IMA Journal of Numerical Analysis (2014) **34**, 800–819
doi:10.1093/imanum/drt022
Advance Access publication on July 19, 2013

Error analysis of discontinuous Galerkin methods for the Stokes problem under minimal regularity

S. BADIA AND R. CODINA
International Center for Numerical Methods in Engineering (CIMNE), Universitat Politècnica de Catalunya, Jordi Girona 1-3, Edifici C1, 08034 Barcelona, Spain
sbadia@cimne.upc.edu ramon.codina@upc.edu

T. GUDI*
Department of Mathematics, Indian Institute of Science, Bangalore 560012, India
*Corresponding author: gudi@math.iisc.ernet.in

AND

J. GUZMÁN
Division of Applied Mathematics, Brown University, Providence, RI 02912, USA
johnny_guzman@brown.edu

[Received on 2 October 2012; revised on 13 March 2013]

In this article, we analyse several discontinuous Galerkin (DG) methods for the Stokes problem under minimal regularity on the solution. We assume that the velocity $\mathbf{u}$ belongs to $[H_0^1(\Omega)]^d$ and the pressure $p \in L_0^2(\Omega)$. First, we analyse standard DG methods assuming that the right-hand side $\mathbf{f}$ belongs to $[H^{-1}(\Omega) \cap L^1(\Omega)]^d$. A DG method that is well defined for $\mathbf{f}$ belonging to $[H^{-1}(\Omega)]^d$ is then investigated. The methods under study include stabilized DG methods using equal-order spaces and inf–sup stable ones where the pressure space is one polynomial degree less than the velocity space.

Keywords: error estimates; finite element; discontinuous Galerkin; Stokes problems; stabilized methods.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded polyhedral domain with boundary $\partial\Omega$. Let $\mathbf{V} = [H_0^1(\Omega)]^d$ and $Q = L_0^2(\Omega) := \{q \in L^2(\Omega) : (q, 1) = 0\}$. Here and throughout, $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product. We also use the notation $(\cdot,\cdot)$ to denote the inner product on $[L^2(\Omega)]^d$.

For a given $\mathbf{f} \in [H^{-1}(\Omega)]^d$, the Stokes problem consists of finding $[\mathbf{u}, p] \in \mathbf{V} \times Q$ such that

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \mathbf{f}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}, \tag{1.1}$$

$$b(\mathbf{u}, q) = 0 \quad \forall q \in Q, \tag{1.2}$$

where

$$a(\mathbf{w}, \mathbf{v}) = (\nabla \mathbf{w}, \nabla \mathbf{v}) \quad \forall \mathbf{w}, \ \mathbf{v} \in \mathbf{V}, \tag{1.3}$$

$$b(\mathbf{v}, q) = -(\nabla \cdot \mathbf{v}, q) \quad \forall \mathbf{v} \in \mathbf{V}, \ q \in Q. \tag{1.4}$$

The space $\mathbf{V}$ is endowed with the norm $\|\cdot\|_{\mathbf{V}}$ defined by $\|\mathbf{v}\|_{\mathbf{V}} = (\nabla \mathbf{v}, \nabla \mathbf{v})^{1/2}$.

Existence and uniqueness of the velocity $\mathbf{u}$ follows from the Lax–Milgram lemma in the space $\mathbf{Z} = \{\mathbf{v} \in \mathbf{V} : b(\mathbf{v}, q) = 0 \ \forall q \in Q\}$. Stability of the pressure is obtained by the well-known inf–sup condition

$$\beta \|q\|_{L^2(\Omega)} \leqslant \sup_{\mathbf{v}\in\mathbf{V}, \mathbf{v}\neq 0} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbf{V}}} \quad \forall q \in Q. \tag{1.5}$$

The discrete analogue of the inf–sup condition leads to stable numerical methods; see, for example, Boffi *et al.* (2008). However, in order to satisfy the discrete inf–sup condition, the discrete pressure and velocity spaces are dependent on each other. For example, as is well known, equal-order spaces are not inf–sup stable. On the other hand, stabilized methods allow one to use discrete spaces that are not inf–sup stable.

There are now many discontinuous Galerkin (DG) methods for Stokes flow using the velocity–pressure formulation; see, for example, Schötzau *et al.* (2003), Cockburn *et al.* (2002) and Hansbo & Larson (2002). Error analyses have been performed for most DG methods in the literature. However, they often assume sufficient regularity of the exact solution. In this paper, we perform an error analysis of some DG methods for the Stokes problem where we only assume minimal regularity $[\mathbf{u}, p] \in \mathbf{V} \times Q$. Moreover, we assume $\mathbf{f} \in [H^{-1}(\Omega) \cap L^1(\Omega)]^d$. We note that DG methods are not well defined for functions $\mathbf{f}$ only belonging to $[H^{-1}(\Omega)]^d$ since DG test functions do not belong to $[H^1(\Omega)]^d$. However, we show how to modify the right-hand side of the DG method so that we can define a method for $\mathbf{f} \in [H^{-1}(\Omega)]^d$. The approach we take in this paper is the one used by Gudi (2010a,b) for elliptic problems where residual estimates are used in the analysis.

Di Pietro & Ern (2010) were the first to address the convergence of DG methods for problems with minimal regularity. There they considered solutions $[\mathbf{u}, p] \in \mathbf{V} \times Q$ and $\mathbf{f} \in [L^p(\Omega)]^d$ for $p > 1$ when $d = 2$, and for $p \geqslant 6/5$ when $d = 3$ (these restrictions are due to Sobolev embeddings). Their approach was to use discrete compactness arguments. Shortly after, Gudi (2010b) considered elliptic problems where he assumed the solution u to be in $H_0^1(\Omega)$ and proved error estimates that give rates of convergence in the case of smoothness. There he assumed that the right-hand side belongs to $L^2(\Omega)$. Finally, Rivière & Wihler (2011) proved optimal error estimates for the Laplace problem in two dimensions using $W^{2,p}$ elliptic regularity for $p > 1$.

As mentioned above, we will apply the techniques in Gudi (2010b) to give error estimates for the Stokes problem. Here we show, following the argument of Gudi (2010b), that if one is a bit more careful, one only needs data belonging to $[H^{-1}(\Omega) \cap L^1(\Omega)]^d$. Our estimates will depend on an oscillation term measuring the sum of local H^{-1} norms of the difference between $\mathbf{f}$ and its L^2 projection onto piecewise polynomials; see Theorem 3.1. Similar oscillation terms were considered by Cohen *et al.* when dealing with data in $[H^{-1}(\Omega)]^d$. Using Sobolev embeddings, we can prove convergence results under the assumptions used in Di Pietro & Ern (2010) while also showing convergence rates when slightly more regular data are used. We stress that our *a priori* analysis does not depend on elliptic regularity.

Pressure estimates require special care. When the pressure polynomial space is one degree less than the velocity space, one only needs to use that the discrete velocity (or the $H(\text{div}; \Omega)$ conforming part of the space) and pressure space form an $H(\text{div}; \Omega) \times L^2(\Omega)$ stable pair (such ideas were used for example in Schötzau *et al.*, 2003). In the equal-order case, one also needs to bound the higher modes of the pressure approximation. We do this by estimating the traces of the error on faces of the simplicial decomposition. In the equal-order case we consider two distinct methods: one that penalizes the jumps of the pressure (see, for example, Schötzau *et al.*, 2003) and a method introduced in Codina *et al.* (2009) and applied to DG methods in Codina & Badia that penalizes the jumps of the total flux. Since penalizing the jumps of the pressure is inconsistent with pressure only belonging to $L^2(\Omega)$, we get the best approximation results in the space $Q_h \cap H^1(\Omega)$, where Q_h is the DG discrete space for the pressure. On the other hand, when we use the DG method in Codina & Badia we get the best approximation in the space Q_h.

Since standard DG methods are not well defined for $\mathbf{f} \in [H^{-1}(\Omega)]^d$ we modify the right-hand side in order to consider this case with the help of enrichment operators. We show that the proposed method converges strongly in $[H^1(\Omega)]^d \times L^2(\Omega)$. Moreover, we show that if one modifies the right-hand side appropriately (i.e., chooses the enrichment operator appropriately), a best approximation property will hold.

All our analysis is based on the assumption that the polynomial degree of the finite element approximation is kept constant. Thus, our results proving convergence under minimal regularity assumptions hold in the case in which the mesh size tends to zero.

The paper is organized as follows. In Section 2 we introduce notation and some preliminary results. We propose the methods to be analysed in Section 3 and prove velocity error estimates. Section 4 is devoted to pressure error estimates. The analysis for a method that penalizes flux jumps is included in Section 5. Finally, the extension of the previous analyses to forcing terms in $[H^{-1}(\Omega)]^d$ is carried out in Section 6.

2. Notation and preliminaries

The following notation will be used throughout the article:

$$
\begin{aligned}
\mathcal{T}_h &= \text{face-to-face (i.e., no hanging nodes), shape-regular simplicial triangulations of } \Omega,\\
T &= \text{a simplex of } \mathcal{T}_h, \quad h_T = \text{diameter of } T, \quad h = \max\{h_T : T \in \mathcal{T}_h\},\\
\mathcal{V}_h^{\mathrm{i}} &= \text{set of all vertices in } \mathcal{T}_h \text{ that are in } \Omega,\\
\mathcal{V}_h^{\mathrm{b}} &= \text{set of all vertices in } \mathcal{T}_h \text{ that are on } \partial\Omega,\\
\mathcal{V}_h &= \mathcal{V}^{\mathrm{i}} \cup \mathcal{V}^{b},\\
\mathcal{E}_h^{\mathrm{i}} &= \text{set of all interior edges of } \mathcal{T}_h,\\
\mathcal{E}_h^{\mathrm{b}} &= \text{set of all boundary edges of } \mathcal{T}_h,\\
\mathcal{E}_h &= \mathcal{E}_h^{\mathrm{i}} \cup \mathcal{E}_h^{b},\\
h_e &= |e|, \text{ the diameter of } e \in \mathcal{E}_h,\\
\mathcal{T}_z &= \text{set of all simplices sharing the vertex } z,\\
\mathcal{T}_e &= \text{patch of two simplices sharing the face } e,\\
\nabla_h &= \text{piecewise (elementwise) gradient},\\
\mathbb{P}_m(T) &= \text{space of polynomials of degree less than or equal to } m \geqslant 0 \text{ and defined on } T,\\
\tilde{\mathbb{P}}_m(T) &= \text{space of homogeneous polynomials of degree } m \text{ defined on } T,\\
I &= \text{ identity matrix of size } d \times d.
\end{aligned}
$$

