{\mu_u} \|\boldsymbol{f}_u\|_{L^1(0,T;L^2(\Omega))}^2 \\ &+ \tau_\eta \|f_\eta\|_{L^2(0,T;L^2(\Omega))}^2 + \tau_u \|\boldsymbol{f}_u\|_{L^2(0,T;L^2(\Omega))}^2. \end{aligned} \tag{4.28}$$

Proof. Using the weak Λ-coercivity for the OSS method (4.6) with Λ as in (4.7) we arrive at

$$\begin{aligned} \| [\eta_h, \boldsymbol{u}_h] \|_{W,h}^2 \lesssim \int_0^T \Big[&(f_\eta, \eta_h + \beta \tau_\eta (\mu_\eta \partial_t \eta_h + P_{\eta,h}(\nabla \cdot \boldsymbol{u}_h))) \\ &+ (\boldsymbol{f}_u, \boldsymbol{u}_h + \beta \tau_u (\mu_u \partial_t \boldsymbol{u}_h + P_{u,h}(\nabla \eta_h))) \\ &+ \left(P_{\eta,h}^\perp(f_\eta), \tau_\eta \nabla \cdot (\boldsymbol{u}_h + \beta \tau_u (\mu_u \partial_t \boldsymbol{u}_h + P_{u,h}(\nabla \eta_h)))\right) \end{aligned}$$

$$
\begin{aligned}
&+ \left(P_{u,h}^{\perp}(\boldsymbol{f}_u), \tau_u \nabla \left(\eta_h + \beta\tau_\eta \left(\mu_\eta \partial_t \eta_h + P_{\eta,h}(\nabla\cdot \boldsymbol{u}_h)\right)\right)\right) \Big] \mathrm{d}t \\
\lesssim &\frac{\alpha_1}{2}\frac{1}{\mu_\eta}\|f_\eta\|^2_{L^1(L^2)} + \frac{1}{2\alpha_1}\mu_\eta\|\eta_h\|^2_{L^\infty(L^2)} \\
&+ \frac{\alpha_2}{2}\tau_\eta\|f_\eta\|^2_{L^2(L^2)} + \frac{\beta^2}{2\alpha_2}\tau_\eta\|\mu_\eta\partial_t\eta_h + \nabla\cdot\boldsymbol{u}_h\|^2_{L^2(L^2)} \\
&+ \frac{\alpha_3}{2}\frac{1}{\mu_u}\|\boldsymbol{f}_u\|^2_{L^1(L^2)} + \frac{1}{2\alpha_3}\mu_u\|\boldsymbol{u}_h\|^2_{L^\infty(L^2)} \\
&+ \frac{\alpha_4}{2}\tau_u\|\boldsymbol{f}_u\|^2_{L^2(L^2)} + \frac{\beta^2}{2\alpha_4}\tau_u\|\mu_u\partial_t\boldsymbol{u}_h + \nabla\eta_h\|^2_{L^2(L^2)} \\
&+ \frac{\alpha_5}{2}\tau_\eta\|P_{\eta,h}^{\perp}(f_\eta)\|^2_{L^2(L^2)} + \frac{\beta^2}{2\alpha_5}\tau_\eta\|P_{\eta,h}^{\perp}(\nabla\cdot\boldsymbol{u}_h)\|^2_{L^2(L^2)} \\
&+ \frac{\alpha_6}{2}\tau_\eta\|P_{\eta,h}^{\perp}(f_\eta)\|^2_{L^2(L^2)} + \frac{\beta^2 C_\tau^2 C_{\mathrm{inv}}^2}{2\alpha_6}\tau_u\|\mu_u\partial_t\boldsymbol{u}_h + P_{u,h}(\nabla\eta_h)\|^2_{L^2(L^2)} \\
&+ \frac{\alpha_7}{2}\tau_u\|P_{u,h}^{\perp}(\boldsymbol{f}_u)\|^2_{L^2(L^2)} + \frac{\beta^2}{2\alpha_7}\tau_u\|P_{u,h}^{\perp}(\nabla\eta_h)\|^2_{L^2(L^2)} \\
&+ \frac{\alpha_8}{2}\tau_u\|P_{u,h}^{\perp}(\boldsymbol{f}_u)\|^2_{L^2(L^2)} + \frac{\beta^2 C_\tau^2 C_{\mathrm{inv}}^2}{2\alpha_8}\tau_\eta\|\mu_\eta\partial_t\eta_h + P_{\eta,h}(\nabla\cdot\boldsymbol{u}_h)\|^2_{L^2(L^2)}.
\end{aligned}
$$

We complete the proof choosing α_i large enough. ∎

THEOREM 4.7 (strong OSS stability). *The solution* $[\eta_h, \boldsymbol{u}_h]$ *of* (3.5) *obtained with the OSS method* (3.9)–(3.10) *satisfies*

$$
\| [\eta_h, \boldsymbol{u}_h] \|^2_{S,h} \lesssim \| [f_\eta, \boldsymbol{f}_u] \|^2_{S,h}, \tag{4.29}
$$

with the norm $\|\cdot\|_{S,h}$ *defined in* (4.12) *and with*

$$
\begin{aligned}
\| [f_\eta, \boldsymbol{f}_u] \|^2_S := &\frac{1}{\mu_\eta}\|f_\eta\|^2_{L^1(0,T;L^2(\Omega))} + \frac{1}{\mu_u}\|\boldsymbol{f}_u\|^2_{L^1(0,T;L^2(\Omega))} \\
&+ \tau_\eta\|f_\eta\|^2_{L^2(0,T;L^2(\Omega))} + \tau_u\|\boldsymbol{f}_u\|^2_{L^2(0,T;L^2(\Omega))} \\
&+ \gamma_\eta\left(\|\partial_t f_\eta\|^2_{L^1(0,T;L^2(\Omega))} + \|f_\eta(0)\|^2\right) \\
&+ \gamma_u\left(\|\partial_t \boldsymbol{f}_u\|^2_{L^1(0,T;L^2(\Omega))} + \|\boldsymbol{f}_u(0)\|^2\right) \\
&+ \frac{\gamma_u}{\mu_u}\tau_\eta\|\partial_t f_\eta\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\gamma_\eta}{\mu_\eta}\tau_u\|\partial_t \boldsymbol{f}_u\|^2_{L^2(0,T;L^2(\Omega))} \\
&+ \gamma_u\tau_\eta\|f_\eta(0)\| + \gamma_\eta\tau_u\|\boldsymbol{f}_u(0)\|,
\end{aligned} \tag{4.30}
$$

and the parameters γ_η *and* γ_u *are given in* (4.15).

Proof. Using the strong Λ-coercivity for the OSS method (4.11) and the definitions of Λ_i from (4.13)–(4.14), we arrive at

$$
\begin{aligned}
\| [\eta_h, \boldsymbol{u}_h] \|^2_{S,h} \lesssim &\int_0^T \mathcal{L}_s\left(\Lambda_1\left([\eta_h, \boldsymbol{u}_h]\right)\right)\mathrm{d}t + \int_0^T \mathcal{L}_t\left(\Lambda_2\left([\eta_h, \boldsymbol{u}_h]\right)\right)\mathrm{d}t \\
&+ \mathcal{L}_s\left(\Lambda_2\left([\eta_h, \boldsymbol{u}_h]\right)\right)\Big|_{t=0},
\end{aligned} \tag{4.31}
$$

where $\mathcal{L}_t$ is $\mathcal{L}_s$ with the time derivative of the forces. The first term of (4.31) can be written and bounded as follows:

$$
\int_0^T \mathcal{L}_s\left(\Lambda_1\left([\eta_h, \boldsymbol{u}_h]\right)\right)\mathrm{d}t
$$

