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STABILITY ANALYSIS OF THE FORWARD EULER SCHEME FOR THE CONVECTION–DIFFUSION EQUATION USING THE SUPG FORMULATION IN SPACE

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SUMMARY

The stability and accuracy of the forward Euler scheme for the semidiscrete problem arising from the space discretization of the convection–diffusion equation using the SUPG formulation are analysed. Both linear and quadratic finite elements are considered. The stability limits are derived for the one-dimensional problem and a heuristic criterion is proposed for the multidimensional equation. Some numerical experiments are conducted in order to assess the performance of different finite elements.

INTRODUCTION AND BACKGROUND

Let $\mathbf{A}$ be an $m \times m$ matrix and $\mathbf{x} = \mathbf{x}(t)$ an m-dimensional vector function of the time variable t. When the numerical solution of a system of linear ordinary differential equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is attempted using a single-step finite difference scheme, the final algebraic system to be solved at each time step will have the form $\mathbf{x}^{n+1} = \mathbf{E}\mathbf{x}^n$, the superscript denoting the time step level and $\mathbf{E}$ being a certain $m \times m$ matrix whose kth power will be denoted by $\mathbf{E}^k$. Since the error $\mathbf{z}$ will also satisfy the difference scheme, we will have that $\mathbf{z}^{n+1} = \mathbf{E}\mathbf{z}^n$. Stability requires that $\|\mathbf{z}^n\|$ must be bounded for any n. Since

$$\|\mathbf{z}^{n+1}\| = \|\mathbf{E}\mathbf{z}^n\| = \|\mathbf{E}^{n+1}\mathbf{z}^0\| \leqslant \|\mathbf{E}\|^{n+1}\|\mathbf{z}^0\|$$

stability will hold whenever

$$\|\mathbf{E}\| \leqslant 1 \tag{1}$$

In fact, one can show that only $\|\mathbf{E}\| \leqslant 1 + O(\Delta t)$ is necessary, [1, 2] where Δt is the time step size. However, in practice one neglects the term $O(\Delta t)$ and what is really checked is condition (1).

The symbol $\|\cdot\|$ used above denotes *any* vector norm when it is applied to a vector and *the induced* matrix norm when it is applied to a matrix. When the differential system comes from the space discretization of a partial differential equation, matrix $\mathbf{A}$ and, hence, $\mathbf{E}$ will depend on the space discretization size h, as well as on the boundary conditions. Checking (1) for any h, Δt, space geometry and boundary conditions is in general intractable. Alternatively, the condition

$$\varrho(\mathbf{E}) \leqslant 1 \tag{2}$$

is verified. Here, $\varrho(\mathbf{E})$ denotes the spectral radius of $\mathbf{E}$ (i.e. the maximum absolute value of the eigenvalues of $\mathbf{E}$). It is known that, for any matrix norm $\|\cdot\|$, $\varrho(\mathbf{E}) \leqslant \|\mathbf{E}\|$. Thus, condition (2) is obviously necessary, but *not sufficient* for stability. It can be proved (see, e.g., Reference 1) that the

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equality holds when $\mathbf{E}$ is symmetric or similar to a symmetric matrix, that is, there exists a non-singular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{EP}$ is symmetric. When the differential system results from the space discretization of the convection–diffusion equation, matrix $\mathbf{E}$ is not symmetric. The use of (2) has caused confusion since it leads to misleading results. See Reference 3 and the discussion originated in References 4 and 5.

Another way to study the stability of finite difference schemes is the von Neumann method, based on a Fourier mode analysis of the error $\mathbf{z}$. It is assumed that this error can be expanded in Fourier series. This requires periodicity of the problem in the space domain. When this condition does not hold, the von Neumann criterion only gives necessary conditions for stability. Nevertheless, experience and also some heuristic considerations show that the necessary condition for stability obtained using the Fourier analysis is much more precise and useful than the one based on the spectral radius (2). An interesting discussion of this fact can be found in References 4 and 5. In the former, several simple cases with different boundary conditions have been studied by checking the stability condition (1) and showing the effectiveness of the von Neumann criterion. It is also argued that the reason why this method works so well in situations where *a priori* it is not sufficient for stability is that instabilities are generated far from the boundary. Thus, boundary conditions do not play a decisive role on the stable or the unstable behaviour of the scheme.

Having these considerations in mind, in this paper we will consider the following initial-boundary-value problem: find a scalar function $\phi = \phi(x, t)$ satisfying the differential equation

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} - k\frac{\partial^2 \phi}{\partial x^2} = 0, \quad 0 < x < l, t > 0 \tag{3}$$

as well as the initial condition

$$\phi(x, 0) = \phi^0(x), \quad 0 < x < l \tag{4}$$

and the periodic boundary conditions

$$\phi(0, t) = \phi(l, t) \quad \text{and} \quad \frac{\partial \phi}{\partial x}(0, t) = \frac{\partial \phi}{\partial x}(l, t) \tag{5}$$

for $t > 0$. The constant diffusion k is positive and, without loss of generality, we will assume the constant velocity u to be also positive. For the sake of clarity, source terms have been omitted. The stability of the forward Euler scheme in time and the SUPG (streamline-upwind/Petrov–Galerkin) method in space will be analysed for both linear and quadratic finite elements using mainly the von Neumann method. This analysis is obviously restricted to this simple one-dimensional problem using a uniform finite element partition. Nevertheless, it gives a critical time step that provides an estimate of what might be used in general situations. The extension to multidimensional problems will be briefly discussed later.

Let $N(x)$ denote a generic shape function. It is known (see, e.g., Reference 6) that the SUPG method for problem (3)–(5) consists in taking the weighting functions as $N(x)$ plus a perturbation $P(x)$ of the form

$$P(x) = \frac{\alpha h}{2}\frac{\mathrm{d}N}{\mathrm{d}x}$$

where α is the upwind function, depending on the Péclet number $\gamma := uh/2k$. In Reference 7, the optimal upwind functions for quadratic elements were derived. In order to simplify the exposition, here we will consider that the asymptotic approximation to these functions is used. Table I summarizes their expressions for linear and quadratic elements, both using the standard shape functions (canonical basis) and the hierarchic approach. For quadratic elements a different

Table I. Upwind functions

Element	Extreme nodes	Central node
Linear	$\alpha = \min(\gamma/3, 1)$	
Quadratic standard	$\alpha = \min(\gamma/12, 1)$	$\beta = \min(\gamma/12, 1/2)$
Quadratic hierarchic	$\alpha = \min(\gamma/15, 1/3)$	$\beta = \min(\gamma/12, 1/2)$

upwind function is needed for the extreme nodes (α) and for the central nodes (β). The modification introduced in Reference 8 to account for the temporal discretization will not be used here.

Let $0 = x_0 < x_1 < \cdots < x_{N_{el}} = l$ be a uniform partition of diameter h of the interval $[0, l]$. Element e, $e = 1, 2, \ldots, N_{el}$, will be defined by the nodes placed at the abscissae x_{e-1} and x_e. If quadratic elements are used, the central node will be placed at $x_{e-1/2} := \frac{1}{2}(x_{e-1} + x_e)$. If $\boldsymbol{\phi} = \boldsymbol{\phi}(t)$ is the vector containing the nodal unknowns of $\phi(x, t)$, the application of the SUPG method to (3)–(5) will lead to an initial value problem of the form

$$\begin{aligned} \mathbf{M}\dot{\boldsymbol{\phi}} + \mathbf{K}\boldsymbol{\phi} &= \mathbf{0} \quad t > 0 \\ \boldsymbol{\phi}(0) &= \boldsymbol{\phi}^0 \end{aligned} \tag{6}$$

If the forward Euler scheme is now employed, once $\boldsymbol{\phi}$ is known at time level t^n, $\boldsymbol{\phi}^{n+1}$ will be found by solving

$$\mathbf{M}\boldsymbol{\phi}^{n+1} = \mathbf{M}\boldsymbol{\phi}^n - \Delta t\mathbf{K}\boldsymbol{\phi}^n \tag{7}$$