Let us define the discrete spaces

$$
\mathbf{V}_h = \{\mathbf{v}_h \in [L^2(\Omega)]^d : \mathbf{v}_h|_T \in [\mathbb{P}_r(T)]^d\} \quad \text{and} \quad Q_h^s = \{q_h \in Q : q_h|_T \in \mathbb{P}_s(T)\}.
$$

We will consider the cases $s = r - 1$ and the equal-order case $s = r$. For the sake of convenience, let us define a broken Sobolev space

$$H^1(\Omega, \mathcal{T}_h) = \{v \in L^2(\Omega) : v|_T \in H^1(T)\ \forall\, T \in \mathcal{T}_h\}.$$

In the problem setting, we require jump and mean definitions of discontinuous functions, vector functions and tensors. For any $e \in \mathcal{E}_h^i$, there are two simplices T_+ and T_- such that $e = \partial T_+ \cap \partial T_-$. Let $\mathbf{n}_+$ be the unit normal of e pointing from T_+ to T_- and let $\mathbf{n}_- = -\mathbf{n}_+$. For any $v \in H^1(\Omega, \mathcal{T}_h)$, we define the jump and mean of v on e by

$$[\![v]\!] = v_+\mathbf{n}_+ + v_-\mathbf{n}_- \quad \text{and} \quad \{v\} = \tfrac{1}{2}(v_+ + v_-), \text{ respectively,}$$

where $v_\pm = v|_{T_\pm}$. For $\mathbf{v} \in [H^1(\Omega, \mathcal{T}_h)]^d$ we define the jump and mean of $\mathbf{v}$ on $e \in \mathcal{E}_h^i$ by

$$[\![\mathbf{v}]\!] = \mathbf{v}_+ \cdot \mathbf{n}_+ + \mathbf{v}_- \cdot \mathbf{n}_- \quad \text{and} \quad \{\mathbf{v}\} = \tfrac{1}{2}(\mathbf{v}_+ + \mathbf{v}_-), \text{ respectively.}$$

We also require the full jump of vector-valued functions. For $\mathbf{v} \in [H^1(\Omega, \mathcal{T}_h)]^d$, we define the full jump by

$$[\![\underline{\mathbf{v}}]\!] = \mathbf{v}_+ \otimes \mathbf{n}_+ + \mathbf{v}_- \otimes \mathbf{n}_-,$$

where for two vectors in Cartesian coordinates $\mathbf{a} = (a_i)$ and $\mathbf{b} = (b_j)$, we define the matrix $\mathbf{a} \otimes \mathbf{b} = [a_i b_j]_{1 \leqslant i,j \leqslant d}$. Similarly, for tensors $\tau \in [H^1(\Omega, \mathcal{T}_h)]^{d\times d}$, the jump and mean on $e \in \mathcal{E}_h^i$ are defined by

$$[\![\tau]\!] = \tau_+\mathbf{n}_+ + \tau_-\mathbf{n}_- \quad \text{and} \quad \{\tau\} = \tfrac{1}{2}(\tau_+ + \tau_-), \text{ respectively.}$$

For notational convenience, we also define the jump and mean on the boundary faces $e \in \mathcal{E}_h^b$ by modifying them appropriately. We use the definition of jump by understanding that $v_- = 0$ (similarly, $\mathbf{v}_- = 0$ and $\tau_- = 0$) and the definition of mean by understanding that $v_- = v_+$ (similarly, $\mathbf{v}_- = \mathbf{v}_+$ and $\tau_- = \tau_+$).

In the analysis of the following sections, we require the existence of an enriching operator $E_h : \mathbf{V}_h \to \mathbf{V}_h \cap [H_0^1(\Omega)]^d$ such that

$$\left(\sum_{T\in\mathcal{T}_h} h_T^{-2} \|E_h\mathbf{v}_h - \mathbf{v}_h\|_{L^2(T)}^2\right)^{1/2} + \|\nabla_h(E_h\mathbf{v}_h - \mathbf{v}_h)\|_{L^2(\Omega)} \leqslant C\|\mathbf{v}_h\|_h, \tag{2.1}$$

where

$$\|\mathbf{v}_h\|_h^2 = \sum_{T\in\mathcal{T}_h} \|\nabla\mathbf{v}_h\|_{L^2(T)}^2 + \sum_{e\in\mathcal{E}_h} \int_e \frac{1}{h_e} [\![\underline{\mathbf{v}_h}]\!]^2. \tag{2.2}$$

It is well known that this type of enriching operator can be constructed by averaging techniques (Brenner, 1996, 1999).

We will also need the following inverse estimates (Brenner & Scott, 2008).

Lemma 2.1 There exists a constant C_m such that for all $v \in \mathbb{P}_m(T)$ one has

$$\|v\|_{H^1(T)} \leqslant C_m h_T^{-1} \|v\|_{L^2(T)} \tag{2.3}$$

and

$$\|v\|_{L^2(\partial T)} \leqslant C_m h_T^{-1/2} \|v\|_{L^2(T)}. \tag{2.4}$$

For the next lemma we need to recall the definition of the $H^{-1}(D)$ norm:

$$\|\mathbf{f}\|_{H^{-1}(D)} = \sup_{\mathbf{w}\in[H_0^1(D)]^d} \frac{\mathbf{f}(\mathbf{w})}{\|\mathbf{w}\|_{H^1(D)}}.$$

For subsequent analyses, we define the restrictions of $\mathbf{f}$ to $H_0^1(T)$ by $\mathbf{f}_T$ and to $H_0^1(\mathcal{T}_e)$ by $\mathbf{f}_e$; that is,

$$\mathbf{f}_T(\mathbf{v}_1) = \mathbf{f}(\tilde{\mathbf{v}}_1) \quad \forall \mathbf{v}_1 \in H_0^1(T), \tag{2.5}$$

$$\mathbf{f}_e(\mathbf{v}_2) = \mathbf{f}(\tilde{\mathbf{v}}_2) \quad \forall \mathbf{v}_2 \in H_0^1(\mathcal{T}_e), \tag{2.6}$$

where $\tilde{\mathbf{v}}_1$(or $\tilde{\mathbf{v}}_2$) is the extension of $\mathbf{v}_1$(or $\mathbf{v}_2$) by zero outside T (or $\mathcal{T}_e$).

The following residual estimates, which resemble local-efficiency estimates, will be crucial for the forthcoming analysis.

LEMMA 2.2 Let $\mathbf{f}_h \in \mathbf{V}_h, \mathbf{v}_h \in \mathbf{V}_h$ and $q_h \in Q_h$ be arbitrary. Then, it holds that

$$h_T\|\mathbf{f}_h + \Delta\mathbf{v}_h - \nabla q_h\|_{L^2(T)} \leqslant C(\|\nabla(\mathbf{u} - \mathbf{v}_h)\|_{L^2(T)} + \|p - q_h\|_{L^2(T)} + \|\mathbf{f}_T - \mathbf{f}_h\|_{H^{-1}(T)}),$$

$$\|h_e^{1/2}[\![\nabla_h\mathbf{v}_h - q_h I]\!]\|_{L^2(e)} \leqslant C(\|\nabla_h(\mathbf{u} - \mathbf{v}_h)\|_{L^2(\mathcal{T}_e)} + \|p - q_h\|_{L^2(\mathcal{T}_e)} + \|\mathbf{f}_e - \mathbf{f}_h\|_{H^{-1}(\mathcal{T}_e)}).$$

Proof. We begin by proving the first estimate. Let $b_T \in H_0^1(T)$ be the polynomial bubble function that takes unit value at the barycentre of T. Then it is obvious that

$$\|b_T(\mathbf{f}_h + \Delta\mathbf{v}_h - \nabla q_h)\|_{L^2(T)} \leqslant \|\mathbf{f}_h + \Delta\mathbf{v}_h - \nabla q_h\|_{L^2(T)},$$

where $\mathbf{f}_h \in \mathbf{V}_h$ is an arbitrary polynomial on T. Using the fact that all norms are equivalent on a finite-dimensional space and a scaling argument, there exists a positive constant C_1 such that

$$C_1\|\mathbf{f}_h + \Delta\mathbf{v}_h - \nabla q_h\|^2_{L^2(T)} \leqslant \|b_T^{1/2}(\mathbf{f}_h + \Delta\mathbf{v}_h - \nabla q_h)\|^2_{L^2(T)}.$$

The constant C_1 depends on the shape of T and the polynomial order, but not on the diameter of T. Let $\mathbf{w}_h = b_T(\mathbf{f}_h + \Delta\mathbf{v}_h - \nabla q_h)$. Clearly $\mathbf{w}_h \in [H_0^1(T)]^d$. Then

$$C_1\int_T(\mathbf{f}_h + \Delta\mathbf{v}_h - \nabla q_h)^2 \leqslant \int_T \mathbf{w}_h\cdot(\mathbf{f}_h + \Delta\mathbf{v}_h - \nabla q_h)$$
$$= \mathbf{f}_T(\mathbf{w}_h) + \int_T \mathbf{w}_h\cdot(\Delta\mathbf{v}_h - \nabla q_h) + (\mathbf{f}_h - \mathbf{f}_T)(\mathbf{w}_h),$$

where $\mathbf{f}_h(\mathbf{w}_h) = \int_T \mathbf{f}_h \cdot \mathbf{w}_h$. Let $\tilde{\mathbf{w}}_h$ be the extension of $\mathbf{w}_h$ by zero outside T. Then using (2.5), (1.1) and integration by parts,

$$\begin{aligned}\mathbf{f}_T(\mathbf{w}_h) + \int_T \mathbf{w}_h \cdot (\Delta \mathbf{v}_h - \nabla q_h) &= \mathbf{f}(\tilde{\mathbf{w}}_h) + \int_T \mathbf{w}_h \cdot (\Delta \mathbf{v}_h - \nabla q_h) \\ &= \int_\Omega \nabla \mathbf{u} : \nabla \tilde{\mathbf{w}}_h - \int_\Omega \nabla \cdot \tilde{\mathbf{w}}_h p \\ &\quad - \int_T \nabla \mathbf{v}_h : \nabla \mathbf{w}_h + \int_T \nabla \cdot \mathbf{w}_h \, q_h \\ &= \int_T \nabla(\mathbf{u} - \mathbf{v}_h) : \nabla \mathbf{w}_h - \int_T \nabla \cdot \mathbf{w}_h (p - q_h).\end{aligned}$$

Therefore,

$$\begin{aligned}C_1 h_T^2 \|\mathbf{f}_h + \Delta \mathbf{v}_h - \nabla q_h\|_{L^2(T)}^2 &\leqslant h_T^2 \|\nabla(\mathbf{u} - \mathbf{v}_h)\|_{L^2(T)} \|\nabla \mathbf{w}_h\|_{L^2(T)} \\ &\quad + h_T^2 \|\nabla \cdot \mathbf{w}_h\|_{L^2(T)} \|p - q_h\|_{L^2(T)} + h_T^2 \|\mathbf{w}_h\|_{H^1(T)} \|\mathbf{f}_T - \mathbf{f}_h\|_{H^{-1}(T)}.\end{aligned}$$

Using an inverse inequality,

$$C h_T \|\mathbf{f}_h + \Delta \mathbf{v}_h - \nabla q_h\|_{L^2(T)} \leqslant \|\nabla(\mathbf{u} - \mathbf{v}_h)\|_{L^2(T)} + \|p - q_h\|_{L^2(T)} + \|\mathbf{f}_T - \mathbf{f}_h\|_{H^{-1}(T)}.$$

It completes the proof of the first inequality.