$$
\begin{aligned}
&= \int_0^T \Big[(f_\eta, \eta_h + \beta_1 \tau_\eta P_{\eta,h}(\nabla\cdot \boldsymbol{u}_h)) + (\boldsymbol{f}_u, \boldsymbol{u}_h + \beta_1 \tau_u P_{u,h}(\nabla \eta_h)) \Big] \mathrm{d}t \\
&\quad + \int_0^T \left(P_{\eta,h}^{\perp}(f_\eta), \tau_\eta \nabla\cdot (\boldsymbol{u}_h + \beta_1 \tau_u P_{u,h}(\nabla\eta_h))\right) \mathrm{d}t \\
&\quad + \int_0^T \left(P_{u,h}^{\perp}(\boldsymbol{f}_u), \tau_u \nabla \left(\eta_h + \beta_1 \tau_\eta P_{\eta,h}(\nabla\cdot \boldsymbol{u}_h)\right)\right) \mathrm{d}t \\
&\le \frac{\alpha_1}{2}\frac{1}{\mu_\eta}\|f_\eta\|^2_{L^1(L^2)} + \left(\frac{\beta_1^2}{2\alpha_2} + \frac{1}{2\alpha_5} + \frac{\beta_1^2 C_\tau^2 C_{\mathrm{inv}}^2}{2\alpha_8}\right)\tau_\eta \|\nabla\cdot \boldsymbol{u}_h\|^2_{L^2(L^2)} \\
&\quad + \frac{\alpha_2+\alpha_5+\alpha_6}{2}\tau_\eta\|f_\eta\|^2_{L^2(L^2)} + \frac{1}{2\alpha_1}\mu_\eta\|\eta_h\|^2_{L^\infty(L^2)} \\
&\quad + \frac{\alpha_3}{2}\frac{1}{\mu_u}\|\boldsymbol{f}_u\|^2_{L^1(L^2)} + \left(\frac{\beta_1^2}{2\alpha_4} + \frac{1}{2\alpha_7} + \frac{\beta_1^2 C_\tau^2 C_{\mathrm{inv}}^2}{2\alpha_6}\right)\tau_u \|\nabla \eta_h\|^2_{L^2(L^2)} \\
&\quad + \frac{\alpha_4+\alpha_7+\alpha_8}{2}\tau_u\|\boldsymbol{f}_u\|^2_{L^2(L^2)} + \frac{1}{2\alpha_3}\mu_u\|\boldsymbol{u}_h\|^2_{L^\infty(L^2)}. \qquad (4.32)
\end{aligned}
$$

The second term of (4.31) can be written and bounded as follows:

$$
\begin{aligned}
&\int_0^T \mathcal{L}_t\left(\Lambda_2\left([\eta_h, \boldsymbol{u}_h]\right)\right)\mathrm{d}t \\
&\quad = \int_0^T \Big[(\partial_t f_\eta, \beta_2 \partial_t \eta_h) + \left(P_{\eta,h}^{\perp}(\partial_t f_\eta), \tau_\eta \nabla\cdot (\beta_2 \partial_t \boldsymbol{u}_h)\right) \Big] \mathrm{d}t \\
&\quad\quad + \int_0^T \Big[(\partial_t \boldsymbol{f}_u, \beta_2 \partial_t \boldsymbol{u}_h) + \left(P_{u,h}^{\perp}(\partial_t \boldsymbol{f}_u), \tau_u \nabla\left(\beta_2 \partial_t \eta_h\right)\right) \Big] \mathrm{d}t \\
&\quad \le \alpha_9 \gamma_\eta \|\partial_t f_\eta\|^2_{L^1(L^2)} + \frac{\gamma_\eta}{\alpha_9}\|\mu_\eta \partial_t \eta_h\|^2_{L^\infty(L^2)} \\
&\quad\quad + \frac{\alpha_{10}\gamma_u}{\mu_u}\tau_\eta\|\partial_t f_\eta\|^2_{L^2(L^2)} + \frac{\gamma_u}{\alpha_{10}\mu_u}\tau_\eta\|P_{\eta,h}^{\perp}(\nabla\cdot \mu_u \partial_t \boldsymbol{u}_h)\|^2_{L^2(L^2)} \\
&\quad\quad + \alpha_{11}\gamma_u\|\partial_t \boldsymbol{f}_u\|^2_{L^1(L^2)} + \frac{\gamma_u}{\alpha_{11}}\|\mu_u \partial_t \boldsymbol{u}_h\|^2_{L^\infty(L^2)} \\
&\quad\quad + \frac{\alpha_{12}\gamma_\eta}{\mu_\eta}\tau_u\|\partial_t \boldsymbol{f}_u\|^2_{L^2(L^2)} + \frac{\gamma_\eta}{\alpha_{12}\mu_\eta}\tau_u\|P_{u,h}^{\perp}(\nabla \mu_\eta \partial_t \eta_h)\|^2_{L^2(L^2)}. \qquad (4.33)
\end{aligned}
$$

The third term of (4.31) can be written and bounded as follows:

$$
\begin{aligned}
&\mathcal{L}_s\left(\Lambda_2\left([\eta_h, \boldsymbol{u}_h]\right)\right)\Big|_{t=0} = (f_\eta(0), \beta_2 \partial_t \eta_h(0)) + (\boldsymbol{f}_u(0), \beta_2 \partial_t \boldsymbol{u}_h(0)) \\
&\quad + \left(P_{\eta,h}^{\perp}(f_\eta(0)), \tau_\eta \nabla\cdot \beta_2 \partial_t \boldsymbol{u}_h(0)\right) + \left(P_{u,h}^{\perp}(\boldsymbol{f}_u(0)), \tau_u \nabla \beta_2 \partial_t \eta_h(0)\right) \\
&\le \alpha_{13}\gamma_\eta\|f_\eta(0)\|^2 + \frac{\gamma_\eta}{\alpha_{13}}\|\mu_\eta \partial_t \eta_h(0)\|^2 + \alpha_{15}\gamma_u\tau_\eta\|f_\eta(0)\|^2 \\
&\quad + \alpha_{14}\gamma_u\|\boldsymbol{f}_u(0)\|^2 + \frac{\gamma_u}{\alpha_{14}}\|\mu_u \partial_t \boldsymbol{u}_h(0)\|^2 + \alpha_{16}\gamma_\eta\tau_u\|\boldsymbol{f}_u(0)\|^2 \\
&\quad + \frac{\gamma_u}{\alpha_{15}}\tau_\eta\|P_{\eta,h}^{\perp}(\nabla\cdot \mu_u \partial_t \boldsymbol{u}_h)(0)\|^2 + \frac{\gamma_\eta}{\alpha_{16}}\tau_u\|P_{u,h}^{\perp}(\nabla \mu_\eta \partial_t \eta_h)(0)\|^2. \qquad (4.34)
\end{aligned}
$$

Finally, combining (4.32), (4.33), and (4.34) and taking α_i large enough, the proof is complete. ∎

Remark 4.1. From Theorems 4.5, 4.6, and 4.7, two conclusions can be drawn. First, in the case of the weaker norm (4.1), both the ASGS and OSS methods are stable, but the latter requires less regularity on the forcing terms than the former.

And, second, the OSS method allows one to obtain convergence in the stronger norm (4.12) (at the expense of more regularity on the forcing terms), and thus to control all the gradient of η_h and all the divergence of $\boldsymbol{u}_h$, not in combination with temporal derivatives.

5. Convergence analysis. In this section we state and prove convergence of the stabilized FE methods proposed: ASGS and OSS. The results obtained apply to any of the variational forms defined in (2.8)–(2.19). We only prove convergence for the variational form I because variational forms II and III only differ in two of the Galerkin terms $(\nabla\eta_h, \boldsymbol{v}_h)$ and $(\nabla\cdot\boldsymbol{u}_h, \xi_h)$ with respect to the variational form I.

Let us define η_I as the $P_{\eta,h}$ projection of the exact solution η on $V_{\eta,h}$, and $\boldsymbol{u}_I$ as the $P_{u,h}$ projection of the exact solution $\boldsymbol{u}$ on $V_{u,h}$.

LEMMA 5.1 (optimality of $P_{\eta,h}$ and $P_{u,h}$). *Let $P_{\eta,h} : V_\eta \longrightarrow V_{\eta,h}$ and $P_{u,h} : V_u \longrightarrow V_{u,h}$ be two projections defined as*

$$(P_{\eta,h}(\xi), \chi_h) = (\xi, \chi_h) \quad \forall \chi_h \in V_{\eta,h}, \tag{5.1}$$

$$P_{\eta,h}(\xi) = 0 \quad \text{on } \Gamma_\eta, \tag{5.2}$$

$$(P_{u,h}(\boldsymbol{v}), \boldsymbol{w}_h) = (\boldsymbol{v}, \boldsymbol{w}_h) \quad \forall \boldsymbol{w}_h \in V_{u,h}, \tag{5.3}$$

$$\boldsymbol{n}\cdot P_{u,h}(\boldsymbol{v}) = 0 \quad \text{on } \Gamma_u. \tag{5.4}$$

Let k and l be the polynomial interpolation order for $V_{\eta,h}$ and $V_{u,h}$, respectively. Then, $P_{\eta,h}$ and $P_{u,h}$ are optimal in $L^2(\Omega)$ and $H^1(\Omega)$. That is,

$$\|\xi - P_{\eta,h}(\xi)\|_0 \lesssim h^{k+1}|\xi|_{k+1}, \qquad \|\xi - P_{\eta,h}(\xi)\|_1 \lesssim h^{k}|\xi|_{k+1}, \tag{5.5}$$

$$\|\boldsymbol{v} - P_{u,h}(\boldsymbol{v})\|_0 \lesssim h^{l+1}|\boldsymbol{v}|_{l+1}, \qquad \|\boldsymbol{v} - P_{u,h}(\boldsymbol{v})\|_1 \lesssim h^{l}|\boldsymbol{v}|_{l+1} \tag{5.6}$$

for smooth enough $\xi \in V_\eta$ and $\boldsymbol{v} \in V_u$.