This scheme is only useful when the matrix $\mathbf{M}$ is approximated by a diagonal matrix $\mathbf{M}_d$. In this case, $\boldsymbol{\phi}^{n+1}$ can be obtained explicitly from $\boldsymbol{\phi}^n$, since the inversion of $\mathbf{M}_d$ is trivial:

$$\boldsymbol{\phi}^{n+1} = \boldsymbol{\phi}^n - \Delta t\mathbf{M}_d^{-1}\mathbf{K}\boldsymbol{\phi}^n \tag{8}$$

Observe that $\mathbf{E} = \mathbf{I} - \Delta t\mathbf{M}_d^{-1}\mathbf{K}$, with the notation introduced earlier. Clearly, matrix $\mathbf{M}_d$ must be non-singular and positive definite, since, otherwise, the analogue of (6) obtained by replacing $\mathbf{M}$ by $\mathbf{M}_d$ would be unstable. Thus, the diagonal entries of $\mathbf{M}_d$ must be positive. This matrix $\mathbf{M}_d$ can be easily obtained by using the classical row-sum lumping technique or through nodal integration when the standard shape functions are used,[9, 10] i.e. when the shape function $N_i(x)$ associated to a node i satisfies $N_i(x_j) = \delta_{ij}$, the Kronecker symbol, when applied to a node of abscissa x_j. In this case $\sum_{e=1}^{N_{el}} N_e(x) \equiv 1$. However, the situation is more delicate when the hierarchic approach is used.

The purpose of what follows is to analyse the stability and accuracy of (8). This a popular scheme due to its reduced computational cost and is very often used in practice. First, linear elements will be considered. The stability condition in this case is well-known (see, e.g., References 4 and 5), but this will serve us as an introduction to the more complicated situation encountered when quadratic elements are treated. For completeness, both standard and hierarchic finite elements will be analysed. Moreover, our interest here is to study the stability when the SUPG method is used, although the analysis could be easily particularized to the Galerkin and Taylor–Galerkin[11] methods. We will not, however, insist on the performance of these formulations, which is well established (see Reference 10 for a comprehensive description of these methods).

The use of quadratic elements deserves an explanation. The best space accuracy one can hope for is an error of order $O(h^3)$ in the L^2-norm (in fact, this error is $O(h^{2\cdot 5})$; see References 12 and 13). On the other hand, the forward Euler scheme is only first-order accurate in time, i.e. errors are of order $O(\Delta t)$. Thus, it is apparent that space and time errors will not be properly compensated unless Δt is very small. Nevertheless, the Euler scheme may be useful to solve a steady-state problem. One may think that the time steps are, in fact, iteration steps of a relaxation procedure.

LINEAR ELEMENTS

When linear elements are used, matrices $\mathbf{M}$ and $\mathbf{K}$ appearing in equation (7) will be obtained by assembling the element matrices $\mathbf{M}^e$ and $\mathbf{K}^e$. Matrix $\mathbf{M}^e$ may be diagonalized by using the row-sum lumping technique (see Reference 14 for different choices of $\mathbf{M}^e$ arising from numerical integration). Once this is done and the element matrices are assembled, a typical algorithmic equation for an internal node m resulting from equation (8) is

$$\phi_m^{n+1} = \phi_m^n + \Delta t\left[\left(\frac{k}{h^2} + \frac{\alpha u}{2h}\right)(\phi_{m+1}^n - 2\phi_m^n + \phi_{m-1}^n) - \frac{u}{2h}(\phi_{m+1}^n - \phi_{m-1}^n)\right] \tag{9}$$

This algorithm is the same as that obtained in a standard finite difference approximation.

Stability and accuracy

The analytical solution of problem (3)–(5) may be expanded in Fourier series, each mode having the form

$$\hat{\phi}(x, t) = a\exp[-(\xi + \mathrm{i}\omega)t]\exp(\mathrm{i}Kx) \tag{10}$$

where a is the amplitude of the mode; K the wavenumber; $\xi := kK^2$, the damping; $\omega := Ku$, the frequency and $\mathrm{i} := \sqrt{-1}$. Let

$$\hat{\phi}_m^n = a\exp[-(\xi^h + \mathrm{i}\omega^h)n\Delta t]\exp(\mathrm{i}Kmh) \tag{11}$$

be the harmonic corresponding to (10) evaluated at $(x, t) = (x_m, t^n) = (mh, n\Delta t)$ for the discrete problem. Here, ξ^h is the algorithmic damping and ω^h the algorithmic frequency. Although only discrete values of K can be reproduced by the discretization, we will consider as usual that K is any real number.

The amplification factor arising from scheme (9) is

$$A^h := \frac{\hat{\phi}_m^{n+1}}{\hat{\phi}_m^n} = 1 + \left(\frac{c}{\gamma} + \alpha c\right)(\cos Kh - 1) - \mathrm{i}c\sin Kh \tag{12}$$

where $c := u\Delta t/h$ is the Courant number.

The von Neumann stability criterion requires that $|A^h| \leqslant 1$ for any K. This leads to

$$c \leqslant \min\left(\frac{\gamma}{1+\alpha\gamma}, \frac{1}{\gamma} + \alpha\right) \tag{13}$$

If α exceeds the critical value

$$\alpha_c := 1 - \frac{1}{\gamma} \tag{14}$$

condition (13) reduces to

$$c \leqslant \frac{\gamma}{1+\alpha\gamma} \tag{15}$$

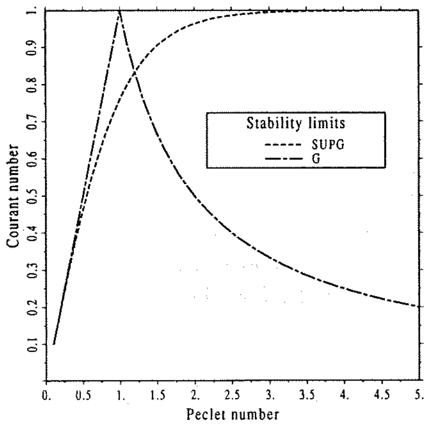

Figure 1. Stability limits for the convection–diffusion equation using linear elements: (G) Galerkin method; (SUPG) streamline-upwind/Petrov–Galerkin method

Since the upwind function given in Table I for linear elements satisfies $\alpha \geqslant \alpha_c$, this is the sought stability condition.

Remarks

(1) Observe that if $\alpha = 0$ (Galerkin method) the algorithm becomes unconditionally unstable when $\gamma \to \infty$, since in that case condition (13) requires $c = 0$.
(2) It can be easily shown that $\alpha \geqslant \alpha_c$, with α_c given by (14), is also the condition under which no oscillations appear in the numerical solution of the stationary equation.[15]
(3) From the expression of α, it follows that (15) reduces to

$$c \leqslant 1 \quad \text{in the advective limit } (\gamma \to \infty) \tag{16}$$

and

$$\Delta t \leqslant \frac{h^2}{2k} \quad \text{in the diffusive limit } (u \to 0)$$

Inequality (16) is the CFL condition. It is the best one can hope for.

The stability region dictated by (13) when $\alpha = 0$ (Galerkin method) and when $\alpha = \min(\gamma/3, 1)$ (SUPG method) has been plotted in Figure 1.