To prove the second inequality, let $b_e \in H_0^1(\mathcal{T}_e)$ be the face bubble function that takes unit value at the barycentre of the face e. Let ψ be the extension of $[\![\nabla_h \mathbf{v}_h - q_h I]\!]$ to $\mathcal{T}_e$ by constants along the lines orthogonal to the edge e. Let $\mathbf{w}_h = b_e \psi$. Then by scaling,

$$\|\mathbf{w}_h\|_{L^2(\mathcal{T}_e)} \leqslant C \|h_e^{1/2} [\![\nabla_h \mathbf{v}_h - q_h I]\!]\|_{L^2(e)}. \tag{2.7}$$

Again using a scaling argument,

$$\begin{aligned}C \|[\![\nabla_h \mathbf{v}_h - q_h I]\!]\|_{L^2(e)}^2 \leqslant \|b_e^{1/2} [\![\nabla_h \mathbf{v}_h - q_h I]\!]\|_{L^2(e)}^2 &= \int_e \mathbf{w}_h \cdot [\![\nabla_h \mathbf{v}_h - q_h I]\!] \\ &= \sum_{T \subset \mathcal{T}_e} \int_T \nabla \mathbf{v}_h : \nabla \mathbf{w}_h - \sum_{T \subset \mathcal{T}_e} \int_T \nabla \cdot \mathbf{w}_h q_h \\ &\quad + \sum_{T \subset \mathcal{T}_e} \int_T (\Delta \mathbf{v}_h - \nabla q_h) \cdot \mathbf{w}_h \\ &= \sum_{T \subset \mathcal{T}_e} \int_T \nabla(\mathbf{v}_h - \mathbf{u}) : \nabla \mathbf{w}_h - \sum_{T \subset \mathcal{T}_e} \int_T \nabla \cdot \mathbf{w}_h (q_h - p) \\ &\quad + \sum_{T \subset \mathcal{T}_e} \int_T (\Delta \mathbf{v}_h - \nabla q_h + \mathbf{f}_h) \cdot \mathbf{w}_h + (\mathbf{f}_e - \mathbf{f}_h)(\mathbf{w}_h),\end{aligned}$$

where $\mathbf{f}_h(\mathbf{w}_h) = \int_{\mathcal{T}_e} \mathbf{f}_h \cdot \mathbf{w}_h$. Using the Cauchy–Schwarz inequality and an inverse inequality,

$$h_e \| [\![\nabla_h \mathbf{v}_h - q_h I]\!] \|^2_{L^2(e)} \leqslant C(\|\nabla_h(\mathbf{v}_h - \mathbf{u})\|_{L^2(\mathcal{T}_e)} + \|p - q_h\|_{L^2(\mathcal{T}_e)}) \|\mathbf{w}_h\|_{L^2(\mathcal{T}_e)} + C\left(\sum_{T \subset \mathcal{T}_e} h_T^2 \|\Delta \mathbf{v}_h - \nabla q_h + \mathbf{f}_h\|^2_{L^2(T)}\right)^{1/2} \|\mathbf{w}_h\|_{L^2(\mathcal{T}_e)} + C\|\mathbf{f}_e - \mathbf{f}_h\|_{H^{-1}(\mathcal{T}_e)} \|\mathbf{w}_h\|_{L^2(\mathcal{T}_e)}.$$

Using (2.7) and the first inequality of this lemma, we complete the proof. □

We will also need to use the Nedelec H(div) spaces (Nedelec, 1986) of index r. The global space is given by

$$\mathbf{V}_h^N = \{\mathbf{v} \in H(\Omega; \mathrm{div}) : \mathbf{v}|_T \in [\mathbb{P}_r(T)]^d \ \forall T \in \mathcal{T}_h\}.$$

The projection $\Pi_h : [H^1(\Omega)]^d \to \mathbf{V}_h^N$ is defined as follows for every $T \in \mathcal{T}_h$:

$$\int_T (\mathbf{v} - \Pi_h \mathbf{v}) \cdot \mathbf{w} = 0 \quad \forall \mathbf{w} \in \mathbf{N}_{r-1}(T), \tag{2.8}$$

$$\int_e (\mathbf{v} - \Pi_h \mathbf{v}) \cdot \mathbf{n} q = 0 \quad \forall q \in \mathbb{P}_r(e), \ \forall e \subset \partial T, \tag{2.9}$$

where $\mathbf{N}_{r-1}(T)$ is the Nedelec space of index $r-1$ given by

$$\mathbf{N}_{r-1}(T) = [\mathbb{P}_{r-2}(T)]^d + \{\mathbf{v} \in [\tilde{\mathbb{P}}_{r-1}(T)]^d : \mathbf{v} \cdot \mathbf{x} = \mathbf{0}\},$$

and $\tilde{\mathbb{P}}_{r-1}(T)$ is the space of homogeneous polynomials of degree $r-1$.

The following commuting property holds (Nedelec, 1986):

$$\nabla \cdot \Pi_h \mathbf{v} = P^{r-1} \nabla \cdot \mathbf{v}, \tag{2.10}$$

where P^{r-1} is the L^2 projection onto discontinuous polynomials of degree r; that is, $P^{r-1} p|_T \in \mathbb{P}_{r-1}(T)$ is defined by

$$\int_T P^{r-1} p q \, \mathrm{d}x = \int_T p q \, \mathrm{d}x \quad \forall q \in \mathbb{P}_{r-1}(T).$$

In addition, the following bound holds:

$$\|\Pi_h \mathbf{v}\|_h \leqslant C \|\mathbf{v}\|_{H^1(\Omega)} \quad \forall \mathbf{v} \in [H_0^1(\Omega)]^d. \tag{2.11}$$

3. DG methods

We now describe the DG methods that we will consider. For simplicity we only consider the symmetric interior penalty formulation for the Laplace operator (see the form $\mathcal{A}_h$ below), but the analysis can readily be applied to other methods found in Arnold *et al.* (2002). Define

$$\mathcal{A}_h(\mathbf{w}_h, \mathbf{v}_h) = \sum_{T \in \mathcal{T}_h} \int_T \nabla \mathbf{w}_h : \nabla \mathbf{v}_h - \sum_{e \in \mathcal{E}_h} \int_e \{\nabla_h \mathbf{w}_h\} : [\![\underline{\mathbf{v}_h}]\!]$$
$$- \sum_{e \in \mathcal{E}_h} \int_e \{\nabla_h \mathbf{v}_h\} : [\![\underline{\mathbf{w}_h}]\!] + \sum_{e \in \mathcal{E}_h} \int_e \frac{\eta}{h_e} [\![\underline{\mathbf{w}_h}]\!] : [\![\underline{\mathbf{v}_h}]\!], \tag{3.1}$$

where η is a stabilization (algorithmic) parameter, and

$$\mathcal{B}_h(\mathbf{v}_h, q_h) = -\sum_{T \in \mathcal{T}_h} \int_T \nabla \cdot \mathbf{v}_h q_h + \sum_{e \in \mathcal{E}_h} \int_e \{q_h\} [\![\mathbf{v}_h]\!]. \tag{3.2}$$

The DG method we consider is to find $[\mathbf{u}_h, p_h] \in \mathbf{V}_h \times Q_h$ such that

$$\mathcal{A}_h(\mathbf{u}_h, \mathbf{v}_h) + \mathcal{B}_h(\mathbf{v}_h, p_h) = \mathbf{f}(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{3.3a}$$
$$\mathcal{B}_h(\mathbf{u}_h, q_h) - \mathcal{S}_h(p_h, q_h) = 0 \quad \forall q_h \in Q_h, \tag{3.3b}$$

where $\mathcal{S}_h$ is a semipositive-definite bilinear form, i.e., $\mathcal{S}_h(q_h, q_h) \geqslant 0$ for all $q_h \in Q_h$. In the case $Q_h = Q_h^{r-1}$, we set $\mathcal{S}_h \equiv 0$. In the case $Q_h = Q_h^r$ (i.e., equal-order case), we set

$$\mathcal{S}_h(q, p) = \sum_{e \in \mathcal{E}_h^i} h_e \int_e [\![q]\!] \cdot [\![p]\!].$$

The reason why extra stabilization is needed for the equal-order case is to control the higher modes of the approximate pressure. In a later section we describe the method where we penalize the total flux. The error estimates will be slightly different.

Note that we are implicitly assuming that $\mathbf{f}(\mathbf{v}_h)$ makes sense for all $\mathbf{v}_h \in \mathbf{V}_h$. This is certainly not the case if we only assume that $\mathbf{f} \in [H^{-1}(\Omega)]^d$. From now on we assume that $\mathbf{f} \in [H^{-1}(\Omega) \cap L^1(\Omega)]^d$.