Proof. Let $Z_{\eta,h}$ and $Z_{u,h}$ be the Scott–Zhang projection operators that satisfy the boundary conditions of V_η and V_u, respectively, [27]. By the definition of $P_{\eta,h}$ in (5.1) we have

$$(\xi - P_{\eta,h}(\xi), \xi - P_{\eta,h}(\xi)) = (\xi - P_{\eta,h}(\xi), \xi - Z_{\eta,h}(\xi)), \tag{5.7}$$

$$\|\xi - P_{\eta,h}(\xi)\| \le \|\xi - Z_{\eta,h}(\xi)\| \lesssim h^{k+1}|\xi|_{k+1}. \tag{5.8}$$

Now, let us consider the approximation error in the H^1-norm:

$$\|\xi - P_{\eta,h}(\xi)\|_1 \le \|\xi - Z_{\eta,h}(\xi)\|_1 + \|Z_{\eta,h}(\xi) - P_{\eta,h}(\xi)\|_1 \tag{5.9}$$

$$\le \|\xi - Z_{\eta,h}(\xi)\|_1 + C_{\text{inv}} h^{-1}\|Z_{\eta,h}(\xi) - P_{\eta,h}(\xi)\|_0 \tag{5.10}$$

$$\le \|\xi - Z_{\eta,h}(\xi)\|_1 + C_{\text{inv}} h^{-1}\left(\|Z_{\eta,h}(\xi) - \xi\|_0 + \|\xi - P_{\eta,h}(\xi)\|_0\right) \tag{5.11}$$

$$\lesssim h^{k}|\xi|_{k+1}, \tag{5.12}$$

where we have used the inverse inequality. The proof for $P_{u,h}$ is similar to the proof for $P_{\eta,h}$, and we omit it. ∎

Let us define two types of error. The error of the approximate solution (obtained using ASGS or OSS) with respect to the projected exact solution is defined as

$$e_\eta := \eta_h - \eta_I, \quad \boldsymbol{e}_u := \boldsymbol{u}_h - \boldsymbol{u}_I, \tag{5.13}$$

and the error of the exact solution with respect to the projected exact solution is defined as

$$\varepsilon_\eta := \eta - \eta_I, \quad \boldsymbol{\varepsilon}_u := \boldsymbol{u} - \boldsymbol{u}_I. \tag{5.14}$$

Notice that $[e_\eta, \boldsymbol{e}_u]$ belongs to the FE space and $[\varepsilon_\eta, \boldsymbol{\varepsilon}_u]$ is orthogonal to the FE space with respect to the $L^2(\Omega)$ inner product. We shall make frequent use of this orthogonality property.

Additionally, let us define the projection error in the $H^i(\Omega)$-norm:

$$\varepsilon_i(\eta) = \|\varepsilon_\eta\|_{H^i(\Omega)}, \quad \varepsilon_i(\boldsymbol{u}) = \|\boldsymbol{\varepsilon}_u\|_{H^i(\Omega)}, \quad i = 0, 1. \tag{5.15}$$

5.1. ASGS method. Let us consider problem (3.5) with $\mathcal{B}_s$ and $\mathcal{L}_s$ defined in (3.7) and (3.8), respectively. The approximate solution to that problem converges as stated in the following result.

THEOREM 5.2 (ASGS convergence). *Let $[\eta, \boldsymbol{u}]$ be the solution of the continuous problem* (2.7), *and let $[\eta_h, \boldsymbol{u}_h]$ be the solution of the stabilized discrete problem* (3.5) *using the ASGS method. Then*

$$\| [\eta - \eta_h, \boldsymbol{u} - \boldsymbol{u}_h] \|_{W,h} \lesssim E_W(h), \tag{5.16}$$

with the norm $\| \cdot \|_{W,h}$ defined in (4.1) *and*

$$\begin{aligned} E_W^2(h) := {} & \mu_\eta \|\varepsilon_\eta\|^2_{L^\infty(0,T;L^2(\Omega))} + \mu_u \|\boldsymbol{\varepsilon}_u\|^2_{L^\infty(0,T;L^2(\Omega))} \\ & + \tau_\eta \|\mu_\eta \partial_t \varepsilon_\eta\|^2_{L^2(0,T;L^2(\Omega))} + \tau_u \|\mu_u \partial_t \boldsymbol{\varepsilon}_u\|^2_{L^2(0,T;L^2(\Omega))} \\ & + \mu_u \tau_\eta \tau_u \|\mu_\eta \partial_t \varepsilon_\eta\|^2_{L^\infty(0,T;L^2(\Omega))} + \mu_\eta \tau_\eta \tau_u \|\mu_u \partial_t \boldsymbol{\varepsilon}_u\|^2_{L^\infty(0,T;L^2(\Omega))} \\ & + \mu_u \tau_\eta \tau_u \|\mu_\eta \partial_{tt} \varepsilon_\eta\|^2_{L^1(0,T;L^2(\Omega))} + \mu_\eta \tau_\eta \tau_u \|\mu_u \partial_{tt} \boldsymbol{\varepsilon}_u\|^2_{L^1(0,T;L^2(\Omega))} \\ & + \frac{1}{\tau_\eta} \|\varepsilon_\eta\|^2_{L^2(0,T;L^2(\Omega))} + \frac{1}{\tau_u} \|\boldsymbol{\varepsilon}_u\|^2_{L^2(0,T;L^2(\Omega))} \\ & + \tau_u \|\nabla \varepsilon_\eta\|^2_{L^2(0,T;L^2(\Omega))} + \tau_\eta \|\nabla \cdot \boldsymbol{\varepsilon}_u\|^2_{L^2(0,T;L^2(\Omega))} \\ & + \mu_\eta \tau_\eta \tau_u \|\nabla \varepsilon_\eta\|^2_{L^\infty(0,T;L^2(\Omega))} + \mu_u \tau_\eta \tau_u \|\nabla \cdot \boldsymbol{\varepsilon}_u\|^2_{L^\infty(0,T;L^2(\Omega))} \\ & + \mu_\eta \tau_\eta \tau_u \|\nabla \partial_t \varepsilon_\eta\|^2_{L^1(0,T;L^2(\Omega))} + \mu_u \tau_\eta \tau_u \|\nabla \cdot \partial_t \boldsymbol{\varepsilon}_u\|^2_{L^1(0,T;L^2(\Omega))}. \end{aligned} \tag{5.17}$$

Proof. Since the ASGS method is consistent (in the sense that the exact solution is solution of the discrete problem) and using (4.2) with Λ given in (4.3), we have

$$\begin{aligned} \| [e_\eta, \boldsymbol{e}_u] \|^2_{W,h} & \lesssim \int_0^T \mathcal{B}_s \left([e_\eta, \boldsymbol{e}_u], \Lambda \left([e_\eta, \boldsymbol{e}_u] \right) \right) \mathrm{d}t = \int_0^T \mathcal{B}_s \left([\varepsilon_\eta, \boldsymbol{\varepsilon}_u], \Lambda \left([e_\eta, \boldsymbol{e}_u] \right) \right) \mathrm{d}t \\ & \lesssim \int_0^T \left(\mu_\eta \partial_t \varepsilon_\eta, e_\eta + \tau_\eta \left(\mu_\eta \partial_t e_\eta + \nabla \cdot \boldsymbol{e}_u \right) + \mu_u \tau_\eta \tau_u \nabla \cdot \partial_t \boldsymbol{e}_u \right) \mathrm{d}t \\ & \quad + \int_0^T \left(\mu_u \partial_t \boldsymbol{\varepsilon}_u, \boldsymbol{e}_u + \tau_u \left(\mu_u \partial_t \boldsymbol{e}_u + \nabla e_\eta \right) + \mu_\eta \tau_\eta \tau_u \nabla \partial_t e_\eta \right) \mathrm{d}t \\ & \quad + \int_0^T \left(\nabla \cdot \boldsymbol{\varepsilon}_u, e_\eta + \tau_\eta \left(\mu_\eta \partial_t e_\eta + \nabla \cdot \boldsymbol{e}_u \right) + \mu_u \tau_\eta \tau_u \nabla \cdot \partial_t \boldsymbol{e}_u \right) \mathrm{d}t \\ & \quad + \int_0^T \left(\nabla \varepsilon_\eta, \boldsymbol{e}_u + \tau_u \left(\mu_u \partial_t \boldsymbol{e}_u + \nabla e_\eta \right) + \mu_\eta \tau_\eta \tau_u \nabla \partial_t e_\eta \right) \mathrm{d}t. \end{aligned} \tag{5.18}$$