In order to determine the formal accuracy of the algorithm, both the exact and the numerical amplification factors, A and A^h, will be expanded in powers of Δt and h. Let $\hat{\phi}^n(x) = \hat{\phi}(x, t^n)$, with $\hat{\phi}$ given by (10). The analytical amplification factor is

$$\begin{aligned} A := \frac{\hat{\phi}^{n+1}(x_m)}{\hat{\phi}^n(x_m)} &= \exp[-(kK^2 + iKu)\Delta t] \\ &= 1 - (kK^2 + iKu)\Delta t + \tfrac{1}{2}(k^2K^4 - K^2u^2 + 2ikK^3u)\Delta t^2 + O(\Delta t^3) \end{aligned} \tag{17}$$

whereas the algorithmic amplification factor given by (12) satisfies

$$A^h = 1 - (kK^2 + iKu)\Delta t - \tfrac{1}{2}\alpha u K^2 h \Delta t + O(h^2 \Delta t) \tag{18}$$

Since $A = \exp[-(\xi + i\omega)\Delta t]$ and $A^h = \exp[-(\xi^h + i\omega^h)\Delta t]$, the damping error and the frequency error will be one order of Δt less than the amplification factor error. Comparing expressions (17) and (18) we see that

- $A - A^h = O(\alpha h \Delta t)$ and, hence, $\xi - \xi^h = O(\alpha h)$ and $\omega - \omega^h = O(\alpha h)$. This formal estimate is clearly pessimistic, since for a properly chosen upwind function α we know that the accuracy is much higher than with $\alpha = 0$. This fact has also been observed in Reference 6 where predictor-corrector algorithms for the Galerkin/least-squares method are studied.
- If $\alpha = 0$ and $h^2 = C\Delta t$, C being a constant, we have that $A - A^h = O(\Delta t^2)$ and, thus, $\xi - \xi^h = O(\Delta t)$ and $\omega - \omega^h = O(\Delta t)$. The algorithm is formally first-order accurate in time, although it suffers from very important spatial oscillations when the Péclet number γ is high.

Algorithmic damping ratio (ADR) and frequency ratio (AFR)

From the practical standpoint, it is important to have a global feeling of how the algorithm behaves for the whole range of space sizes h. The algorithmic damping ratio (ADR) and the algorithmic frequency ratio (AFR) are dimensionless numbers defined by

$$ADR := \frac{\xi^h}{\xi} \quad \text{and} \quad AFR := \frac{\omega^h}{\omega}$$

respectively. The ADR gives a measure of the damping error and the AFR of the phase error. These quantities are functions of the dimensionless wave number $\bar{K} := Kh$. The numerical method can only reproduce values $0 \leqslant \bar{K} \leqslant \pi$, the upper bound corresponding to two elements per wavelength. For accuracy, it is commonly argued that at least ten elements per wavelength are needed,[6,16] corresponding to $\bar{K} \approx 0{\cdot}6$.

In Reference 8, the ADR and the AFR are plotted only for the pure convection problem, whereas the convection–diffusion case is considered in Reference 16, although not for the forward Euler scheme. We will do this here, considering also the relative importance of convection in the problem.

Given a diffusion k and a velocity u, let us write the Péclet number γ as

$$\gamma = \frac{u}{2kK} Kh =: \gamma_0 \bar{K}$$

The coefficient γ_0 is proportional to the global Péclet number. We will call it *convection factor*. Low values of γ_0 will indicate that diffusion dominates, whereas convection will be dominant for high values of γ_0.

On the other hand, the Courant number will be taken as

$$c = c_0 \frac{\gamma}{1 + \alpha\gamma} \tag{19}$$

For stability, $c_0 \leqslant 1$. This value c_0 will be called *security factor*.

Having introduced γ_0 and c_0, the analytical amplification factor and the algorithmic amplification factor will be

$$A = \exp\left(-\frac{c}{2}\frac{\bar{K}}{\gamma_0} - ic\bar{K}\right)$$

$$A^h = 1 + c_0(\cos\bar{K} - 1) - ic\sin\bar{K}$$

with c given by (19), $\gamma = \gamma_0 \bar{K}$ and $\alpha = \min(\gamma/3, 1)$. The ADR and the AFR will be a function of the convection factor γ_0 and the security factor c_0.

We have considered the cases $\gamma_0 = 0{\cdot}1$, 1 and 10 as representative of problems with different importance of convection. For each case, the ADR and the AFR have been plotted for $c_0 = 0{\cdot}25$, 0·5, 0·75 and 0·95. Results are shown in Figure 2. Since the sign of ω^h only affects the imaginary part of A^h, the absolute value of AFR has been plotted.

The conclusions that may be drawn from these plots can be predicted considering the mode $\bar{K} = \pi$. In this case

$$|A| = \exp\left(-\frac{\pi}{2\gamma_0} \frac{c_0 \gamma_0 \pi}{1 + \gamma_0 \pi} \right) \quad (\text{for } \alpha = 1)$$

$$|A^h| = |1 - 2c_0|$$

We see that when $\gamma_0 \to \infty$, then $|A| \to 1$ and, hence, $\xi \to 0$. In order to obtain values of $ADR \geqslant 1$ (for precluding oscillations), security factors close to 1 (in which case also ξ^h is small) may be used. But if γ_0 is fixed ($< \infty$) and $c_0 = 1$, the mode $\bar{K} = \pi$ is not damped in the numerical solution and $ADR = 0$. Oscillations or unphysical behaviour may be expected if the analytical solution exhibits this mode.

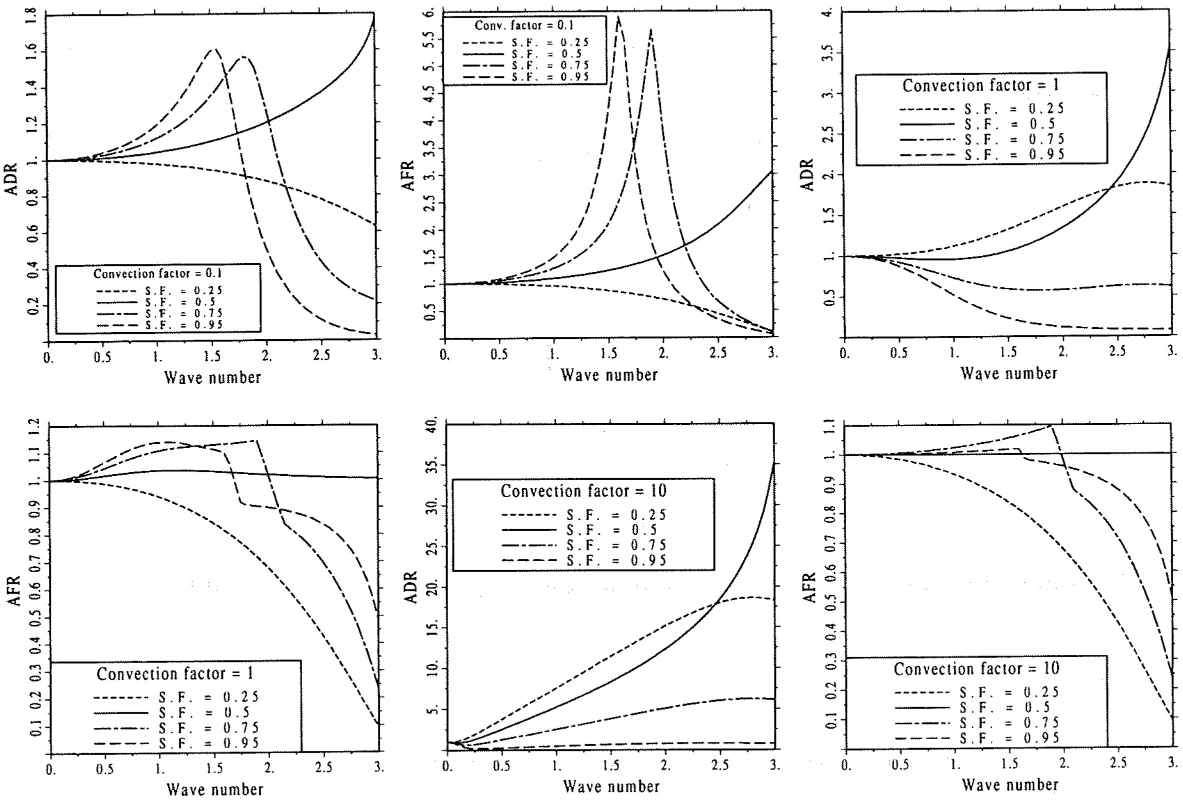

Figure 2. Algorithmic damping ratio (ADR) and algorithmic frequency ratio (AFR) for the SUPG method using linear elements and for different values of the security factor (SF) c_0 and the convection factor γ_0

Since for $c_0 = 0{\cdot}5$ it is $|A^h| = 0$, for $\gamma_0 < \infty$ we will have that $ADR \to \infty$ as $\bar{K} \to \pi$. From Figure 2 it is seen that $c_0 = 0{\cdot}5$ gives an important damping for the whole range of $\bar{K}$ and different values of γ_0. We conclude that this choice of c_0 is 'safe' and especially useful if only the steady-state solution is of interest. Moreover, the AFR is close to one, showing that the phase (or dispersion) errors will be small. This fact will be confirmed by the numerical experiments presented later.