For the rest of the analysis, we define $\mathbf{Z}_h = \{\mathbf{v}_h \in \mathbf{V}_h : \mathcal{B}_h(\mathbf{v}_h, q_h) = 0 \, \forall q_h \in Q_h\}$. Since $\nabla \cdot \mathbf{u} \equiv 0$ from (2.10) we see that $\nabla \cdot \Pi_h \mathbf{u} \equiv 0$ and so $\Pi_h \mathbf{u}$ belongs to $\mathbf{Z}_h$. Of course, we also used that $\Pi_h \mathbf{u}$ is in $H(\mathrm{div})$ and its normal components vanish on $\partial\Omega$.

Therefore, if $\mathbf{u} \in [H^{1+s}(\Omega)]^d$ for any $s \geqslant 0$ we have

$$\inf_{\mathbf{v}_h \in \mathbf{Z}_h} \|\mathbf{u} - \mathbf{v}_h\|_h \leqslant \|\mathbf{u} - \Pi_h \mathbf{u}\|_h \leqslant C h^s \|\mathbf{u}\|_{H^{1+s}(\Omega)}. \tag{3.4}$$

We also need the following coercivity estimate. For a sufficiently large stabilizing parameter $\eta > 0$, it holds that

$$C \|\mathbf{v}_h\|_h^2 \leqslant \mathcal{A}_h(\mathbf{v}_h, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{3.5}$$

In order to describe the error for the velocity, we need to define $\mathrm{Osc}(\mathbf{f}, \Omega)$. Let $P : [L^1(\Omega)]^d \to \mathbf{V}_h$ be the L^2 orthogonal projection onto $\mathbf{V}_h$; that is,

$$\int_\Omega (P\mathbf{f} - \mathbf{f}) \cdot \mathbf{w} = 0, \quad \mathbf{w} \in \mathbf{V}_h.$$

Note that this is well defined for functions $\mathbf{f} \in [L^1(\Omega)]^d$. Let us also define

$$\mathrm{Osc}(\mathbf{f}, \Omega) := \left(\sum_{e \in \mathcal{E}_h} \|\mathbf{f}_e - P\mathbf{f}\|^2_{H^{-1}(\mathcal{T}_e)} \right)^{1/2}.$$

The following obvious estimate is useful to note for the subsequent analysis:

$$\sum_{T \subset \mathcal{T}_e} \|\mathbf{f}_T - P\mathbf{f}\|^2_{H^{-1}(T)} \leqslant C \|\mathbf{f}_e - P\mathbf{f}\|^2_{H^{-1}(\mathcal{T}_e)}.$$

We can now prove an estimate for the velocity.

THEOREM 3.1 It holds that

$$\|\mathbf{u} - \mathbf{u}_h\|_h + \mathcal{S}_h(p_h, p_h)^{1/2} \leqslant C \left(\inf_{\mathbf{v}_h \in \mathbf{Z}_h} \|\mathbf{u} - \mathbf{v}_h\|_h + \mathrm{Osc}(\mathbf{f}, \Omega) \right) + C \inf_{q_h \in Q_h} (\|p - q_h\|_{L^2(\Omega)} + \mathcal{S}_h(q_h, q_h)^{1/2}).$$

Proof. Let $\mathbf{v}_h \in \mathbf{Z}_h$ be arbitrary. Set $\mathbf{w}_h = \mathbf{u}_h - \mathbf{v}_h$. Then by using (3.5) and (3.3b),

$$\begin{aligned} C\|\mathbf{w}_h\|_h^2 &\leqslant \mathcal{A}_h(\mathbf{u}_h, \mathbf{w}_h) - \mathcal{A}_h(\mathbf{v}_h, \mathbf{w}_h) \\ &= \mathbf{f}(\mathbf{w}_h) - \mathcal{B}_h(\mathbf{w}_h, p_h) - \mathcal{A}_h(\mathbf{v}_h, \mathbf{w}_h) \\ &= \mathbf{f}(\mathbf{w}_h) - \mathcal{B}_h(\mathbf{w}_h, q_h) - \mathcal{A}_h(\mathbf{v}_h, \mathbf{w}_h) - \mathcal{S}_h(p_h, p_h) + \mathcal{S}_h(p_h, q_h). \end{aligned}$$

Therefore,

$$\begin{aligned} C\|\mathbf{w}_h\|_h^2 + \mathcal{S}_h(p_h, p_h) &\leqslant \mathbf{f}(\mathbf{w}_h) - \mathcal{B}_h(\mathbf{w}_h, q_h) - \mathcal{A}_h(\mathbf{v}_h, \mathbf{w}_h) + \mathcal{S}_h(p_h, q_h) \\ &= \mathbf{f}(\mathbf{w}_h - E_h\mathbf{w}_h) - \mathcal{B}_h(\mathbf{w}_h - E_h\mathbf{w}_h, q_h) - \mathcal{A}_h(\mathbf{v}_h, \mathbf{w}_h - E_h\mathbf{w}_h) \\ &\quad + \mathbf{f}(E_h\mathbf{w}_h) - \mathcal{B}_h(E_h\mathbf{w}_h, q_h) - \mathcal{A}_h(\mathbf{v}_h, E_h\mathbf{w}_h) + \mathcal{S}_h(p_h, q_h) \\ &= \mathbf{f}(\mathbf{w}_h - E_h\mathbf{w}_h) - \mathcal{B}_h(\mathbf{w}_h - E_h\mathbf{w}_h, q_h) - \mathcal{A}_h(\mathbf{v}_h, \mathbf{w}_h - E_h\mathbf{w}_h) \\ &\quad + a(\mathbf{u}, E_h\mathbf{w}_h) + b(E_h\mathbf{w}_h, p) - \mathcal{B}_h(E_h\mathbf{w}_h, q_h) - \mathcal{A}_h(\mathbf{v}_h, E_h\mathbf{w}_h) \\ &\quad + \mathcal{S}_h(p_h, q_h). \end{aligned}$$

First note that

$$|a(\mathbf{u}, E_h\mathbf{w}_h) - \mathcal{A}_h(\mathbf{v}_h, E_h\mathbf{w}_h)| \leqslant C\|\mathbf{u} - \mathbf{v}_h\|_h \|\mathbf{w}_h\|_h, \tag{3.6}$$

$$|b(E_h\mathbf{w}_h, p) - \mathcal{B}_h(E_h\mathbf{w}_h, q_h)| \leqslant C\|p - q_h\|_{L^2(\Omega)} \|\mathbf{w}_h\|_h, \tag{3.7}$$

$$|\mathcal{S}_h(p_h, q_h)| \leqslant \mathcal{S}_h(p_h, p_h)^{1/2} \mathcal{S}_h(q_h, q_h)^{1/2}. \tag{3.8}$$

Using integration by parts,

$$-\mathcal{B}_h(\mathbf{w}_h - E_h\mathbf{w}_h, q_h) = -\sum_{T \in \mathcal{T}_h} \int_T \nabla q_h \cdot (\mathbf{w}_h - E_h\mathbf{w}_h) + \sum_{e \in \mathcal{E}_h^i} \int_e [\![q_h]\!] \cdot \{\mathbf{w}_h - E_h\mathbf{w}_h\} \tag{3.9}$$

and

$$-\mathcal{A}_h(\mathbf{v}_h, \mathbf{w}_h - E_h\mathbf{w}_h) = \sum_{T\in\mathcal{T}_h}\int_T \Delta\mathbf{v}_h \cdot (\mathbf{w}_h - E_h\mathbf{w}_h) - \sum_{e\in\mathcal{E}_h^i}\int_e [\![\nabla_h\mathbf{v}_h]\!] \cdot \{\mathbf{w}_h - E_h\mathbf{w}_h\}$$
$$+ \sum_{e\in\mathcal{E}_h}\int_e \{\nabla_h(\mathbf{w}_h - E_h\mathbf{w}_h)\} : [\![\underline{\mathbf{v}_h}]\!] - \sum_{e\in\mathcal{E}_h}\int_e \frac{\eta}{h_e}[\![\underline{\mathbf{v}_h}]\!] : [\![\underline{\mathbf{w}_h - E_h\mathbf{w}_h}]\!]. \quad (3.10)$$

Hence,

$$\begin{aligned}
&\mathbf{f}(\mathbf{w}_h - E_h\mathbf{w}_h) - \mathcal{B}_h(\mathbf{w}_h - E_h\mathbf{w}_h, q_h) - \mathcal{A}_h(\mathbf{v}_h, \mathbf{w}_h - E_h\mathbf{w}_h)\\
&= \sum_{T\in\mathcal{T}_h}\int_T (P\mathbf{f} + \Delta\mathbf{v}_h - \nabla q_h)\cdot(\mathbf{w}_h - E_h\mathbf{w}_h)\\
&\quad - \sum_{e\in\mathcal{E}_h^i}\int_e [\![\nabla_h\mathbf{v}_h - q_h I]\!] \cdot \{\mathbf{w}_h - E_h\mathbf{w}_h\}\\
&\quad + \sum_{e\in\mathcal{E}_h}\int_e \{\nabla_h(\mathbf{w}_h - E_h\mathbf{w}_h)\} : [\![\underline{\mathbf{v}_h}]\!] - \sum_{e\in\mathcal{E}_h}\int_e \frac{\eta}{h_e}[\![\underline{\mathbf{v}_h}]\!] : [\![\underline{\mathbf{w}_h - E_h\mathbf{w}_h}]\!].
\end{aligned}$$

By using Cauchy–Schwarz, inequality (2.1) and Lemma 2.2, we find

$$\begin{aligned}
&|\mathbf{f}(\mathbf{w}_h - E_h\mathbf{w}_h) - \mathcal{B}_h(\mathbf{w}_h - E_h\mathbf{w}_h, q_h) - \mathcal{A}_h(\mathbf{v}_h, \mathbf{w}_h - E_h\mathbf{w}_h)|\\
&\quad \leqslant C(\|\mathbf{u} - \mathbf{v}_h\|_h + \|p - q_h\|_{L^2(\Omega)} + \mathrm{Osc}(\mathbf{f}, \Omega))\|\mathbf{w}_h\|_h.
\end{aligned} \quad (3.11)$$

Altogether, we complete the proof. □

A few remarks are in order. In the equal-order case we see that

$$\|\mathbf{u} - \mathbf{u}_h\|_h + \mathcal{S}_h(p_h, p_h)^{1/2} \leqslant C\left(\inf_{\mathbf{v}_h\in\mathbf{Z}_h}\|\mathbf{u} - \mathbf{v}_h\|_h + \mathrm{Osc}(\mathbf{f}, \Omega)\right)$$
$$+ C\inf_{q_h\in Q_h\cap H^1(\Omega)}\|p - q_h\|_{L^2(\Omega)}. \quad (3.12)$$

Note that in the last term we are taking the infimum over $Q_h \cap H^1(\Omega)$. In the case $Q_h = Q_h^{r-1}$ we can take the infimum over Q_h. In the next section we modify the method in the equal-order case in order to improve this result.