Now we can bound $\| [e_\eta, \boldsymbol{e}_u] \|_{W,h}$ in terms of the projection error $[\varepsilon_\eta, \boldsymbol{\varepsilon}_u]$. We will show parts of the right-hand side of (5.18) and how they are bounded. The first term of this expression can be bounded as

$$\int_0^T \left(\mu_\eta \partial_t \varepsilon_\eta, e_\eta + \tau_\eta \left(\mu_\eta \partial_t e_\eta + \nabla \cdot \boldsymbol{e}_u \right) + \mu_u \tau_\eta \tau_u \nabla \cdot \partial_t \boldsymbol{e}_u \right) \mathrm{d}t$$

$$\leq \frac{\alpha_1}{2}\tau_\eta\|\mu_\eta\partial_t\varepsilon_\eta\|^2_{L^2(L^2)} + \frac{1}{2\alpha_1}\tau_\eta\|\mu_\eta\partial_t e_\eta + \nabla\cdot\boldsymbol{e}_u\|^2_{L^2(L^2)}$$
$$+ \frac{\alpha_2}{2}\mu_u\tau_\eta\tau_u\|\mu_\eta\partial_t\varepsilon_\eta\|^2_{L^\infty(L^2)} + \frac{1}{2\alpha_2}\mu_u\tau_\eta\tau_u\|\nabla\cdot\boldsymbol{e}_u\|^2_{L^\infty(L^2)}$$
$$(5.19)\qquad + \frac{\alpha_3}{2}\mu_u\tau_\eta\tau_u\|\mu_\eta\partial_{tt}\varepsilon_\eta\|^2_{L^1(L^2)} + \frac{1}{2\alpha_3}\mu_u\tau_\eta\tau_u\|\nabla\cdot\boldsymbol{e}_u\|^2_{L^\infty(L^2)}.$$

The third term of (5.18) can be bounded as

$$\int_0^T (\nabla\cdot\boldsymbol{\varepsilon}_u, e_\eta + \tau_\eta(\mu_\eta\partial_t e_\eta + \nabla\cdot\boldsymbol{e}_u) + \mu_u\tau_\eta\tau_u\nabla\cdot\partial_t\boldsymbol{e}_u)\,\mathrm{d}t$$
$$\leq \frac{\alpha_4}{2}\tau_u^{-1}\|\boldsymbol{\varepsilon}_u\|^2_{L^2(L^2)} + \frac{1}{2\alpha_4}\tau_u\|\mu_u\partial_t\boldsymbol{e}_u + \nabla e_\eta\|^2_{L^2(L^2)}$$
$$+ \frac{\alpha_5}{2}\tau_\eta\|\nabla\cdot\boldsymbol{\varepsilon}_u\|^2_{L^2(L^2)} + \frac{1}{2\alpha_5}\tau_\eta\|\mu_\eta\partial_t e_\eta + \nabla\cdot\boldsymbol{e}_u\|^2_{L^2(L^2)}$$
$$+ \frac{\alpha_6}{2}\mu_u\tau_\eta\tau_u\|\nabla\cdot\boldsymbol{\varepsilon}_u\|^2_{L^\infty(L^2)} + \frac{1}{2\alpha_6}\mu_u\tau_\eta\tau_u\|\nabla\cdot\boldsymbol{e}_u\|^2_{L^\infty(L^2)}$$
$$(5.20)\qquad + \frac{\alpha_7}{2}\mu_u\tau_\eta\tau_u\|\nabla\cdot\partial_t\boldsymbol{\varepsilon}_u\|^2_{L^1(L^2)} + \frac{1}{2\alpha_7}\mu_u\tau_\eta\tau_u\|\nabla\cdot\boldsymbol{e}_u\|^2_{L^\infty(L^2)}.$$

The second and fourth terms of (5.18) can be bounded similarly as the first and third terms as shown in (5.19)–(5.20). Taking α_i big enough, it follows that $|\!|\!|\,[e_\eta,\boldsymbol{e}_u]\,|\!|\!|^2_{W,h} \lesssim E^2_W(h)$. Additionally, $|\!|\!|\,[\eta-\eta_h,\boldsymbol{u}-\boldsymbol{u}_h]\,|\!|\!|^2_{W,h} \lesssim |\!|\!|\,[e_\eta,\boldsymbol{e}_u]\,|\!|\!|^2_{W,h} + |\!|\!|\,[\varepsilon_\eta,\boldsymbol{\varepsilon}_u]\,|\!|\!|^2_{W,h}$. Furthermore, by definition $|\!|\!|\,[\varepsilon_\eta,\boldsymbol{\varepsilon}_u]\,|\!|\!|^2_{W,h} \lesssim E^2_W(h)$, which completes the proof. ◻

The inequality $|\!|\!|\,[\varepsilon_\eta,\boldsymbol{\varepsilon}_u]\,|\!|\!|^2_{W,h} \lesssim E^2_W(h)$ is precisely the way of determining $E^2_W(h)$.

5.2. OSS method. As we have defined two norms for the OSS method, one weaker than the other, we will prove convergence in both, starting with the weaker one.

THEOREM 5.3 (OSS convergence in the weak norm). *Let $[\eta,\boldsymbol{u}]$ be the solution of the continuous problem* (2.7)*, and let $[\eta_h,\boldsymbol{u}_h]$ be the solution of the stabilized discrete problem* (3.5) *using the OSS method. Then*

$$(5.21)\qquad |\!|\!|\,[\eta-\eta_h,\boldsymbol{u}-\boldsymbol{u}_h]\,|\!|\!|_{W,h} \lesssim E_W(h),$$

with the norm $|\!|\!|\cdot|\!|\!|_{W,h}$ defined in (4.1) *and*

$$E^2_W(h) := \mu_\eta\|\varepsilon_\eta\|^2_{L^\infty(0,T;L^2(\Omega))} + \mu_u\|\boldsymbol{\varepsilon}_u\|^2_{L^\infty(0,T;L^2(\Omega))}$$
$$+ \tau_\eta\|\mu_\eta\partial_t\varepsilon_\eta\|^2_{L^2(0,T;L^2(\Omega))} + \tau_u\|\mu_u\partial_t\boldsymbol{\varepsilon}_u\|^2_{L^2(0,T;L^2(\Omega))}$$
$$+ \frac{1}{\tau_\eta}\|\varepsilon_\eta\|^2_{L^2(0,T;L^2(\Omega))} + \frac{1}{\tau_u}\|\boldsymbol{\varepsilon}_u\|^2_{L^2(0,T;L^2(\Omega))}$$
$$(5.22)\qquad + \tau_u\|\nabla\varepsilon_\eta\|^2_{L^2(0,T;L^2(\Omega))} + \tau_\eta\|\nabla\cdot\boldsymbol{\varepsilon}_u\|^2_{L^2(0,T;L^2(\Omega))}.$$