QUADRATIC ELEMENTS I: CANONICAL BASIS

Suppose now that the spatial discretization of the problem is performed using quadratic elements. Here we will consider that the standard shape functions are used. Once again, the element matrix $\mathbf{M}^e$ may be diagonalized by using the row-sum lumping technique.

The situation is now more involved than for linear elements. Two different typical algorithmic equations will be found for the internal nodes of the finite element partition, one for the extreme nodes and another one for the central nodes. The stability of each set of equations has to be studied separately, as well as its accuracy. These equations are:

- Central nodes:

$$\phi_{m+1/2}^{n+1} = \left[c(1+2\beta) + 2\frac{c}{\gamma}\right]\phi_m^n + \left[1 - 4\left(\frac{c}{\gamma} + \beta c\right)\right]\phi_{m+1/2}^n + \left[-c(1-2\beta) + 2\frac{c}{\gamma}\right]\phi_{m+1}^n \tag{20}$$

- Extreme nodes:

$$\begin{aligned}\phi_m^{n+1} = {} & \left[-3\frac{c}{\gamma}\left(\frac{1}{6} - \alpha\right) - 3c\left(\frac{1}{6} + \frac{1}{6}\alpha\right)\right]\phi_{m-1}^n \\ & + \left[3\frac{c}{\gamma}\left(\frac{4}{3} - 2\alpha\right) + 3c\left(\frac{2}{3} + \frac{4}{3}\alpha\right)\right]\phi_{m-1/2}^n \\ & + \left[1 - 7\left(\frac{c}{\gamma} + \alpha c\right)\right]\phi_m^n \\ & + \left[3\frac{c}{\gamma}\left(\frac{4}{3} + 2\alpha\right) - 3c\left(\frac{2}{3} - \frac{4}{3}\alpha\right)\right]\phi_{m+1/2}^n \\ & + \left[-3\frac{c}{\gamma}\left(\frac{1}{6} + \alpha\right) + 3c\left(\frac{1}{6} - \frac{1}{6}\alpha\right)\right]\phi_{m+1}^n\end{aligned} \tag{21}$$

The Courant number used in these expressions is also $u\Delta t/h$, where h is the total length of the elements.

Stability and accuracy

Define $\hat{\phi}(x, t)$ and $\hat{\phi}_{m+q}^n$ as before, with $q = 0$ or $q = \frac{1}{2}$. We first consider the stability and accuracy for equation (20) (central nodes), i.e. $q = \frac{1}{2}$. The amplification factor in this case will be denoted by A_c^h. It is given by

$$A_c^h := \frac{\hat{\phi}_{m+1/2}^{n+1}}{\hat{\phi}_{m+1/2}^n} = 1 + 4\left(\frac{c}{\gamma} + \beta c\right)\left(\cos K\frac{h}{2} - 1\right) - \mathrm{i}2c\sin K\frac{h}{2}$$

Obtaining the von Neumann stability condition in this case is an easy task. It can be done exactly as for linear elements. Omitting the details, the stability limit is found to be

$$c \leqslant \frac{\gamma}{4(1 + \beta\gamma)} \tag{22}$$

This is the stability requirement for the central nodes. It holds whenever $\beta \geqslant 1/2 - 1/\gamma$.

The expansion of A_c^h in powers of Δt and h yields

$$A_c^h = 1 - (kK^2 + \mathrm{i}Ku)\Delta t + O(\beta h \Delta t) + O(h^2 \Delta t) \tag{23}$$

The analytical amplification factor is given again by (17). Exactly the same comments concerning the accuracy of linear elements are valid for this case. As it could be expected, the accuracy of the algorithm is driven by the time discretization. No improvement is obtained because of the use of quadratic elements.

Consider now the algorithmic equation for the extreme nodes (21). The corresponding amplification factor will be denoted by A_e^h. Its expression is

$$\begin{aligned} A_e^h := \frac{\hat{\phi}_m^{n+1}}{\hat{\phi}_m^n} &= 1 - 7\left(\frac{c}{\gamma} + \alpha c\right) + 8\left(\frac{c}{\gamma} + \alpha c\right)\cos K\frac{h}{2} - \left(\frac{c}{\gamma} + \alpha c\right)\cos Kh \\ &\quad + \mathrm{i}\left[-\left(6\frac{c}{\gamma}\alpha - c\right)\sin Kh + 4\left(3\frac{c}{\gamma}\alpha - c\right)\sin K\frac{h}{2}\right] \end{aligned} \tag{24}$$

It is proved in the Appendix that the von Neumann criterion leads to

$$c \leqslant \min\left(\frac{\gamma}{8(1 + \alpha\gamma)}, \frac{1}{\gamma} + \alpha\right) = \frac{\gamma}{8(1 + \alpha\gamma)} \tag{25}$$

for the upwind function α given in Table I. This is the sought stability limit for the extreme nodes.

The algorithm will be stable only if both (22) and (25) hold. Since $\alpha \geqslant \beta$, we have that (25) is more restrictive than (22). Therefore, we finally obtain that (25) is the necessary and sufficient condition needed to satisfy the von Neumann stability criterion.

Remarks

(1) In Reference 7 it is shown that the choice $\alpha = \beta = \min(\gamma/6, 1/2)$ may also be used. For this unique upwind function also (25) is the stability condition.

(2) From (25) it follows that the Galerkin method ($\alpha = \beta = 0$) becomes unconditionally unstable when $\gamma \to \infty$.

(3) From the expression given for α, condition (25) reduces to

$$c \leqslant \frac{1}{8} \quad \text{in the advective limit } (\gamma \to \infty) \tag{26}$$

and

$$\Delta t \leqslant \frac{h^2}{16k} \quad \text{in the diffusive limit } (u \to 0) \tag{27}$$

These conditions are clearly far from being optimal. Instead of (26) one would hope $c \leqslant \frac{1}{2}$ for the definition of the Courant number we have used. This limit depends on the upwind function α and it could be thought that this lack of 'optimality' is due to the choice of this function. However, (27) is independent of α and is also suboptimal if the result obtained for linear elements is taken as a reference. In particular, for a given set of nodes, the critical time step for stability will be higher using linear elements than quadratic elements (twice or four times, according to (27) or (26)). Neverthless the numerical results presented later show that the steady state is reached in a similar number of time steps using linear and quadratic elements.

Figure 3 shows the stability limits dictated by (22) and (25). It is observed that the stability restriction imposed by the extreme nodes is much more severe than the one imposed by the central nodes.

So far, we have considered only *necessary* conditions for stability. However, we can prove (see the Appendix) that the diffusive limit (27) is also sufficient. So, we have the following proposition.

Proposition 1. *Consider the forward Euler scheme defined by the algorithmic equations* (20) *and* (21). *Let the upwind functions be* $\alpha = \min(\gamma/12, 1)$ and $\beta = \min(\gamma/12, 1/2)$. *Then, the algorithm satisfies the von Neumann stability condition if, and only if,*

$$c \leqslant \frac{\gamma}{8(1+\alpha\gamma)}$$

Moreover, for $u = 0$ *the condition*

$$\Delta t \leqslant \frac{h^2}{16k}$$

is both necessary and sufficient for stability.