In order to prove convergence of the DG methods under consideration we will have to argue that $\mathrm{Osc}(\mathbf{f}, \Omega)$ approaches zero as h tends to zero. We, however, are not able to show this under the assumption that $\mathbf{f} \in [H^{-1}(\Omega) \cap L^1(\Omega)]^d$. Therefore, we prove convergence requiring more regularity on $\mathbf{f}$.

To this end, we establish the following inequality:

$$\mathrm{Osc}(\mathbf{f}, \Omega) \leqslant C\|\mathbf{f} - P\mathbf{f}\|_{H^{-1}(\Omega)}. \quad (3.13)$$

A similar result was proved in Cohen *et al.* Here, we give an alternative proof for completeness. In order to prove (3.13), we recall the following theorem (Grisvard, 1985).

THEOREM 3.2 For $\tilde{\mathbf{f}} \in [H^{-1}(\Omega)]^d$, there exists $\mathbf{F} = [F_{ij}] \in [L^2(\Omega)]^{d\times d}$ such that

$$\tilde{\mathbf{f}}(\mathbf{v}) = \int_\Omega \mathbf{F} : \nabla \mathbf{v} \quad \forall \mathbf{v} \in \mathbf{V},$$

with

$$\|\tilde{\mathbf{f}}\|_{H^{-1}(\Omega)} = \|\mathbf{F}\|_{L^2(\Omega)}.$$

Proof of (3.13). Let $\mathbf{v} \in H_0^1(\mathcal{T}_e)$ and let $\tilde{\mathbf{v}}$ be the extension of $\mathbf{v}$ by zero outside $\mathcal{T}_e$. Using Theorem 3.2 with $\tilde{\mathbf{f}} = \mathbf{f} - P\mathbf{f}$,

$$(\mathbf{f}_e - P\mathbf{f})(\mathbf{v}) = (\mathbf{f} - P\mathbf{f})(\tilde{\mathbf{v}}) = \int_\Omega \mathbf{F} : \nabla \tilde{\mathbf{v}} = \int_{\mathcal{T}_e} \mathbf{F} : \nabla \mathbf{v} \quad \forall \mathbf{v} \in [H_0^1(\mathcal{T}_e)]^d,$$

we find

$$\|\mathbf{f}_e - P\mathbf{f}\|^2_{H^{-1}(\mathcal{T}_e)} \leqslant \|\mathbf{F}\|^2_{L^2(\mathcal{T}_e)}.$$

Therefore, the proof of (3.13) follows by summing over $e \in \mathcal{E}_h$ and using Theorem 3.2. □

Below we prove that $\|\mathbf{f} - P\mathbf{f}\|_{H^{-1}(\Omega)}$ approaches 0 as h tends to 0 whenever $\mathbf{f} \in [L^p(\Omega)]^d$ for $p > 1$ when $d = 2$, and for $p \geqslant 6/5$ when $d = 3$; these are exactly the assumptions made in Di Pietro & Ern (2010).

First of all note that $\|P\mathbf{f}\|_{L^p(\Omega)} \leqslant C\|\mathbf{f}\|_{L^p(\Omega)}$. Then

$$\|\mathbf{f} - P\mathbf{f}\|_{H^{-1}(\Omega)} = \sup_{\mathbf{v}\in[H_0^1(\Omega)]^d, \mathbf{v}\neq 0} \frac{(\mathbf{f} - P\mathbf{f})(\mathbf{v})}{\|\mathbf{v}\|_{H_0^1(\Omega)}}$$

and

$$\begin{aligned}(\mathbf{f} - P\mathbf{f})(\mathbf{v}) = (\mathbf{f} - P\mathbf{f})(\mathbf{v} - P\mathbf{v}) &\leqslant \|\mathbf{f} - P\mathbf{f}\|_{L^p(\Omega)}\|\mathbf{v} - P\mathbf{v}\|_{L^q(\Omega)} \\ &\leqslant C\|\mathbf{f} - P\mathbf{f}\|_{L^p(\Omega)} h^{1-d(1/2-1/q)}\|\mathbf{v}\|_{H_0^1(\Omega)},\end{aligned}$$

where p and q are such that $1/p + 1/q = 1$. Therefore,

$$\|\mathbf{f} - P\mathbf{f}\|_{H^{-1}(\Omega)} \leqslant Ch^{1-d(1/2-1/q)}\|\mathbf{f} - P\mathbf{f}\|_{L^p(\Omega)}.$$

Thus,

$$\mathrm{Osc}(\mathbf{f}, \Omega) \leqslant Ch^{1-d(1/2-1/q)}\|\mathbf{f} - P\mathbf{f}\|_{L^p(\Omega)}. \tag{3.14}$$

The following corollary is a simple consequence of the previous results.

COROLLARY 3.3 Assume $\mathbf{f} \in [L^p(\Omega)]^d$ for $p > 1$ when $d = 2$, and for $p > 6/5$ when $d = 3$. Then, $\|\mathbf{u} - \mathbf{u}_h\|_h + \mathcal{S}_h(p_h, p_h)^{1/2}$ converges to zero as h approaches zero. Moreover, if we assume $[\mathbf{u}, p] \in H^{1+s}(\Omega) \times H^s(\Omega)$ for $0 \leqslant s \leqslant r$ and $\mathbf{f} \in W^{\ell,p}(\Omega)$ for an integer $0 \leqslant \ell \leqslant r + 1$, then we have

$$\|\mathbf{u} - \mathbf{u}_h\|_h + \mathcal{S}_h(p_h, p_h)^{1/2} \leqslant Ch^s(\|\mathbf{u}\|_{H^{1+s}(\Omega)} + \|p\|_{H^s(\Omega)}) + Ch^{1+\ell-d(1/2-1/q)}\|\mathbf{f}\|_{W^{\ell,p}(\Omega)}. \tag{3.15}$$

Proof. Using (3.12) and (3.14) we have

$$\|\mathbf{u}-\mathbf{u}_h\|_h + \mathcal{S}_h(p_h,p_h)^{1/2} \leqslant C \inf_{\mathbf{v}_h\in\mathbf{Z}_h} \|\mathbf{u}-\mathbf{v}_h\|_h + C \inf_{q_h\in Q_h} (\|p-q_h\|_{L^2(\Omega)} + S_h(q_h,q_h)^{1/2}) + Ch^{1-d(1/2-1/q)}\|\mathbf{f}-P\mathbf{f}\|_{L^p(\Omega)}.$$

The inequality (3.15) follows using approximation properties of $\mathbf{V}_h$ and Q_h (or $Q_h \cap H_0^1$ when $Q_h = Q_h^r$). □

4. Pressure error estimates

Next, we derive the error estimate in the approximation of pressure.

THEOREM 4.1 If $Q_h = Q_h^{r-1}$, it holds

$$\|p-p_h\|_{L_2(\Omega)} \leqslant C\left(\inf_{\mathbf{v}\in\mathbf{Z}_h}\|\mathbf{u}-\mathbf{v}_h\|_h + \|p-P^{r-1}p\|_{L^2(\Omega)} + \operatorname{Osc}(\mathbf{f},\Omega)\right), \tag{4.1}$$

and if $Q_h = Q_h^r$, it holds

$$\|p-p_h\|_{L_2(\Omega)} \leqslant C\left(\inf_{\mathbf{v}\in\mathbf{Z}_h}\|\mathbf{u}-\mathbf{v}_h\|_h + \|p-P^{r-1}p\|_{L^2(\Omega)} + \operatorname{Osc}(\mathbf{f},\Omega)\right) + C\inf_{q_h\in Q_h}(\|p-q_h\|_{L^2(\Omega)} + \mathcal{S}_h(q_h,q_h)^{1/2}), \tag{4.2}$$

where P^{r-1} is the L^2 projection onto the space of piecewise polynomials of degree $r-1$.

Proof. **Step 1:** We first prove

$$\|P^{r-1}(p-p_h)\|_{L_2(\Omega)} \leqslant C(\|\mathbf{u}-\mathbf{u}_h\|_h + \|p-P^{r-1}p\|_{L^2(\Omega)} + \operatorname{Osc}(\mathbf{f},\Omega)). \tag{4.3}$$

Note that this will prove the theorem in the case of $Q_h = Q_h^{r-1}$ (i.e., (4.1)).

To this end, let $m_h = P^{r-1}p$ and let $\theta_h = P^{r-1}(p-p_h)$. Then it is well known (see, for example, Acosta *et al.*, 2006) that there exists $\mathbf{v}\in[H_0^1(\Omega)]^d$ such that

$$\nabla\cdot\mathbf{v} = \theta_h \quad \text{in } \Omega, \tag{4.4}$$

$$\|\mathbf{v}\|_{H^1(\Omega)} \leqslant C\|\theta_h\|_{L_2(\Omega)}. \tag{4.5}$$

By (2.10) we have that $\nabla\cdot\Pi_h\mathbf{v} = \theta_h$. In addition, $\Pi_h\mathbf{v}\cdot\mathbf{n} = \mathbf{0}$ on $\partial\Omega$ since $\mathbf{v}$ vanishes on $\partial\Omega$.