Proof. Since the OSS method is consistent (in the sense that the exact solution is the solution of the discrete problem) because the OSS method uses the projections $P_{\eta,h}$ and $P_{u,h}$, we can use the weak Λ-coercivity of the OSS method (4.6) with Λ given in (4.7) and we get

$$|\!|\!|\,[e_\eta,\boldsymbol{e}_u]\,|\!|\!|^2_{W,h} \lesssim \int_0^T (\mu_\eta\partial_t\varepsilon_\eta + \nabla\cdot\boldsymbol{\varepsilon}_u, e_\eta + \beta\tau_\eta(\mu_\eta\partial_t e_\eta + P_{\eta,h}(\nabla\cdot\boldsymbol{e}_u)))\,\mathrm{d}t$$

$$+ \int_0^T (\mu_u \partial_t \boldsymbol{\varepsilon}_u + \nabla \varepsilon_\eta, \boldsymbol{e}_u + \beta \tau_u (\mu_u \partial_t \boldsymbol{e}_u + P_{u,h}(\nabla e_\eta)))\,\mathrm{d}t$$
$$+ \int_0^T \left(\tau_\eta P_{\eta,h}^\perp(\nabla \cdot \boldsymbol{\varepsilon}_u), \nabla \cdot (\boldsymbol{e}_u + \beta \tau_u (\mu_u \partial_t \boldsymbol{e}_u + P_{u,h}(\nabla e_\eta)))\right)\mathrm{d}t$$
$$(5.23)\qquad + \int_0^T \left(\tau_u P_{u,h}^\perp(\nabla \varepsilon_\eta), \nabla (e_\eta + \beta \tau_\eta (\mu_\eta \partial_t e_\eta + P_{\eta,h}(\nabla \cdot \boldsymbol{e}_u)))\right)\mathrm{d}t.$$

Now we can bound $\| [e_\eta, \boldsymbol{e}_u] \|_{W,h}$ in terms of the projection error $[\varepsilon_\eta, \boldsymbol{\varepsilon}_u]$. We will show parts of the right-hand side of (5.23) and how they are bounded. The first term of this expression can be bounded as

$$\int_0^T (\mu_\eta \partial_t \varepsilon_\eta + \nabla \cdot \boldsymbol{\varepsilon}_u, e_\eta + \beta \tau_\eta (\mu_\eta \partial_t e_\eta + P_{\eta,h}(\nabla \cdot \boldsymbol{e}_u)))\,\mathrm{d}t$$
(5.24)
$$\leq \frac{\alpha_1}{2}\tau_\eta \|\mu_\eta \partial_t \varepsilon_\eta + P_{\eta,h}(\nabla \cdot \boldsymbol{\varepsilon}_u)\|_{L^2(L^2)}^2 + \frac{\alpha_2}{2}\tau_u^{-1}\|\boldsymbol{\varepsilon}_u\|_{L^2(L^2)}^2$$
$$+ \frac{\beta^2}{2\alpha_1}\tau_\eta \|\mu_\eta \partial_t e_\eta + P_{\eta,h}(\nabla \cdot \boldsymbol{e}_u)\|_{L^2(L^2)}^2 + \frac{\tau_u}{2\alpha_2}\|\mu_u \partial_t \boldsymbol{e}_u + P_{u,h}(\nabla e_\eta)\|_{L^2(L^2)}^2.$$

The third term of (5.23) can be bounded as

$$\int_0^T \left(\tau_\eta P_{\eta,h}^\perp(\nabla \cdot \boldsymbol{\varepsilon}_u), \nabla \cdot (\boldsymbol{e}_u + \beta \tau_u (\mu_u \partial_t \boldsymbol{e}_u + P_{u,h}(\nabla e_\eta)))\right)\mathrm{d}t$$
$$\leq \frac{\alpha_3}{2}\tau_\eta \|P_{\eta,h}^\perp(\nabla \cdot \boldsymbol{\varepsilon}_u)\|_{L^2(L^2)}^2 + \frac{1}{2\alpha_3}\tau_\eta \|P_{\eta,h}^\perp(\nabla \cdot \boldsymbol{e}_u)\|_{L^2(L^2)}^2$$
$$(5.25)\qquad + \frac{\beta^2}{2\alpha_3}\tau_u C_\tau^2 C_{\mathrm{inv}}^2 \|\mu_u \partial_t \boldsymbol{e}_u + P_{u,h}(\nabla e_\eta)\|_{L^2(L^2)}^2.$$

The second and fourth terms of (5.23) can be bounded similarly as we did for the first and third terms in (5.24)–(5.25). Taking α_i big enough, it follows that $\| [e_\eta, \boldsymbol{e}_u] \|_{W,h}^2 \lesssim E_W^2(h)$. Using the same reasoning as for the ASGS method, we complete the proof for the OSS method. ∎

Let us examine each term of $E_{W,h}^2$ from (5.22) with respect to $\|\eta - \eta_h\|_{L^\infty(0,T;L^2(\Omega))}^2$ (or the equivalent norm for $\boldsymbol{u}$). The first and second terms are optimal for any of the variational forms I, II, and III (2.8)–(2.17). The third and fourth terms are at least optimal for I, II, or III. The fifth and seventh terms are quasi-optimal for I, optimal for II, and suboptimal for III. The sixth and eighth terms are quasi-optimal for I, suboptimal for II, and optimal for III. A similar analysis can be carried out for the error of $\nabla \eta_h$ and $\nabla \cdot \boldsymbol{u}_h$. Results are summarized in Table 2.

Now, let us examine the convergence of the OSS method in the strong norm (4.12).

THEOREM 5.4 (OSS convergence in the strong norm). *Let $[\eta, \boldsymbol{u}]$ be the solution of the continuous problem* (2.7)*, and let $[\eta_h, \boldsymbol{u}_h]$ be the solution of the stabilized discrete problem* (3.5) *using the OSS method. Then*

$$(5.26)\qquad \| [\eta - \eta_h, \boldsymbol{u} - \boldsymbol{u}_h] \|_{S,h} \lesssim E_S(h),$$

with the norm $\| \cdot \|_{S,h}$ defined in (4.12) *and*

$$E_S^2(h) := \mu_\eta \|\varepsilon_\eta\|_{L^\infty(0,T;L^2(\Omega))}^2 + \mu_u \|\boldsymbol{\varepsilon}_u\|_{L^\infty(0,T;L^2(\Omega))}^2$$

$$
\begin{aligned}
&+ \tau_u \|\nabla \varepsilon_\eta\|_{L^2(0,T;L^2(\Omega))} + \tau_\eta \|\nabla \cdot \boldsymbol{\varepsilon}_u\|_{L^2(0,T;L^2(\Omega))} \\
&+ \tau_\eta \|\mu_\eta \partial_t \varepsilon_\eta\|^2_{L^2(0,T;L^2(\Omega))} + \tau_u \|\mu_u \partial_t \boldsymbol{\varepsilon}_u\|^2_{L^2(0,T;L^2(\Omega))} \\
&+ \frac{1}{\tau_\eta} \|\varepsilon_\eta\|^2_{L^2(0,T;L^2(\Omega))} + \frac{1}{\tau_u} \|\boldsymbol{\varepsilon}_u\|^2_{L^2(0,T;L^2(\Omega))} \\
&+ \gamma_\eta \|\mu_\eta \partial_{tt} \varepsilon_\eta + \nabla \cdot \partial_t \boldsymbol{\varepsilon}_u\|^2_{L^1(0,T;L^2(\Omega))} + \gamma_u \|\mu_u \partial_{tt} \boldsymbol{\varepsilon}_u + \nabla \partial_t \varepsilon_\eta\|^2_{L^1(0,T;L^2(\Omega))} \\
&+ \beta_2 \tau_\eta \|\nabla \cdot \partial_t \boldsymbol{\varepsilon}_u\|^2_{L^2(0,T;L^2(\Omega))} + \beta_2 \tau_u \|\nabla \partial_t \varepsilon_\eta\|^2_{L^2(0,T;L^2(\Omega))} \\
&+ \gamma_\eta \|\mu_\eta \partial_t \varepsilon_\eta(0)\|^2 + \gamma_u \|\mu_u \partial_t \boldsymbol{\varepsilon}_u(0)\|^2. \qquad (5.27)
\end{aligned}
$$

Proof. Since the OSS method is consistent (in the sense that the exact solution is the solution of the discrete problem), using the strong Λ-coercivity (4.11) with Λ_i given in (4.13)–(4.14), $i = 1, 2$, we get

$$
\begin{aligned}
\|\!| [e_\eta, \boldsymbol{e}_u] |\!\|^2_{W,h} \lesssim& \int_0^T \mathcal{B}_s \left([\varepsilon_\eta, \boldsymbol{\varepsilon}_u], \Lambda_1 \left([e_\eta, \boldsymbol{e}_u] \right) \right) \mathrm{d}t \\
&+ \int_0^T \mathcal{B}_s \left([\partial_t \varepsilon_\eta, \partial_t \boldsymbol{\varepsilon}_u], \Lambda_2 \left([e_\eta, \boldsymbol{e}_u] \right) \right) \mathrm{d}t \\
&+ \mathcal{B}_s \left([\varepsilon_\eta, \boldsymbol{\varepsilon}_u], \Lambda_2 \left([e_\eta, \boldsymbol{e}_u] \right) \right) \Big|_{t=0}. \qquad (5.28)
\end{aligned}
$$