Not much new can be said about the accuracy when the extreme nodes are considered. The expansion of the amplification factor A_e^h in powers of Δt and h is

$$A_e^h = 1 - (kK^2 + \mathrm{i}Ku)\Delta t + O(\alpha h \Delta t) + O(h^2 \Delta t)$$

that has the same form as equation (23). What was said for the central nodes also applies here.

Algorithmic damping ratio (*ADR*) *and frequency ratio* (*AFR*)

As for linear elements, the *ADR* and the *AFR* have been plotted for values $c_0 = 0{\cdot}25, 0{\cdot}5, 0{\cdot}75$ and 0·95 of the security factor and $\gamma_0 = 0{\cdot}1$, 1 and 10 for the convection factor. Here, the *ADR* and

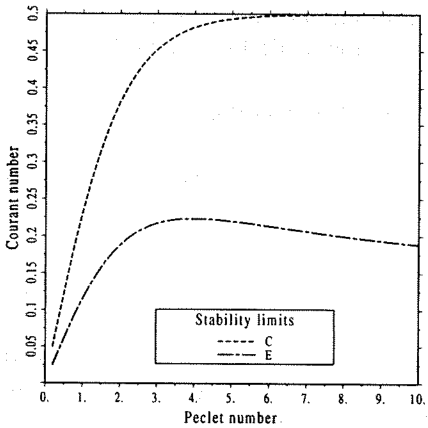

Figure 3. Stability limits for the convection–diffusion equation using quadratic elements: (C) central nodes; (E) extreme nodes

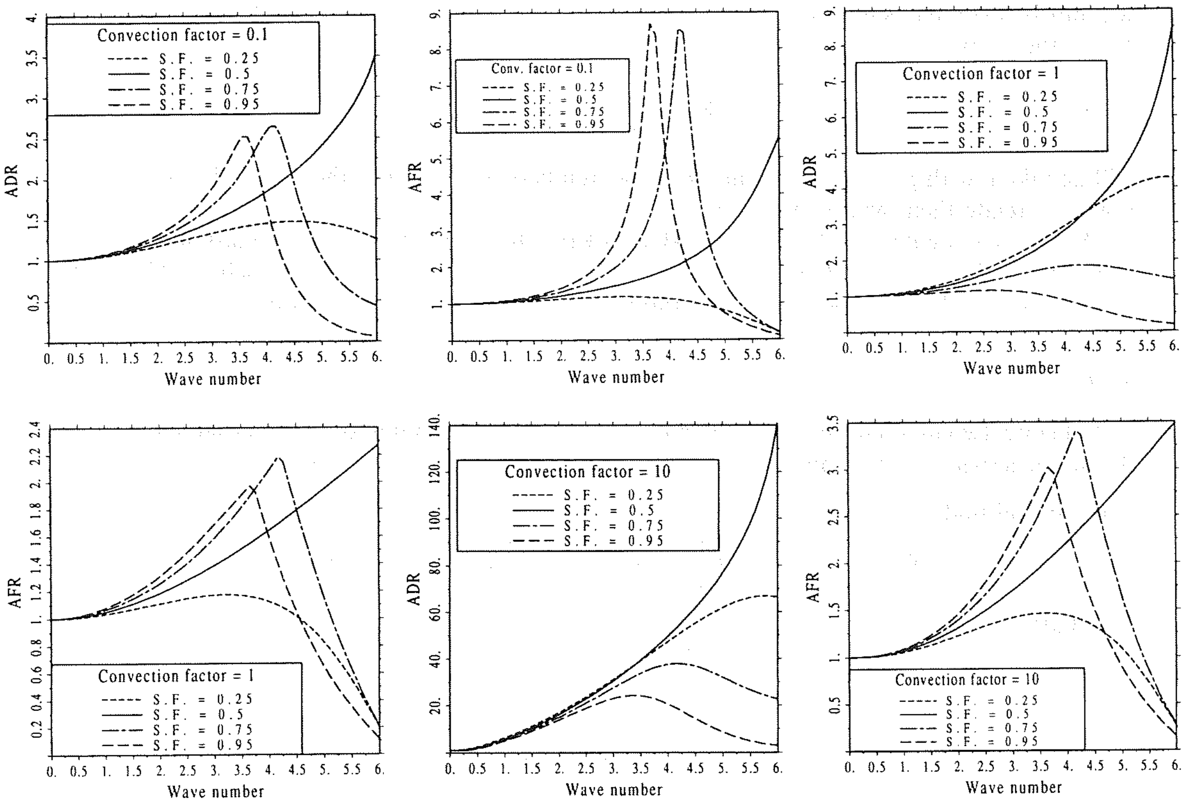

Figure 4. Algorithmic damping ratio (ADR) and algorithmic frequency ratio (AFR) for the SUPG method using quadratic elements and for different values of the security factor (SF) c_0 and the convection factor γ_0

the AFR are referred to the extreme nodes. Now the numerical method can reproduce values of $\bar{K}$ in the whole interval $[0, 2\pi]$, the upper bound corresponding to a single element per wavelength (three nodes).

From the plots shown in Figure 4, it is seen that the choice $c_0 = 0{\cdot}5$ is also 'safe', as it happened for linear elements. The ADR is always $\geqslant 1$. However, now the phase errors for this value of the security factor are higher than for linear elements. The salient point of these results is that $c_0 = 0{\cdot}95$ gives values of the ADR higher than 1 in a range much wider than for linear elements, especially for $\gamma_0 = 10$. This indicates that the algorithm will have an important amount of numerical damping. Although this fact has a negative connotation when the time accuracy is crucial, it is beneficial if only the steady state is sought. This explains why the number of time steps needed to reach the stationary solution are similar for linear and quadratic elements, even though the critical time step for stability is smaller using quadratic than linear interpolations.

QUADRATIC ELEMENTS II: HIERARCHIC APPROACH

Now we consider that the quadratic finite element interpolation is done using the hierarchic basis, i.e. the linear basis enriched with piecewise quadratic shape functions correponding to the central nodes. In this case, it is not clear how to obtain a diagonal matrix $\mathbf{M}_{\mathrm{d}}^{e}$ that approximates the element mass matrix $\mathbf{M}^{e}$. Now the property $\sum_{e=l}^{N_{\mathrm{el}}} N_e(x) \equiv 1$ does *not* hold and the row-sum lumping technique does not make sense. It a nodal quadrature rule is used, the resulting matrix will not be diagonal, since now $N_i(x_j) \neq \delta_{ij}$, with the notation used earlier. Therefore, a matrix M_d^e of the form

$$\mathrm{M}_\mathrm{d}^e = \frac{h}{3}\,\mathrm{diag}(\mu, \mu', \mu)$$

will be taken, with μ and μ' being any *positive* numbers. In principle, they are independent, but a way to relate them will be discussed.

The transient evolution of the system $\mathbf{M}_\mathrm{d}\dot{\boldsymbol{\phi}} + \mathbf{K}\boldsymbol{\phi} = \mathbf{0}$ will not be an approximation to the real problem (6). Only the steady state of both problems will (hopefully) coincide. Scheme (8) can be thought of as a Jacobi-type iterative method to reach this stationary solution.

Stability

As before, two different sets of algorithmic equations will be found for the internal nodes. If we define the scaled Courant numbers $\bar{c}' := c/\mu'$ and $\bar{c} := c/2\mu$, these equations are:

- Central nodes:

$$\phi_{m+1/2}^{n+1} = 2\bar{c}'\phi_m^n + \left[1 - 8\left(\frac{\bar{c}'}{\gamma} + \beta\bar{c}'\right)\right]\phi_{m+1/2}^n - 2\bar{c}'\phi_{m+1}^n \tag{28}$$

- Extreme nodes:

$$\begin{aligned}\phi_m^{n+1} = {} & \left[\frac{3}{2}\frac{\bar{c}}{\gamma} + \frac{3}{2}\bar{c}(\alpha + 1)\right]\phi_{m-1}^n + \left[-6\alpha\frac{\bar{c}}{\gamma} + 2\bar{c}\right]\phi_{m-1/2}^n \\ & + \left[1 - 3\frac{\bar{c}}{\gamma} - 3\alpha\bar{c}\right]\phi_m^n + \left[6\alpha\frac{\bar{c}}{\gamma} - 2\bar{c}\right]\phi_{m+1/2}^n \\ & + \left[\frac{3}{2}\frac{\bar{c}}{\gamma} + \frac{3}{2}\bar{c}(\alpha - 1)\right]\phi_{m+1}^n\end{aligned} \tag{29}$$

Using the same techniques as in the previous section, the following result can be proved (see Reference 17 for details).