We also set $I_h : [H_0^1(\Omega)]^d \to \mathbf{V}_h \cap [H_0^1(\Omega)]^d$ to be the Scott–Zhang interpolant of degree r (Scott & Zhang, 1990). We will also use the bound

$$\|I_h\mathbf{v}\|_{H^1(\Omega)} \leqslant C\|\mathbf{v}\|_{H^1(\Omega)} \quad \forall\, \mathbf{v}\in[H_0^1(\Omega)]^d. \tag{4.6}$$

Then,

$$\|\theta_h\|^2_{L^2(\Omega)} = \int_\Omega (\nabla\cdot\Pi_h\mathbf{v})\theta_h = -\mathcal{B}_h(\Pi_h\mathbf{v},\theta_h) = \mathcal{B}_h(\Pi_h\mathbf{v},p_h) - \mathcal{B}_h(\Pi_h\mathbf{v},m_h)$$

$$
\begin{aligned}
&= -\mathcal{A}_h(\mathbf{u}_h, \Pi_h\mathbf{v}) + \mathbf{f}(\Pi_h\mathbf{v}) - \mathcal{B}_h(\Pi_h\mathbf{v}, m_h) \\
&= -\mathcal{A}_h(\mathbf{u}_h, \Pi_h\mathbf{v}) + \mathbf{f}(\Pi_h\mathbf{v}) - \mathbf{f}(I_h\mathbf{v}) + a(\mathbf{u}, I_h\mathbf{v}) + b(I_h\mathbf{v}, p) - \mathcal{B}_h(\Pi_h\mathbf{v}, m_h) \\
&= -\mathcal{A}_h(\mathbf{u}_h, \Pi_h\mathbf{v}) + a(\mathbf{u}, I_h\mathbf{v}) - \mathbf{f}(I_h\mathbf{v} - \Pi_h\mathbf{v}) - \mathcal{B}_h(\Pi_h\mathbf{v} - I_h\mathbf{v}, m_h),
\end{aligned}
$$

where we used that $b(I_h\mathbf{v}, p - m_h) = 0$ since $\nabla \cdot I_h\mathbf{v}$ is a piecewise polynomial of degree $r-1$. Using the definition of $\mathcal{A}_h$ we get

$$
\begin{aligned}
-\|\theta_h\|^2_{L^2(\Omega)} &= \sum_{T\in\mathcal{T}_h} \int_T \nabla\mathbf{u}_h : \nabla\Pi_h\mathbf{v} - \sum_{e\in\mathcal{E}_h} \int_e \{\nabla_h\mathbf{u}_h\} : [\![\underline{\Pi_h\mathbf{v}}]\!] \\
&\quad - \sum_{e\in\mathcal{E}_h} \int_e \{\nabla_h\Pi_h\mathbf{v}\} : [\![\underline{\mathbf{u}_h}]\!] + \sum_{e\in\mathcal{E}_h} \int_e \frac{\eta}{h_e} [\![\underline{\mathbf{u}_h}]\!] : [\![\underline{\Pi_h\mathbf{v}}]\!] \\
&\quad - a(\mathbf{u}, I_h\mathbf{v}) + \mathbf{f}(I_h\mathbf{v} - \Pi_h\mathbf{v}) + \mathcal{B}_h(\Pi_h\mathbf{v} - I_h\mathbf{v}, m_h) \\
&= \sum_{T\in\mathcal{T}_h} \int_T \nabla\mathbf{u}_h : \nabla(\Pi_h\mathbf{v} - I_h\mathbf{v}) - \sum_{e\in\mathcal{E}_h} \int_e \{\nabla_h\mathbf{u}_h\} : [\![\underline{\Pi_h\mathbf{v} - I_h\mathbf{v}}]\!] \\
&\quad - \sum_{e\in\mathcal{E}_h} \int_e \{\nabla_h\Pi_h\mathbf{v}\} : [\![\underline{\mathbf{u}_h}]\!] + \sum_{e\in\mathcal{E}_h} \int_e \frac{\eta}{h_e} [\![\underline{\mathbf{u}_h}]\!] : [\![\underline{\Pi_h\mathbf{v}}]\!] \\
&\quad + (\nabla_h(\mathbf{u}_h - \mathbf{u}), \nabla(I_h\mathbf{v})) + \mathbf{f}(I_h\mathbf{v} - \Pi_h\mathbf{v}) + \mathcal{B}_h(\Pi_h\mathbf{v} - I_h\mathbf{v}, m_h).
\end{aligned}
$$

Using integration by parts we see that

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_h} \int_T \nabla\mathbf{u}_h : \nabla(\Pi_h\mathbf{v} - I_h\mathbf{v}) - \sum_{e\in\mathcal{E}_h} \int_e \{\nabla_h\mathbf{u}_h\} : [\![\underline{\Pi_h\mathbf{v} - I_h\mathbf{v}}]\!] \\
&\quad + P\mathbf{f}(I_h\mathbf{v} - \Pi_h\mathbf{v}) + \mathcal{B}_h(\Pi_h\mathbf{v} - I_h\,\mathbf{v}, m_h) \\
&= -\sum_{T\in\mathcal{T}_h} \int_T (P\mathbf{f} + \Delta\mathbf{u}_h - \nabla m_h) \cdot (\Pi_h\mathbf{v} - I_h\mathbf{v})\,\mathrm{d}x \\
&\quad + \sum_{e\in\mathcal{E}_h^i} \int_e [\![\nabla_h\mathbf{u}_h - m_h I]\!] \cdot \{I_h\mathbf{v} - \Pi_h\mathbf{v}\}\,\mathrm{d}s.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
-\|\theta_h\|^2_{L^2(\Omega)} &= -\sum_{T\in\mathcal{T}_h} \int_T (P\mathbf{f} + \Delta\mathbf{u}_h - \nabla m_h) \cdot (\Pi_h\mathbf{v} - I_h\,\mathbf{v})\,\mathrm{d}x \\
&\quad + \sum_{e\in\mathcal{E}_h^i} \int_e [\![\nabla_h\mathbf{u}_h - m_h I]\!] \cdot \{I_h\mathbf{v} - \Pi_h\mathbf{v}\}\,\mathrm{d}s \\
&\quad - \sum_{e\in\mathcal{E}_h} \int_e \{\nabla_h\Pi_h\mathbf{v}\} : [\![\underline{\mathbf{u}_h}]\!] + \sum_{e\in\mathcal{E}_h} \int_e \frac{\eta}{h_e} [\![\underline{\mathbf{u}_h}]\!] : [\![\underline{\Pi_h\mathbf{v}}]\!].
\end{aligned}
$$

Using inverse estimates, Lemma 2.2 and the bounds (2.11), (4.5) and (4.6), we arrive at (4.3).

Step 2: We next prove that if $e \in \mathcal{E}_h$ and $e \subset T$ with $T \in \mathcal{T}_h$, then

$$h_e^{1/2}\|(p_h - P^{r-1}p_h)|_T\|_{L^2(e)} \leqslant C\left(\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L^2(T)} + \frac{\eta}{h_T^{1/2}}\|[\![\underline{\mathbf{u}_h}]\!]\|_{L^2(\partial T)}\right) + C(\|p - P^{r-1}p\|_{L^2(T)} + \|\mathbf{f}_e - P\mathbf{f}\|_{H^{-1}(\mathcal{T}_e)}) + C(\|p - q_h\|_{L^2(\mathcal{T}_e)} + h_e^{1/2}\|[\![p_h - q_h]\!]\|_{L^2(e)}), \tag{4.7}$$

where $q_h \in Q_h$ is arbitrary.

Fix an $e \in \mathcal{E}_h$ and find one $T \in \mathcal{T}_h$ such that $e \subset \partial T$. We define $\mathbf{v} \in \mathbf{V}_h$ in the following way so that $\mathbf{v}|_{\Omega\setminus T} \equiv 0$. Using the degrees of freedom of the Nedelec (1986) space, define $\mathbf{v}|_T \in [\mathbb{P}_r(T)]^d$ so that

$$\int_e (\mathbf{v}\cdot\mathbf{n} - (p_h - P^{r-1}p_h))q = 0 \quad \forall\, q \in \mathbb{P}_r(e),$$

$$\int_{e'} \mathbf{v}\cdot\mathbf{n}q = 0 \quad \forall\, q \in \mathbb{P}_r(e')\forall \text{ edges } e' \subset T \text{ and } e' \neq e,$$

$$\int_T \mathbf{v}\cdot\mathbf{w} = 0 \quad \forall\, \mathbf{w} \in \mathbf{N}^{r-1}(T),$$

where (by abuse of notation) in the integrations over ∂T we consider the traces of the piecewise discontinuous functions taken from the interior of T.

A scaling argument gives

$$\|\mathbf{v}\|_{L^2(T)} \leqslant C h_T^{1/2}\|p_h - P^{r-1}p_h\|_{L^2(e)}, \tag{4.8}$$

with the constant C depending on r but not on h. Note that

$$\begin{aligned}
\mathcal{B}_h(p_h, \mathbf{v}) &= -\int_T p_h \nabla\cdot\mathbf{v} + \int_e \{p_h\}\mathbf{v}\cdot\mathbf{n} \\
&= -\int_T P^{r-1}p_h\nabla\cdot\mathbf{v} + \int_e p_h\mathbf{v}\cdot\mathbf{n} - \frac{1}{2}\int_e [\![p_h]\!]\cdot\mathbf{n}(\mathbf{v}\cdot\mathbf{n}) \\
&= \int_T \nabla(P^{r-1}p_h)\cdot\mathbf{v} + \int_e (p_h - P^{r-1}p_h)\mathbf{v}\cdot\mathbf{n} - \frac{1}{2}\int_e [\![p_h]\!]\cdot\mathbf{n}(\mathbf{v}\cdot\mathbf{n}) \\
&= \|p_h - P^{r-1}p_h\|^2_{L^2(e)} + \int_T \nabla(P^{r-1}p_h)\cdot\mathbf{v} - \frac{1}{2}\int_e [\![p_h]\!]\cdot\mathbf{n}(\mathbf{v}\cdot\mathbf{n}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|p_h - P^{r-1}p_h\|^2_{L^2(e)} &= -\mathcal{A}_h(\mathbf{u}_h, \mathbf{v}) + \mathbf{f}(\mathbf{v}) - \int_T \nabla(P^{r-1}p_h)\cdot\mathbf{v} + \frac{1}{2}\int_e [\![p_h]\!]\cdot\mathbf{v} \\
&= \int_T (P\mathbf{f} + \Delta\mathbf{u}_h - \nabla(P^{r-1}p_h))\cdot\mathbf{v} - \frac{1}{2}\int_e [\![\nabla_h\mathbf{u}_h - q_h I]\!]\cdot\mathbf{v} \\
&\quad + \int_{\partial T}\{\nabla_h\mathbf{v}\} : [\![\underline{\mathbf{u}_h}]\!] - \sum_{g\subset\partial T}\int_g \frac{\eta}{h_g}[\![\underline{\mathbf{v}}]\!] : [\![\underline{\mathbf{u}_h}]\!] - \frac{1}{2}\int_e [\![p_h - q_h]\!]\cdot\mathbf{v},
\end{aligned}$$

where $q_h \in Q_h$ is arbitrary. Applying inverse estimates we find

$$\|p_h - P^{r-1}p_h\|^2_{L^2(e)} \leqslant C\left(\|P\mathbf{f} + \Delta\mathbf{u}_h - \nabla(P^{r-1}p_h)\|_{L^2(T)} + h_e^{-1/2}\|[\![\nabla_h\mathbf{u}_h - q_h I]\!]\|_{L^2(e)}\right.$$
$$\left. + \frac{\eta}{h_T^{3/2}}\|[\![\underline{\mathbf{u}_h}]\!]\|_{L^2(\partial T)} + h_e^{-1/2}\|[\![p_h - q_h]\!]\|_{L^2(e)}\right)\|\mathbf{v}\|_{L^2(T)}.$$

Using (4.8) and Lemma 2.2 proves (4.7).