Now we can bound each term of (5.28) in terms of the interpolation error $[\varepsilon_\eta, \boldsymbol{\varepsilon}_u]$. The first term of (5.28) can be written as

$$
\begin{aligned}
\int_0^T \mathcal{B}_s \left([\varepsilon_\eta, \boldsymbol{\varepsilon}_u], \Lambda_1 \left([e_\eta, \boldsymbol{e}_u] \right) \right) \mathrm{d}t =& \int_0^T \left(\mu_\eta \partial_t \varepsilon_\eta + \nabla \cdot \boldsymbol{\varepsilon}_u, e_\eta + \beta_1 \tau_\eta P_{\eta,h}(\nabla \cdot \boldsymbol{e}_u) \right) \mathrm{d}t \\
&+ \int_0^T \left(\mu_u \partial_t \boldsymbol{\varepsilon}_u + \nabla \varepsilon_\eta, \boldsymbol{e}_u + \beta_1 \tau_u P_{u,h}(\nabla e_\eta) \right) \mathrm{d}t \\
&+ \int_0^T \left(P^\perp_{\eta,h}(\nabla \cdot \boldsymbol{\varepsilon}_u), \tau_\eta \nabla \cdot (\boldsymbol{e}_u + \beta_1 \tau_u P_{u,h}(\nabla e_\eta)) \right) \mathrm{d}t \\
&+ \int_0^T \left(P^\perp_{u,h}(\nabla \varepsilon_\eta), \tau_u \nabla (e_\eta + \beta_1 \tau_\eta P_{\eta,h}(\nabla \cdot \boldsymbol{e}_u)) \right) \mathrm{d}t. \qquad (5.29)
\end{aligned}
$$

The first term (and similarly the second term) of (5.29) can be bounded as

$$
\begin{aligned}
&\int_0^T \left(\mu_\eta \partial_t \varepsilon_\eta + \nabla \cdot \boldsymbol{\varepsilon}_u, e_\eta + \beta_1 \tau_\eta P_{\eta,h}(\nabla \cdot \boldsymbol{e}_u) \right) \mathrm{d}t \\
&\quad \le \frac{\alpha_1}{2} \tau_u^{-1} \|\boldsymbol{\varepsilon}_u\|^2_{L^2(L^2)} + \frac{1}{2\alpha_1} \tau_u \|\nabla e_\eta\|^2_{L^2(L^2)} \\
&\qquad + \frac{\alpha_2}{2} \tau_\eta \|\mu_\eta \partial_t \varepsilon_\eta + \nabla \cdot \boldsymbol{\varepsilon}_u\|^2_{L^2(L^2)} + \frac{\beta_1^2}{2\alpha_2} \tau_\eta \|\nabla \cdot \boldsymbol{e}_u\|^2_{L^2(L^2)}. \qquad (5.30)
\end{aligned}
$$

The third term (and similarly the fourth term) of (5.29) can be bounded as

$$
\begin{aligned}
&\int_0^T \left(P^\perp_{\eta,h}(\nabla \cdot \boldsymbol{\varepsilon}_u), \tau_\eta \nabla \cdot (\boldsymbol{e}_u + \beta_1 \tau_u P_{u,h}(\nabla e_\eta)) \right) \mathrm{d}t \\
&\le \frac{\alpha_3}{2} \tau_\eta \|\nabla \cdot \boldsymbol{\varepsilon}_u\|^2_{L^2(L^2)} + \frac{1}{2\alpha_3} \tau_\eta \|\nabla \cdot \boldsymbol{e}_u\|^2_{L^2(L^2)} + \frac{\beta_1^2}{2\alpha_3} C_\tau^2 C_{\mathrm{inv}}^2 \tau_u \|\nabla e_\eta\|^2_{L^2(L^2)}. \qquad (5.31)
\end{aligned}
$$

The second term of (5.28) can be written and bounded as

$$\begin{aligned}
&\int_0^T \mathcal{B}_s\left([\partial_t\varepsilon_\eta, \partial_t\boldsymbol{\varepsilon}_u], \Lambda_2([e_\eta, \boldsymbol{e}_u])\right) \mathrm{d}t \\
&\quad = \int_0^T \Big[(\mu_\eta \partial_{tt}\varepsilon_\eta + \nabla\cdot\partial_t\boldsymbol{\varepsilon}_u, \beta_2\partial_t e_\eta) + (\mu_u\partial_{tt}\boldsymbol{\varepsilon}_u + \nabla\partial_t\varepsilon_\eta, \beta_2\partial_t\boldsymbol{e}_u)\Big]\mathrm{d}t \\
&\qquad + \int_0^T \Big[\big(P^\perp_{\eta,h}(\nabla\cdot\partial_t\boldsymbol{\varepsilon}_u), \tau_\eta\nabla\cdot(\beta_2\partial_t\boldsymbol{e}_u)\big) + \big(P^\perp_{u,h}(\nabla\partial_t\varepsilon_\eta), \tau_u\nabla(\beta_2\partial_t e_\eta)\big)\Big]\mathrm{d}t \\
&\quad \le \alpha_4\Big(\gamma_\eta\|\mu_\eta\partial_{tt}\varepsilon_\eta + \nabla\cdot\partial_t\boldsymbol{\varepsilon}_u\|^2_{L^1(L^2)} + \gamma_u\|\mu_u\partial_{tt}\boldsymbol{\varepsilon}_u + \nabla\partial_t\varepsilon_\eta\|^2_{L^1(L^2)}\Big) \\
&\qquad + \frac{1}{\alpha_4}\Big(\gamma_\eta\|\mu_\eta\partial_t e_\eta\|^2_{L^\infty(L^2)} + \gamma_u\|\mu_u\partial_t\boldsymbol{e}_u\|^2_{L^\infty(L^2)}\Big) \\
&\qquad + \frac{\alpha_5}{2}\Big(\beta_2\tau_\eta\|\nabla\cdot\partial_t\boldsymbol{\varepsilon}_u\|^2_{L^2(L^2)} + \beta_2\tau_u\|\nabla\partial_t\varepsilon_\eta\|^2_{L^2(L^2)}\Big) \\
&\qquad + \frac{1}{2\alpha_5}\Big(\beta_2\tau_\eta\|P^\perp_{\eta,h}(\nabla\cdot\partial_t\boldsymbol{e}_u)\|^2_{L^2(L^2)} + \beta_2\tau_u\|P^\perp_{u,h}(\nabla\partial_t e_\eta)\|^2_{L^2(L^2)}\Big).
\end{aligned} \tag{5.32}$$

The third term of (5.28) can be written and bounded as

$$\begin{aligned}
&\mathcal{B}_s\left([\varepsilon_\eta, \boldsymbol{\varepsilon}_u], \Lambda_2([e_\eta, \boldsymbol{e}_u])\right)\big|_{t=0} \\
&= (\mu_\eta\partial_t\varepsilon_\eta(0) + \nabla\cdot\boldsymbol{\varepsilon}_u(0), \beta_2\partial_t e_\eta(0)) + (\mu_u\partial_t\boldsymbol{\varepsilon}_u(0) + \nabla\varepsilon_\eta(0), \beta_2\partial_t\boldsymbol{e}_u(0)) \\
&\quad + \big(P^\perp_{\eta,h}(\nabla\cdot\boldsymbol{\varepsilon}_u(0)), \tau_\eta\nabla\cdot(\beta_2\partial_t\boldsymbol{e}_u(0))\big) + \big(P^\perp_{u,h}(\nabla\varepsilon_\eta(0)), \tau_u\nabla(\beta_2\partial_t e_\eta(0))\big) \\
&\le \frac{\alpha_6}{2}\left(\gamma_\eta\|\mu_\eta\partial_t\varepsilon_\eta(0)\|^2 + \gamma_u\|\mu_u\partial_t\boldsymbol{\varepsilon}_u(0)\|^2\right) \\
&\quad + \frac{1}{2\alpha_6}\left(\gamma_\eta\|\mu_\eta\partial_t e_\eta(0)\|^2 + \gamma_u\|\mu_u\partial_t\boldsymbol{e}_u(0)\|^2\right).
\end{aligned} \tag{5.33}$$

Combining all bounds (5.30)–(5.33) and taking α_i big enough, it follows that $\|\!|[e_\eta, \boldsymbol{e}_u]|\!\|_h^2 \lesssim E_S^2(h)$. Using the same reasoning as for the ASGS method, we complete the proof for the OSS method. ∎

5.3. Accuracy of ASGS and OSS methods. Let us define as k the order of η-interpolation and as l the order of $\boldsymbol{u}$-interpolation. Analyzing the a priori error estimates for the ASGS and the OSS methods from (5.16) and (5.21) and assuming regular enough solutions, we can summarize the convergence rates of the formulations as shown in Table 2. When the convergence rate is the same as that of the interpolation error we call it *optimal*, when the gap is 1/2 *quasi-optimal*, and when the gap is 1 *suboptimal*.