Proposition 2. *Consider the forward Euler scheme defined by the algorithmic equations* (28) *and* (29). *Let the upwind functions be* $\alpha = \min(\gamma/15, 1/3)$ and $\beta = \min(\gamma/12, 1/2)$, *and* γ_c *the smallest root of the equation*

$$9\alpha^2\gamma^2 + 30\alpha\gamma - 8\gamma^2 + 9 = 0$$

(*which is* $\approx 1{\cdot}23$). *Then, the algorithm satisfies the von Neumann stability condition if, and only if,*

$$c \leqslant \begin{cases} \min(\mu' c_1, 2\mu c_2) & \text{if } \gamma > \gamma_c \\ \min(\mu' c_1, 2\mu c_3) & \text{if } \gamma \leqslant \gamma_c \end{cases}$$

where c_1, c_2 *and* c_3 *are given by*

$$c_1 := \gamma\frac{1 + \beta\gamma}{\gamma^2 + 4(1 + \beta\gamma)^2}$$

$$c_2 := \frac{3\gamma(1 + \alpha\gamma)}{4(3\alpha - 2\gamma)^2}$$

$$c_3 := \gamma\frac{9(1 + \alpha\gamma)^2 - 4\gamma^2}{27(1 + \alpha\gamma)^3 + 12(1 + \alpha\gamma)[(3\alpha - \gamma)^2 - \gamma^2]}$$

Moreover, for $u = 0$ the condition

$$\Delta t \leqslant \min\left(\frac{\mu' h^2}{8k}, \frac{\mu h^2}{3k}\right)$$

is both necessary and sufficient for stability.

Remarks

(1) Clearly, the magnitude of μ' affects proportionately the time step size. What will be important is the ratio μ'/μ, and not μ or μ' themselves (see equation (8)).

(2) The values of μ and μ' have been considered independent, but there is a simple criterion to relate them in order to speed up the convergence of the algorithm towards the steady state. With the notation used in Proposition 2, we could take μ and μ' such that

$$\mu' c_1 = \begin{cases} 2\mu c_2 & \text{if } \gamma > \gamma_c \\ 2\mu c_3 & \text{if } \gamma \leqslant \gamma_c \end{cases}$$

This choice will ensure that both the equations for the central nodes and for the extreme nodes advance in 'time' as fast as possible. In particular, for the advective limit it is easy to see that this yields $\mu = 2\mu'$, and for the diffusive limit $\mu = (3/8)\mu'$. From numerical experiments we have found that this reduces the number of required time steps by about 10 per cent for convection dominated problems. Clearly, this technique could also be applied to the standard quadratic elements if only the steady state were important. The mass matrix should be properly scaled.

(3) The discrete modes $\hat{\phi}^n_m$ given by (11) have to be considered from the series expansion *of the error*, since now $\phi^n_{m+1/2}$ are not the nodal values of the unknown function.

EXTENSION TO MULTIDIMENSIONAL PROBLEMS

The Fourier analysis of a difference scheme in a general multidimensional mesh is a difficult, if not impossible, goal. The expression of the critical time step in these cases is necessarily based on heuristic criteria. In Reference 4 the following partial result was proved for the finite difference method. Assume that the domain is two-dimensional (for simplicity), discretized using a uniform grid and that the centred five-point stencil is used to approximate both the first and the second spatial derivatives. Define

$$\gamma_x := \frac{u_x h_x}{2k_x}, \quad \gamma_y := \frac{u_y h_y}{2k_y}$$

$$c_x := \frac{u_x \Delta t}{h_x}, \quad c_y := \frac{u_y \Delta t}{h_y}$$

where subscripts x and y refer to the Cartesian directions. Under these conditions, the forward Euler scheme satisfies the von Neumann stability condition if and only if

$$c_x\gamma_x + c_y\gamma_y \leqslant 1 \quad \text{and} \quad \frac{c_x}{\gamma_x} + \frac{c_y}{\gamma_y} \leqslant 1 \tag{30}$$

Let Δt_x and Δt_y be the critical time steps that would be found if the problem were one-dimensional along the x and y directions, respectively. Observe that (30) can be written as

$$\Delta t \Delta t_x^{-1} + \Delta t \Delta t_y^{-1} \leqslant 1 \tag{31}$$

Now suppose that a local system of co-ordinates σ and ν is taken at each point, σ following the streamline and ν normal to it. If we make the assumption that (31) still holds true in this new co-ordinate system, Δt must satisfy

$$\Delta t \leqslant \frac{\Delta t_\sigma \Delta t_\nu}{\Delta t_\sigma + \Delta t_\nu} \tag{32}$$

The problem is now reduced to computation of Δt_σ and Δt_ν. Since the velocity follows the σ-direction, Δt_ν is calculated using the diffusive stability limits and Δt_σ using the general expressions obtained for one-dimensional convection–diffusion.

Another question that arises using finite elements is the way a diagonal approximation to the mass matrix is obtained. We have used standard nodal quadrature rules in the following examples. Since the weights of the classical second-order rule for the quadratic triangle are zero for the corner nodes, this element has been split into four linear triangles and the weights for these subelements have been utilized.

In general situations what we do is the following. Let Δt^e be the critical time step computed using (32) for element e, i.e. using its characteristic diffusion and element length and the Euclidian norm of the characteristic velocity within this element. The global time step is then taken as

$$\Delta t = f_t \min_e (\Delta t^e) \tag{33}$$

where f_t acts as a safety factor. In the numerical results presented hereafter, we have found $f_t = 1$ effective in all the cases except for the six-noded triangular element, where $f_t < 1$ is needed.

NUMERICAL EXAMPLES

Example 1

This example has been taken from Reference 18 and is useful to check the accuracy of the temporal discretization. Problem given by equations (3) and (4) has been solved with boundary conditions $\phi(0, t) = 0$ and $\partial\phi(l, t)/\partial x = 0$. The initial condition is

$$\phi^0(x) = \exp[-(x-u)^2/4k]$$

This problem has an analytical solution given by

$$\phi(x,t) = \frac{1}{\sqrt{1+t}} \exp\{-[x - u(t+1)]^2/4k(t+1)\}$$

The data are now $l = 2$, $u = 0{\cdot}25$ and $k = 0{\cdot}00125$. The space discretization has been done using 81 equally spaced nodal points. Both linear and quadratic elements have been considered (80 and 40, respectively). The resulting Péclet number is 2·5 for linear elements and 5 for quadratic elements. When the Crank–Nicolson scheme has been used, the time step (Δt) has been taken as 0·1, yielding a Courant number $c = 0{\cdot}5$ for quadratic elements and $c = 1$ for linear elements.

Figure 5(a) shows the solution obtained using linear elements and the forward Euler scheme in time with a security factor $c_0 = 1$, as well as the analytical solution for $t = 0{\cdot}197$ and 3·946. As it was already explained, some modes of the Fourier series expansion of the analytical solution are not properly damped by the numerical algorithm for this choice of c_0. The underdiffusive behaviour of the numerical solution is evident, showing that the method is potentially oscillatory in more complicated situations. The results obtained under the same conditions but with $c_0 = 0{\cdot}5$ are depicted in Figure 5(b). As expected, the numerical answers show a much higher dissipation.