Step 3: Now we combine the previous two steps to finish the proof. By the triangle inequality we have

$$\|p_h - p\|_{L^2(T)} \leqslant \|p - P^{r-1}p\|_{L^2(T)} + \|P^{r-1}(p - p_h)\|_{L^2(T)} + \|P^{r-1}p_h - p_h\|_{L^2(T)}.$$

From a scaling argument we see that

$$\|P^{r-1}p_h - p_h\|_{L^2(T)} \leqslant Ch_T^{1/2}\|P^{r-1}p_h - p_h\|_{L^2(e)},$$

for any edge e of T, with the constant C depending on r but not on h_T. This follows from the fact that $P^{r-1}p_h - p_h \in \mathbb{P}_r(T)$ and that its moments up to degree $r-1$ vanish. If we use (4.3) and (4.7), we see that

$$\|p - p_h\|_{L_2(\Omega)} \leqslant C(\|\mathbf{u} - \mathbf{u}_h\|_h + S_h(p_h, p_h)^{1/2} + \|p - P^{r-1}p\|_{L^2(\Omega)} + \operatorname{Osc}(\mathbf{f}, \Omega))$$
$$+ C\inf_{q_h\in Q_h}(\|p - q_h\|_{L^2(\Omega)} + \mathcal{S}_h(q_h, q_h)^{1/2}).$$

We arrive at (4.2) if we apply Theorem 3.1. □

5. Stabilizing with the full flux in the equal-order case

We consider the equal-order case $Q_h = Q_h^r$ and modify the method (3.3). Define $[\mathbf{u}_h, p_h] \in \mathbf{V}_h \times Q_h$ as the solution to

$$\mathcal{A}_h(\mathbf{u}_h, \mathbf{v}_h) + \mathcal{B}_h(\mathbf{v}_h, p_h) + \sum_{e\in\mathcal{E}_h^i} h_e \int_e [\![\nabla_h\mathbf{u}_h - p_h I]\!] \cdot [\![\nabla_h\mathbf{v}_h]\!] = \mathbf{f}(\mathbf{v}_h) \quad \forall\mathbf{v}_h \in \mathbf{V}_h, \tag{5.1a}$$

$$\mathcal{B}_h(\mathbf{u}_h, q_h) - \sum_{e\in\mathcal{E}_h^i} h_e \int_e [\![\nabla_h\mathbf{u}_h - p_h I]\!] \cdot [\![q_h I]\!] = 0 \quad \forall q_h \in Q_h. \tag{5.1b}$$

THEOREM 5.1 Consider the method defined by (5.1). It holds that

$$\|\mathbf{u} - \mathbf{u}_h\|_h + \left(\sum_{e\in\mathcal{E}_h^i} h_e \int_e [\![\nabla_h\mathbf{u}_h - p_h I]\!]^2\right)^{1/2} \leqslant C\left(\inf_{\mathbf{v}_h\in\mathbf{Z}_h}\|\mathbf{u} - \mathbf{v}_h\|_h + \inf_{q_h\in Q_h}\|p - q_h\|_{L^2(\Omega)}\right)$$
$$+ C\operatorname{Osc}(\mathbf{f}, \Omega).$$

Proof. If we let $\mathbf{w}_h = \mathbf{u}_h - \mathbf{v}_h$ for an arbitrary $\mathbf{v}_h \in \mathbf{Z}_h$ and follow similar arguments as the proof of Theorem 3.1, we arrive at

$$\begin{aligned} C\|\mathbf{w}_h\|_h^2 + \sum_{e\in\mathcal{E}_h^i} h_e \int_e [\![\nabla_h \mathbf{u}_h - p_h I]\!]^2 &\leqslant \mathbf{f}(\mathbf{w}_h - E_h\mathbf{w}_h) - \mathcal{B}_h(\mathbf{w}_h - E_h\mathbf{w}_h, q_h) - \mathcal{A}_h(\mathbf{v}_h, \mathbf{w}_h - E_h\mathbf{w}_h) \\ &\quad + a(\mathbf{u}, E_h\mathbf{w}_h) + b(E_h\mathbf{w}_h, p) - \mathcal{B}_h(E_h\mathbf{w}_h, q_h) - \mathcal{A}_h(\mathbf{v}_h, E_h\mathbf{w}_h) \\ &\quad + \sum_{e\in\mathcal{E}_h^i} h_e \int_e [\![\nabla_h \mathbf{u}_h - p_h I]\!] \cdot [\![\nabla_h \mathbf{v}_h - q_h I]\!]. \end{aligned}$$

Using the bounds in the proof of Theorem 3.1 (i.e., (3.6), (3.7) and (3.11)) we obtain

$$\begin{aligned} C\|\mathbf{w}_h\|_h^2 + \sum_{e\in\mathcal{E}_h^i} h_e \int_e [\![\nabla_h \mathbf{u}_h - p_h I]\!]^2 &\leqslant C(\|\mathbf{u} - \mathbf{v}_h\|_h + \|p - q_h\|_{L^2(\Omega)} + \mathrm{Osc}(\mathbf{f}, \Omega))\|\mathbf{w}_h\|_h \\ &\quad + \sum_{e\in\mathcal{E}_h^i} h_e \int_e [\![\nabla_h \mathbf{v}_h - q_h I]\!]^2. \end{aligned}$$

We complete the proof after applying the second inequality of Lemma 2.2 and the triangle inequality. □

The proof of the following pressure estimate follows a similar argument as in the proof of Theorem 4.1. We leave the details to the reader.

THEOREM 5.2 Consider the method defined by (5.1). The following estimate holds:

$$\|p - p_h\|_{L_2(\Omega)} \leqslant C\left(\inf_{\mathbf{v}_h\in\mathbf{Z}_h} \|\mathbf{u} - \mathbf{v}_h\|_h + \|p - P^{r-1}p\|_{L^2(\Omega)} + \mathrm{Osc}(\mathbf{f}, \Omega)\right).$$

6. A DG method for $\mathbf{f} \in [H^{-1}(\Omega)]^d$

It is clear that the DG method (3.3) is not well defined for every $\mathbf{f} \in [H^{-1}(\Omega)]^d$. In order to have a well-defined method we modify it in the following way:

$$\mathcal{A}_h(\mathbf{u}_h, \mathbf{v}_h) + \mathcal{B}_h(\mathbf{v}_h, p_h) = \mathbf{f}(E_h\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{6.1a}$$

$$\mathcal{B}_h(\mathbf{u}_h, q_h) - \mathcal{S}_h(p_h, q_h) = 0 \quad \forall q_h \in Q_h, \tag{6.1b}$$

where we recall $E_h : V_h \to V_h \cap H_0^1(\Omega)$ satisfies (2.1). There are many choices for E_h, which are normally easy to compute.

We prove the following error estimate.

THEOREM 6.1 Consider the method (6.1). There holds

$$\|\mathbf{u}-\mathbf{u}_h\|_h + \mathcal{S}_h(p_h,p_h)^{1/2} \leqslant C \inf_{\mathbf{v}_h\in\mathbf{Z}_h} \|\mathbf{u}-\mathbf{v}_h\|_h + C\left(\sum_{e\in\mathcal{E}_h} \|\mathbf{f}_e\|^2_{H^{-1}(\mathcal{T}_e)}\right)^{1/2} + C \inf_{q_h\in Q_h} (\|p-q_h\|_{L^2(\Omega)} + \mathcal{S}_h(q_h,q_h)^{1/2}).$$

Note that the difference in this estimate compared with Theorem 3.1 is that $P\mathbf{f}$ does not appear. This will allow us to prove convergence assuming $\mathbf{f}\in[H^{-1}(\Omega)]^d$. Indeed, given $\epsilon>0$ there exists $\tilde{\mathbf{f}}\in[L^2(\Omega)]^d$ such that

$$\|\mathbf{f}-\tilde{\mathbf{f}}\|_{H^{-1}(\Omega)} \leqslant \epsilon.$$

Then,

$$\left(\sum_{e\in\mathcal{E}_h} \|\mathbf{f}_e\|^2_{H^{-1}(\mathcal{T}_e)}\right) \leqslant \left(\sum_{e\in\mathcal{E}_h} \|\mathbf{f}_e-\tilde{\mathbf{f}}\|^2_{H^{-1}(\mathcal{T}_e)}\right) + \left(\sum_{e\in\mathcal{E}_h} \|\tilde{\mathbf{f}}\|^2_{H^{-1}(\mathcal{T}_e)}\right) \leqslant C\|\mathbf{f}-\tilde{\mathbf{f}}\|^2_{H^{-1}(\Omega)} + \left(\sum_{e\in\mathcal{E}_h} \|\tilde{\mathbf{f}}\|^2_{H^{-1}(\mathcal{T}_e)}\right),$$

where we used (3.13). Using Friedrichs' inequality (see, for example, Gilbarg & Trudinger, 1977) while using the fact that $\operatorname{diam}(\mathcal{T}_e)\leqslant Ch$ we have

$$\|\tilde{\mathbf{f}}\|_{H^{-1}(\mathcal{T}_e)} \leqslant Ch\|\tilde{\mathbf{f}}\|_{L^2(\mathcal{T}_e)},$$

and hence we have

$$\left(\sum_{e\in\mathcal{E}_h} \|\tilde{\mathbf{f}}\|^2_{H^{-1}(\mathcal{T}_e)}\right)^{1/2} \leqslant Ch\|\tilde{\mathbf{f}}\|_{L^2(\Omega)}.$$

Therefore, we have

$$\left(\sum_{e\in\mathcal{E}_h} \|\mathbf{f}_e\|^2_{H^{-1}(\mathcal{T}_e)}\right)^{1/2} \leqslant C(\epsilon + h\|\tilde{\mathbf{f}}\|_{L^2(\Omega)}).$$

There exists $h_0>0$ such that for $h<h_0$ we have $h\|\tilde{\mathbf{f}}\|_{L^2(\Omega)}\leqslant\epsilon$.