Now, let us just consider the OSS method and the error estimate in the strong norm from (5.26). The convergence rates of each of the variational forms in the strong norm can be summarized as shown in Table 3.

5.4. Numerical tests. Let us consider a two dimensional transient problem with analytical solution to investigate the convergence properties of the stabilized FE formulations proposed. We take Ω as the unit square $(0,1)\times(0,1)$, the time interval is taken as $[0, 0.01]$, the physical properties are taken as $\mu_\eta = 10.0$ and $\mu_u = 10.0$, and the forcing terms f_η and $\boldsymbol{f}_u$ are taken such that the exact solution is

$$\eta = \sin(3\pi x)\sin(3\pi y)\sin(2\pi t), \quad \boldsymbol{u} = [\eta, \eta]. \tag{5.34}$$

Table 2
Convergence rates according to the variational forms for the ASGS and OSS methods in the weak norm.

Variational form	I	II	III
$\Vert\eta-\eta_h\Vert_{L^\infty(0,T;L^2(\Omega))}$	$h^{k+1/2}+h^{l+1/2}$ quasi-optimal	$h^{k+1}+h^{l}$ suboptimal	$h^{k}+h^{l+1}$ suboptimal
$\Vert\boldsymbol{u}-\boldsymbol{u}_h\Vert_{L^\infty(0,T;L^2(\Omega))}$	$h^{k+1/2}+h^{l+1/2}$ quasi-optimal	$h^{k+1}+h^{l}$ suboptimal	$h^{k}+h^{l+1}$ suboptimal
$\Vert\mu_u\partial_t(\boldsymbol{u}-\boldsymbol{u}_h)+\nabla(\eta-\eta_h)\Vert_{L^2(0,T;L^2(\Omega))}$	$h^{k}+h^{l}$ optimal	$h^{k}+h^{l-1}$ suboptimal	$h^{k}+h^{l+1}$ optimal
$\Vert\mu_\eta\partial_t(\eta-\eta_h)+\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\Vert_{L^2(0,T;L^2(\Omega))}$	$h^{k}+h^{l}$ optimal	$h^{k+1}+h^{l}$ optimal	$h^{k-1}+h^{l}$ suboptimal
k, l optimal	$k=l$	$k+1=l$	$k=l+1$

Table 3
Convergence rates according to the variational forms for the OSS method in the strong norm.

Variational form	I	II	III
$\Vert\nabla(\eta-\eta_h)\Vert_{L^2(0,T;L^2(\Omega))}$	$h^{k}+h^{l}$ optimal	$h^{k-1}+h^{l-1}$ suboptimal	$h^{k}+h^{l}$ optimal
$\Vert\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\Vert_{L^2(0,T;L^2(\Omega))}$	$h^{k}+h^{l}$ optimal	$h^{k}+h^{l}$ optimal	$h^{k-1}+h^{l-1}$ suboptimal
k, l optimal	$k=l$	$k=l$	$k=l$

This exact solution is zero on the boundary $\partial\Omega$, so the boundary conditions of the problem are satisfied. We denote as BC1 the imposition (weak or strong) of $\eta=0$ on $\partial\Omega$. Additionally, we denote as BC2 the imposition of $\boldsymbol{n}\cdot\boldsymbol{u}=0$ on $\partial\Omega$.

For the spatial discretization, we have used four uniform FE meshes with $h=0.010$, $h=0.005$, $h=0.002$, and $h=0.001$. The elements used are $P1$ (three-node triangular elements) and $P2$ (six-node triangular elements).

Figure 1 shows the mesh for $h=0.10$. The other meshes are isotropic refinements of that one. Anisotropic meshes are not encompassed in the analysis presented. This would require the use of the analysis techniques introduced in [14].

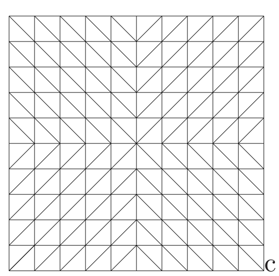

Fig. 1. *Mesh sample.*

The stabilization parameters are computed with the algorithmic constant $C_\tau=0.01$ for $P1$ and with $C_\tau=0.4$ for $P2$. The characteristic domain length was taken as $\mathrm{L}_0=\sqrt[d]{\mathrm{meas}(\Omega)}=1$. The time integration scheme is Crank–Nicolson with a time step size of 10^{-5}. We have used a very small time step to avoid any interference of the time marching algorithm into the spatial error since we are only interested in this spatial error. With that time step size the difference between the error of the scalar unknown in the norm $\|\cdot\|_{L^\infty(0,T;L^2(\Omega))}$ and the error in the same norm with a time step size twice as big was less than 1% in the finest mesh. This, of course, depends on the time interval chosen ($T = 0.01$), which has been taken small enough to avoid long term effects which would require a carefully designed time integration scheme, but large enough to obtain significant spatial errors.

In Tables 4 to 7 the experimental convergence rates for the ASGS and the OSS methods are shown. BC1 and BC2 are the results obtained using BC1 and BC2 as boundary conditions and min stands for the minimum expected convergence rate based on theoretical analysis. All of these numerical results match with the convergence rates predicted theoretically, in the sense that the convergence rate is always at least as fast as the worst predicted by the analysis. It would perhaps be possible to improve the convergence rates obtained using some sort of duality arguments, although we have not pursued this in this work (see [5]). It is noteworthy that the convergence rates for both the ASGS and the OSS method are surprisingly similar, although the absolute errors are not. For example, for BC1 and the variational form III using linear elements, the slope of the convergence curve of η_h in $L^\infty(0,T;L^2(\Omega))$ is 2.13 for both the ASGS and the OSS methods, but the absolute errors are different. For $h = 0.1$, 0.05, and 0.025 the errors for the ASGS method are 2.92×10^{-2}, 6.60×10^{-3}, and 1.38×10^{-3}, respectively, whereas for the OSS method these are 2.61×10^{-2}, 6.15×10^{-3}, and 1.36×10^{-3}.

Table 4
Experimental convergence rates for the ASGS method using $P1/P1$ interpolation.

Variational form	I			II			III		
Boundary cond.	BC1	BC2	min	BC1	BC2	min	BC1	BC2	min
$\|\eta-\eta_h\|_{L^\infty(0,T;L^2(\Omega))}$	2.15	2.38	1.5	1.99	1.79	1	2.13	2.20	1
$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^\infty(0,T;L^2(\Omega))}$	2.19	2.09	1.5	1.43	1.59	1	1.84	1.79	1
$\|\nabla(\eta-\eta_h)\|_{L^2(0,T;L^2(\Omega))}$	1.03	1.18	1	0.99	0.84	0	1.04	1.07	1
$\|\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2(0,T;L^2(\Omega))}$	1.48	1.10	1	1.10	1.13	1	0.81	0.71	0

Table 5
Experimental convergence rates for the OSS method using $P1/P1$ interpolation.

Variational form	I			II			III		
Boundary cond.	BC1	BC2	min	BC1	BC2	min	BC1	BC2	min
$\|\eta-\eta_h\|_{L^\infty(0,T;L^2(\Omega))}$	2.16	2.43	1.5	1.99	1.79	1	2.13	2.20	1
$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^\infty(0,T;L^2(\Omega))}$	2.18	2.18	1.5	1.41	1.59	1	1.84	1.79	1
$\|\nabla(\eta-\eta_h)\|_{L^2(0,T;L^2(\Omega))}$	1.04	1.18	1	0.99	0.84	0	1.04	1.07	1
$\|\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2(0,T;L^2(\Omega))}$	1.48	1.10	1	1.10	1.13	1	0.81	0.71	0

Table 6
Experimental convergence rates for the ASGS method using $P2/P2$ interpolation.