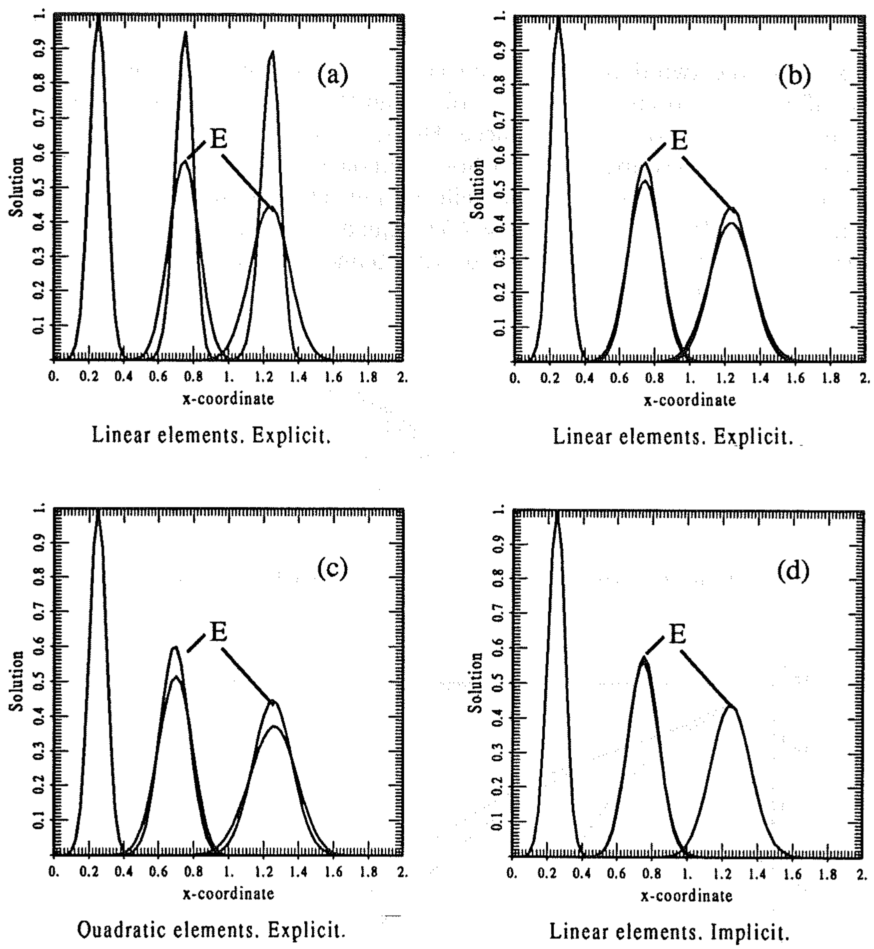

Figure 5. Results for example 1. (a): linear elements and forward Euler with $c_0 = 1$. (b): linear elements and forward Euler with $c_0 = 0{\cdot}5$. (c): quadratic elements and forward Euler with $c_0 = 1$. (d): linear elements and Crank–Nicolson

In fact, they are a little overdiffusive. It is important to point out that the phase accuracy is very good, as it was predicted from the behaviour of the algorithmic frequency ratio. If quadratic elements are used, this phase accuracy is not so high, as it can be observed from Figure 5(c), where the solution for $t = 1{\cdot}761$ and $4{\cdot}006$ has been represented. Nevertheless, now a security factor $c_0 = 1$ can be used without suffering from underdiffusive behaviour. All these observations confirm the theoretical predictions that had been obtained.

The results obtained using the Crank–Nicolson scheme with linear elements are plotted in Figure 5(d) for $t = 2$ and 4, showing a much higher accuracy than the forward Euler method (the Crank–Nicolson scheme is second-order accurate, but requires the solution of a linear algebraic system). If quadratic elements are used in this case, the numerical solution is almost the same. Also, if the Galerkin formulation is employed, only small amplitude oscillations are present (not shown), since the exact solution is now very smooth.

Example 2

In this example, the two-dimensional convection–diffusion equation is solved in the unit square. The diffusion has been taken as $k = 10^{-2}$ and the velocity field $\mathbf{u} = (\cos(\pi/4), \sin(\pi/4))$. A source term $f = 5$ has also been introduced. Homogeneous boundary conditions of Dirichlet type ($\phi = 0$) have been prescribed on the whole boundary.

The domain has been discretized using a uniform finite element mesh with 21×21 nodes in all the cases. The resulting Péclet number is $\gamma = 2{\cdot}5$ for quadratic elements ($h \approx 0{\cdot}1$) and $\gamma = 1{\cdot}25$ for linear elements ($h \approx 0{\cdot}05$). Results obtained with bilinear elements are shown in Figure 6. A similar numerical solution is obtained using the linear and quadratic triangles and the

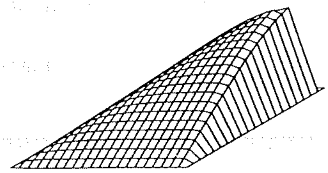

Figure 6. Numerical solution of example 2 using bilinear elements

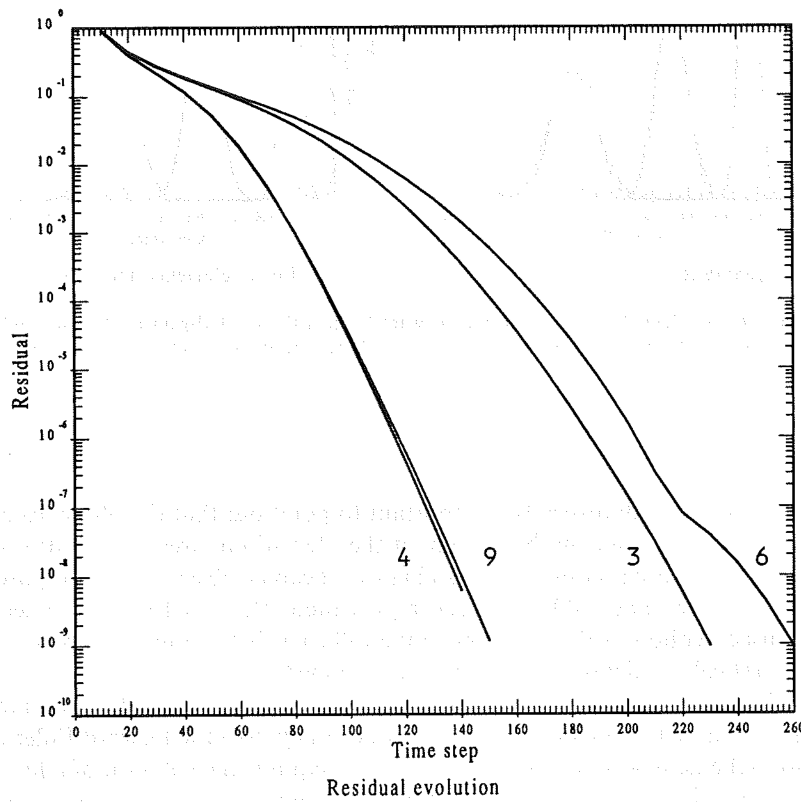

Figure 7. Evolution towards the steady state for example 2. The number of nodes of the elements is indicated on each curve

biquadratic element (not shown; see Reference 7). For quadratic elements, the unique upwind function $\alpha = \beta = \min(\gamma/6, 1/2)$ has been chosen.

We are now interested in the evolution of the residual $\max_m |\phi_m^{n+1} - \phi_m^n|$ as time advances. The subscript refers to a nodal point. The results obtained for elements of 3, 4, 6 and 9 nodes are shown in Figure 7. Formula (33) has been used to compute the critical time step. In all the cases except for the six-noded triangular element, $f_t = 1$ has been used. For the quadratic triangle, $f_t = 0{\cdot}9$ has been needed, since the time marching scheme has been found to be unstable for $f_t = 1$.

It is seen that the steady state is reached faster for quadrilateral elements than for triangles. The bilinear element 'converges' slightly faster than the biquadratic one. The difference is more pronounced for the 3- and 6-noded elements.