This shows that $(\sum_{e\in\mathcal{E}_h}\|\mathbf{f}_e\|^2_{H^{-1}(\mathcal{T}_e)})^{1/2}$ converges to zero as h approaches zero. The other terms in Theorem 6.1 approach zero as h approaches zero, again using density arguments, and therefore, we can conclude that method (6.1) converges.

We now prove Theorem 6.1.

Proof of Theorem 6.1. Following the proof of Theorem 3.1 we get

$$C\|\mathbf{w}_h\|_h^2 + \mathcal{S}_h(p_h,p_h) \leqslant -\mathcal{B}_h(\mathbf{w}_h - E_h\mathbf{w}_h, q_h) - \mathcal{A}_h(\mathbf{v}_h, \mathbf{w}_h - E_h\mathbf{w}_h) + a(\mathbf{u}, E_h\mathbf{w}_h) + b(E_h\mathbf{w}_h, p) - \mathcal{B}_h(E_h\mathbf{w}_h, q_h) - \mathcal{A}_h(\mathbf{v}_h, E_h\mathbf{w}_h) + \mathcal{S}_h(p_h, q_h).$$

Using (3.9) and (3.10) we have

$$
\begin{aligned}
-\mathcal{B}_h(\mathbf{w}_h - E_h\mathbf{w}_h, q_h) - \mathcal{A}_h(\mathbf{v}_h, \mathbf{w}_h - E_h\mathbf{w}_h) &= \sum_{T\in\mathcal{T}_h}\int_T (\Delta\mathbf{v}_h - \nabla q_h)\cdot(\mathbf{w}_h - E_h\mathbf{w}_h) \\
&\quad - \sum_{e\in\mathcal{E}_h^i}\int_e [\![\nabla_h\mathbf{v}_h - q_h I]\!]\cdot\{\mathbf{w}_h - E_h\mathbf{w}_h\} \\
&\quad + \sum_{e\in\mathcal{E}_h}\int_e \{\nabla_h(\mathbf{w}_h - E_h\mathbf{w}_h)\} : [\![\mathbf{v}_h]\!] \\
&\quad - \sum_{e\in\mathcal{E}_h}\int_e \frac{\eta}{h_e}[\![\mathbf{v}_h]\!] : [\![\mathbf{w}_h - E_h\mathbf{w}_h]\!].
\end{aligned}
$$

The proof is completed by applying Lemma 2.2 with $\mathbf{f}_h = 0$, (3.6) and (3.7). □

Note that Theorem 6.1 only gives at most an $\mathcal{O}(h)$ convergence rate, even if $\mathbf{f}$ is smooth due to the term $(\sum_{e\in\mathcal{E}_h}\|\mathbf{f}_e\|^2_{H^{-1}(\mathcal{T}_e)})^{1/2}$.

In order to remove this term altogether one can use instead an enrichment operator $E_h : \mathbf{V}_h \to \mathbf{V}_h^s \cap [H_0^1(\Omega)]^d$, where

$$
\mathbf{V}_h^s = \{\mathbf{v}_h \in [L^2(\Omega)]^d : \mathbf{v}_h|_T \in [\mathbb{P}_s(T)]^d\},
$$

with $s > r$. For example, suppose $Q_h = Q_h^{r-1}$ and suppose $d = 2$. Then, we can define $E_h : \mathbf{V}_h \to \mathbf{V}_h^{r+1} \cap [H_0^1(\Omega)]^d$ as follows:

$$
\begin{aligned}
\int_T (E_h\mathbf{w}_h - \mathbf{w}_h)\cdot\mathbf{v} = 0 &\quad \forall\, \mathbf{v} \in [\mathbb{P}_{r-2}(T)]^d \forall\, T \in \mathcal{T}_h, \\
\int_e (E_h\mathbf{w}_h - \{\mathbf{w}_h\})\cdot\mathbf{v} = 0 &\quad \forall\, \mathbf{v} \in [\mathbb{P}_{r-1}(e)]^d \forall \text{ edges } e \subset \mathcal{E}_h^i, \\
E_h\mathbf{w}_h(z) = \frac{1}{|\mathcal{T}_z|}\sum_{T\in\mathcal{T}_z}\mathbf{w}_h|_T(z) &\quad \forall \text{ vertices } z \in \mathcal{V}_h^i, \\
\int_e E_h\mathbf{w}_h\cdot\mathbf{v} = 0 &\quad \forall\, \mathbf{v} \in [\mathbb{P}_{r-1}(e)]^d \forall \text{ edges } e \in \mathcal{E}_h^b, \\
E_h\mathbf{w}_h(z) = 0 &\quad \forall \text{ vertices } z \in \mathcal{V}_h^b,
\end{aligned}
$$

where $|\mathcal{T}_z|$ denotes the cardinality of the set $\mathcal{T}_z$ and $\mathbb{P}_{-1}(T) = \{0\}$ for all $T \in \mathcal{T}_h$. Further, it is not difficult to see that E_h also satisfies (2.1).

In this case, we have

$$
\begin{aligned}
-\mathcal{B}_h(\mathbf{w}_h - E_h\mathbf{w}_h, q_h) - \mathcal{A}_h(\mathbf{v}_h, \mathbf{w}_h - E_h\mathbf{w}_h) &= \sum_{e\in\mathcal{E}_h}\int_e \{\nabla_h(\mathbf{w}_h - E_h\mathbf{w}_h)\} : [\![\mathbf{v}_h]\!] \\
&\quad - \sum_{e\in\mathcal{E}_h}\int_e \frac{\eta}{h_e}[\![\mathbf{v}_h]\!] : [\![\mathbf{w}_h - E_h\mathbf{w}_h]\!].
\end{aligned}
$$

Hence, following the proof of Theorem 6.1 we get the following estimate:

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leqslant C \left(\inf_{\mathbf{v}_h \in \mathbf{Z}_h} \|\mathbf{u} - \mathbf{v}_h\|_h + \inf_{q_h \in Q_h} \|p - q_h\|_{L^2(\Omega)} \right).$$

This shows that best approximation error estimates (modulo a constant) are obtained for DG methods (6.1) by choosing the enrichment operator E_h carefully.

Acknowledgements

T.G. and J.G. are grateful to the Indo-US Virtual Institute of Mathematical and Statistical Sciences for sponsoring a short term visit.

Funding

S.B. was funded by the European Research Council under the FP7 Programme 'Ideas' through the Starting Grant No. 258443 - COMFUS: Computational Methods for Fusion Technology.

References

Acosta, G., Durán, R. & Muschietti, M. A. (2006) Solutions of the divergence operator on John domains. *Adv. Math.*, **206**, 373–401.

Arnold, D. N., Brezzi, F., Cockburn, B. & Marini, L. D. (2002) Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, **39**, 1749–1779.

Boffi, D., Brezzi, F. & Fortin, M. (2008) Finite elements for the Stokes problem. *Mixed Finite Elements, Compatibility Conditions, and Applications* (D. Boffi & L. Gastaldi eds). Lectures given at the C.I.M.E. Summer School held in Cetraro, Italy, 26 June–1 July 2006. Lecture Notes in Mathematics 1939. Berlin: Springer, pp. 45–100.

Brenner, S. C. (1996) Two-level additive Schwarz preconditioners for nonconforming finite element methods. *Math. Comp.*, **65**, 897–921.

Brenner, S. C. (1999) Convergence of nonconforming multigrid methods without full elliptic regularity. *Math. Comp.*, **68**, 25–53.

Brenner, S. C. & Scott, L. R. (2008) *The Mathematical Theory of Finite Element Methods*, 3rd edn. New York: Springer.

Cockburn, B., Kanschat, G., Schötzau, D. & Schwab, C. (2002) Local discontinuous Galerkin methods for the Stokes system. *SIAM J. Numer. Anal.*, **40**, 319–343.

Codina, R. & Badia, S. (2013) On the design of discontinuous Galerkin methods for elliptic problems based on hybrid formulations. *Comput. Meth. Appl. Mech. Eng.*, **263**, 158–168.

Codina, R., Principe, J. & Baiges, J. (2009) Subscales on the element boundaries in the variational two-scale finite element method. *Comput. Methods Appl. Mech. Engrg.*, **198**, 838–852.

Cohen, A., Devore, R. & Nochetto R.H. (2012) Convergence rates for AFEM with H_1 data. *Found. Comput. Math.*, **12**, 671–718.

Di Pietro, D. A. & Ern, A. (2010) Discrete functional analysis tools for discontinuous Galerkin methods with application to the incompressible Navier–Stokes equations. *Math. Comp.*, **79**, 1303–1330.

Gilbarg, D. & Trudinger, N. S. (1977) *Elliptic Partial Differential Equations of Second Order*. Grundlehren der Mathematischen Wissenschaften 224. Berlin: Springer.

Grisvard, P. (1985) *Elliptic Problems in Nonsmooth Domains*. Boston: Pitman.

Gudi, T. (2010a) Some nonstandard error analysis of discontinuous Galerkin methods for elliptic problems. *Calcolo*, **47**, 239–261.

GUDI, T. (2010b) A new error analysis for discontinuous finite element methods for linear elliptic problems. *Math. Comp.*, **79**, 2169–2189.

HANSBO, P. & LARSON, M. G. (2002) Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitsche's method. *Comput. Methods Appl. Mech. Engrg.*, **191**, 1895–1908.

NEDELEC, J.-C. (1986) A new family of mixed finite elements in $\mathbb{R}^3$. *Numer. Math.*, **50**, 57–81.

RIVIÈRE, B. & WIHLER, T. P. (2011) Discontinuous Galerkin methods for second-order elliptic PDE with low-regularity solutions. *J. Sci. Comput.*, **46**, 151–165.

SCHÖTZAU, D., SCHWAB, C. & TOSSELLI, A. (2003) Mixed *hp*-DGFEM for incompressible flows. *SIAM J. Numer. Anal.*, **40**, 2171–2194.

SCOTT, R. & ZHANG, S. (1990) Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, **54**, 483–493.