Variational form	I			II			III		
Boundary cond.	BC1	BC2	min	BC1	BC2	min	BC1	BC2	min
$\|\eta-\eta_h\|_{L^\infty(0,T;L^2(\Omega))}$	3.03	3.03	2.5	2.67	2.67	2	3.41	3.40	2
$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^\infty(0,T;L^2(\Omega))}$	3.01	3.01	2.5	2.50	2.54	2	2.76	2.76	2
$\|\nabla(\eta-\eta_h)\|_{L^2(0,T;L^2(\Omega))}$	2.06	2.07	2	1.79	1.79	1	2.54	2.54	2
$\|\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2(0,T;L^2(\Omega))}$	2.07	2.06	2	2.55	2.55	2	1.79	1.79	1

Table 7
Experimental convergence rates for the OSS method using $P2/P2$ interpolation.

Variational form	I			II			III		
Boundary cond.	BC1	BC2	min	BC1	BC2	min	BC1	BC2	min
$\|\eta - \eta_h\|_{L^\infty(0,T;L^2(\Omega))}$	3.03	3.02	2.5	2.66	2.66	2	3.30	3.29	2
$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^\infty(0,T;L^2(\Omega))}$	2.99	3.00	2.5	2.48	2.52	2	2.76	2.76	2
$\|\nabla(\eta - \eta_h)\|_{L^2(0,T;L^2(\Omega))}$	2.06	2.06	2	1.79	1.79	1	2.53	2.53	2
$\|\nabla \cdot (\boldsymbol{u} - \boldsymbol{u}_h)\|_{L^2(0,T;L^2(\Omega))}$	2.06	2.06	2	2.55	2.54	2	1.79	1.79	1

6. Conclusions. In the present work, we have presented two stabilized FE methods (ASGS and OSS) for the wave equation in mixed form. Additionally, these stabilized methods have been applied to three variational forms of the problem. We normally use the stabilized FE formulations for equal interpolation of the unknowns, but the analysis is not restricted to that and allows any continuous interpolation pair. Extension to discontinuous approximations would be easy to analyze, but this would require the introduction of terms evaluated on the interelement boundaries.

Length scales related to the unknowns were introduced in order to treat all variational forms in a unified manner. The way length scales are computed is determined by the norm $|\!|\!| \cdot |\!|\!|$ in order to have control over the gradient of the scalar unknown or the divergence of the vector unknown if it is the case. Furthermore, length scales obviously influence the stability and accuracy of the methods.

Stability was proven for both the ASGS and the OSS methods applied to the wave equation in mixed form. According to those results, we have control over the L^2-norm of these unknowns. Moreover, depending on the variational form under consideration, additional control over the gradient of the scalar unknown and/or over the divergence of the vector unknown is attained; therefore, we can guarantee that no numerical point-to-point oscillations will occur.

Theoretical convergence rates were found for both the ASGS and OSS methods, and for all the variational forms analyzed. Both stabilization methods exhibit at least the same convergence properties based on theoretical analysis. Additionally, there is no numerical evidence that the ASGS method has convergence superiority over the OSS method or vice-versa. One interesting feature of the convergence analysis applied to the three variational forms of the wave equation in mixed form is the guideline given as to which variational form to use. That said, for a balanced accuracy for both unknowns and equal interpolation, variational form I is the best choice.

REFERENCES

[1] T. Arbogast and D. Brunson, *A computational method for approximating a Darcy-Stokes system governing a vuggy porous medium*, Comput. Geosci., 11 (2007), pp. 207–218.
[2] D. Arnold, F. Brezzi, and M. Fortin, *A stable finite element for the Stokes equations*, Calcolo, 21 (1984), pp. 337–344.
[3] S. Badia, *On stabilized finite element methods based on the Scott–Zhang projector. Circumventing the inf–sup condition for the Stokes problem*, Comput. Methods Appl. Mech. Engrg., 247/248 (2012), pp. 65–72.
[4] S. Badia and R. Codina, *Analysis of a stabilized finite element approximation of the transient convection-diffusion equation using an ALE framework*, SIAM J. Numer. Anal., 44 (2006), pp. 2159–2197.
[5] S. Badia and R. Codina, *Unified stabilized finite element formulations for the Stokes and the Darcy problems*, SIAM J. Numer. Anal., 47 (2009), pp. 1971–2000.
[6] S. Badia and R. Codina, *Stabilized continuous and discontinuous Galerkin techniques for Darcy flow*, Comput. Methods Appl. Mech. Engrg., 199 (2010), pp. 1654–1667.

[7] S. Badia and R. Codina, *Stokes, Maxwell and Darcy: A single finite element approximation for three model problems*, Appl. Numer. Math., 62 (2012), pp. 246–263.

[8] E. Bécache, P. Joly, and C. Tsogka, *An analysis of new mixed finite elements for the approximation of wave propagation problems*, SIAM J. Numer. Anal., 37 (2000), pp. 1053–1084.

[9] E. Bécache, P. Joly, and C. Tsogka, *A new family of mixed finite elements for the linear elastodynamic problem*, SIAM J. Numer. Anal., 39 (2002), pp. 2109–2132.

[10] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.

[11] F. Brezzi, J. Douglas, Jr., and L. D. Marini, *Two families of mixed finite elements for second order elliptic problems*, Numer. Math., 47 (1985), pp. 217–235.

[12] R. Codina, *Stabilization of incompressibility and convection through orthogonal sub-scales in finite element methods*, Comput. Methods Appl. Mech. Engrg., 190 (2000), pp. 1579–1599.

[13] R. Codina, *Stabilized finite element approximation of transient incompressible flows using orthogonal subscales*, Comput. Methods Appl. Mech. Engrg., 191 (2002), pp. 4295–4321.

[14] R. Codina, *Analysis of a stabilized finite element approximation of the Oseen equations using orthogonal subscales*, Appl. Numer. Math., 58 (2008), pp. 264–283.

[15] R. Codina, *Finite element approximation of the hyperbolic wave equation in mixed form*, Comput. Methods Appl. Mech. Engrg., 197 (2008), pp. 1305–1322.

[16] R. Codina, J. Blasco, G.C. Buscaglia, and A. Huerta, *Implementation of a stabilized finite element formulation for the incompressible Navier-Stokes equations based on a pressure gradient projection*, Internat. J. Numer. Methods Fluids, 37 (2001), pp. 419–444.

[17] D. A. French, *A space-time finite element method for the wave equation*, Comput. Methods Appl. Mech. Engrg., 107 (1993), pp. 145–157.

[18] T. Geveci, *On the application of mixed finite element methods to the wave equations*, RAIRO Modél. Math. Anal. Numér., 22 (1988), pp. 243–250.

[19] T. J. R. Hughes, *Multiscale phenomena: Green's functions, the Dirichlet-to-Neumann formulation, subgrid scale models, bubbles and the origins of stabilized methods*, Comput. Methods Appl. Mech. Engrg., 127 (1995), pp. 387–401.

[20] T. J. R. Hughes, G. R. Feijóo, L. Mazzei, and J.-B. Quincy, *The variational multiscale method—A paradigm for computational mechanics*, Comput. Methods Appl. Mech. Engrg., 166 (1998), pp. 3–24.

[21] E. W. Jenkins, B. Rivière, and M. F. Wheeler, *A priori error estimates for mixed finite element approximations of the acoustic wave equation*, SIAM J. Numer. Anal., 40 (2002), pp. 1698–1715.

[22] P. Joly, *Variational methods for time-dependent wave propagation problems*, in Topics in Computational Wave Propagation, Lect. Notes Comput. Sci. Eng. 31, M. Ainsworth, P. Davies, D. Duncan, B. Rynne, P. Martin, T. J. Barth, M. Griebel, D. E. Keyes, R. M. Nieminen, D. Roose, and T. Schlick, eds., Springer, Berlin, 2003, pp. 201–264.

[23] O. Karakashian and C. Makridakis, *Convergence of a continuous Galerkin method with mesh modification for nonlinear wave equations*, Math. Comp., 74 (2005), pp. 85–102.

[24] C. G. Makridakis, *On mixed finite element methods for linear elastodynamics*, Numer. Math., 61 (1992), pp. 235–260.

[25] K.-A. Mardal, X.-C. Tai, and R. Winther, *A robust finite element method for Darcy–Stokes flow*, SIAM J. Numer. Anal., 40 (2002), pp. 1605–1631.

[26] J.-C. Nédélec, *A new family of mixed finite elements in* $\mathbb{R}^3$, Numer. Math., 50 (1986), pp. 57–81.

[27] L. R. Scott and S. Zhang, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp., 54 (1990), pp. 483–493.