At this point, it is interesting to make the following observation. Quadratic elements are often blamed to be more expensive than linear elements. The main argument is that the bandwidth of the final 'stiffness' matrix is larger. On the other hand, the total number of numerical quadrature points for a mesh with a given number of nodes is smaller. For example, if a 2×2 Gauss–Legendre quadrature rule is used for the bilinear element and a 3×3 rule for the biquadratic one, the ratio of total quadrature points of the former to the latter is 16/9. The important fact is that if an iterative method for solving the algebraic system of equations is used, the problem of the large bandwidth disappears and quadratic elements could be cheaper. In order to verify this hypothesis in this particular problem, the CPU time required on a CONVEX-C120 computer has been calculated. The results are the following:

No. of nodes	No. of integration points	CPU (s)
3	3	114·15
4	4 (2 × 2)	32·73
6	4	34·22
9	9 (3 × 3)	28·87

It is observed that the CPU time is smaller for quadratic elements than for linear elements, even though more time steps have to be performed to reach the steady state.

CONCLUSIONS

The stability and accuracy of the forward Euler scheme has been studied, using both linear and quadratic finite elements. The interest of this method relies basically on the fact that it allows us to obtain stationary solutions via an iterative procedure. This technique is ubiquitous in computational fluid dynamics, especially when the numerical simulation of compressible flow problems is attempted. Thus, its analysis has an inherent interest.

For linear elements, it has been shown that the upwind function yields optimal stability limits, in the sense that both the advective and diffusive limit cases reduce to conditions known to be optimal. Concerning the accuracy, a methodology has been proposed in order to determine the effect of convection and the time step size. The method is based on the representation of the *ADR* and the *AFR* for different values of the security factor c_0 and the convection factor γ_0 introduced here. Using this technique, it has been shown that if linear elements are used, the choice $c_0 = 1$ may lead to unphysical results for the transient evolution of convection–diffusion problems. This drawback is circumvented if $c_0 = 0{\cdot}5$ is selected. Moreover, an excellent phase accuracy is obtained for this value of c_0. These facts have been corroborated through numerical experiments.

The stability limits for quadratic elements have been derived using the standard and the hierarchic shape functions. If standard quadratic elements are employed, the value of the security factor $c_0 = 1$ yields dissipative results for a wide range of Fourier modes. The problem encountered with linear elements does not appear here. This compensates the fact that the critical time step is smaller for quadratic elements.

Finally, an *ad hoc* extension to multidimensional problems has been described. The method has proved to work well in practice.

ACKNOWLEDGEMENTS

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APPENDIX

We proceed to prove Proposition 1. In order to see that condition (25) ensures $|A_e^h| \leqslant 1$, with A_e^h given by (24), let us introduce the abbreviations

$$z := \cos K\frac{h}{2}; \quad b := \frac{c}{\gamma} + \alpha c; \quad d := 6\frac{c}{\gamma} - \alpha - c$$

which after using some elementary trigonometric relations allow one to write $|A_e^h|^2$ as

$$|A_e^h|^2(z) = 1 - 4b(z-1)(z-3) + 4b^2(z-1)^2(z-3)^2 - (z-1)(z+1)[2(d-c) - 2dz]^2$$

The von Neumann criterion $|A_e^h|^2 \leqslant 1$ for any $z \in [-1, 1]$ leads to the inequality $p(z) \geqslant 0$ in the interval $[-1, 1]$, where $p(z)$ is the third-degree polynomial

$$p(z) := -b(z-3) + b^2(z-1)(z-3)^2 - (z+1)(d-c-dz)^2 \tag{34}$$

There are two obvious necessary conditions for having $p(z) \geqslant 0$ in $[-1, 1]$, viz.

$$p(1) \geqslant 0 \Leftrightarrow c^2 \leqslant b \tag{35}$$

$$p(-1) \geqslant 0 \Leftrightarrow b \leqslant 1/8 \tag{36}$$

We now prove that (35) and (36) are also sufficient. Let us write $p(z)$ as $p(z) = a_0 z^3 + a_1 z^2 + a_2 z + a_3$. Suppose that $a_0 \neq 0$. This polynomial may have two local extrema, located at the abscissae z_1 and z_2 given by

$$z_1 = \zeta + \sigma \quad \text{and} \quad z_2 = \zeta - \sigma \tag{37}$$

where the notation

$$\zeta := -\frac{a_1}{3a_0}; \quad \sigma := \frac{1}{3a_0}(a_1^2 - 3a_0 a_2)^{1/2}$$

has been introduced. Assume now that the following two conditions hold: (i) $a_0 > 0$, (ii) $-a_1 \geqslant 3a_0$.

If $a_0 > 0$, then $p(z) \to +\infty$ as $z \to +\infty$ and $p(z) \to -\infty$ as $z \to -\infty$. Hence, if $p(z)$ has a local minimum located at z_m and a local maximum located at z_M then it must be $z_m \geqslant z_M$. From (37) it follows that $z_1 = z_m$ and $z_2 = z_M$. If condition (ii) holds then $\zeta \geqslant 1$. Since $z_1 \geqslant \zeta$, we have that if $p(z)$ has a local minimum it is located out of the interval $[-1, 1]$ and conditions (35) and (36) will suffice for the von Neumann stability condition.

It only remains to check inequalities (i) and (ii). Expanding the polynomial $p(z)$ given by (32), it is found that

$$a_0 = b^2 + d^2 > 0$$

$$-a_1 - 3a_0 = \frac{4c^2}{\gamma^2}[(1+\alpha\gamma)^2 + 6\alpha(\gamma - 6\alpha)]$$

Since the upwind function α is $\leqslant \gamma/6$, we have that $-a_1 - 3a_0 \geqslant 0$, i.e. condition (ii) holds.

Inequalities (35) and (36) can be written as (25) indicates. The last equality in this equation holds for

$$\alpha \geqslant \alpha_c := \frac{\sqrt{2}}{4} - \frac{1}{\gamma}$$

The upwind function α given in Table I satisfies $\alpha \geqslant \alpha_c$.

To prove that the diffusive limit (27) is also sufficient for stability, we apply now the matrix method. Since for $\gamma = 0$ (and hence $\alpha = \beta = 0$) matrices $\mathbf{M}_d$ and $\mathbf{K}$ are symmetric, the spectral radius of $\mathbf{I} - \Delta t \mathbf{M}_d^{-1}\mathbf{K}$ is equal to its L^2-matrix norm. The necessary and sufficient condition for stability will be $\varrho_0 \leqslant 1$. Since $\varrho_0 = |1 - \Delta t \varrho(\mathbf{M}_d^{-1}\mathbf{K})|$ this leads to

$$\Delta t \leqslant \frac{2}{\varrho(\mathbf{M}_d^{-1}\mathbf{K})} \tag{38}$$

Applying Irons' Theorem and solving an elementary eigenvalue problem, we obtain

$$\begin{aligned}\varrho(\mathbf{M}_d^{-1}\mathbf{K}) &= \max_{\lambda}\{\lambda \,|\, \det(\mathbf{K} - \lambda\mathbf{M}_d) = 0\} \\ &\leqslant \max_{e}\max_{\lambda^e}\{\lambda^e \,|\, \det(\mathbf{K}^e - \lambda^e\mathbf{M}_d^e) = 0\} \\ &= \max\left\{0, \frac{12k}{h^2}, \frac{32k}{h^2}\right\} = \frac{32k}{h^2}\end{aligned}$$

where $\mathbf{M}_d^e = (h/2)\mathrm{diag}(\frac{1}{3}, \frac{4}{3}, \frac{1}{3})$ comes from the 'lumping' on $\mathbf{M}^e$. Since

$$\frac{2}{\varrho(\mathbf{M}_d^{-1}\mathbf{K})} \geqslant \frac{h^2}{16k}$$

it follows that (27) implies (38). Stability is ensured.

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