Mathematical Modelling and Numerical Analysis Will be set by the publisher

Modélisation Mathématique et Analyse Numérique

STABILITY ANALYSIS OF TIME DEPENDENT LINEARIZED VISCOELASTIC FLOW PROBLEMS USING A STABILIZED FINITE ELEMENT FORMULATION IN SPACE

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Abstract. In this paper we present the stability analysis of a numerical approximation of the linearized viscoelastic flow equations using finite elements in space and a finite difference scheme in time. Both a standard and a log-conformation reformulation of the constitutive equations are considered. The space approximation consists of a stabilized finite element formulation that allows one to use equal interpolation for all variables and to deal with convection dominated flows. It is based on the Variational-Multiscale concept, allowing the sub-grid scales arising in the formulation to be time dependent. This permits to refine independently space and time, avoiding the small time step instability.

1991 Mathematics Subject Classification. 65N12, 76A10, 76M10.

The dates will be set by the publisher.

1. Introduction

The numerical approximation of the equations governing the flow of viscoelastic fluids presents several numerical challenges. We will focus our attention on finite element (FE) approximations in space and finite differences in time. Obviously, the difficulties associated to the approximation of the incompressible Navier-Stokes equations are inherited, and these include the choice of the velocity-pressure interpolation and the treatment of convection dominated flows and, in particular, the complex flow behavior encountered for high Reynolds number. On top of that, now (part of) the stress needs to be considered as an independent variable, as it is solution of an evolution equation, and this poses the question of how to interpolate it. Stress-velocity-pressure interpolations that render the discrete problem stable using the standard Galerkin method are rare and difficult to implement. Our option in this case has been to resort from the Galerkin to stabilized FE methods, which allow one to use arbitrary interpolations. Moreover, these formulations also serve to deal

Keywords and phrases: Viscoelastic fluids, log-conformation reformulation, high Weissenberg number problem, VMS formulation, time dependent sub-grid scales

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with high Reynolds number problems and, to some extend, with high Weissenberg number problems, thus allowing one to approximate flows with high inertia and high elasticity.

In fact, perhaps the most relevant issue in the approximation of viscoelastic flows is precisely the high Weissenberg number problem (HWNP), which consists in the failure of iterative schemes to solve the discrete problem when elasticity is beyond a threshold. To alleviate this issue, the log-conformation reformulation (LCR) of the constitutive equation was introduced [1]. It essentially consists of a change of unknowns, replacing the elastic stress by the logarithm of the conformation tensor. Thus, two sets of equations can be considered when approximating viscoelastic flows, the standard one and the LCR.

In our group, we have designed FE methods for both the standard and the LCR form of the viscoelastic flow equations. For the stationary problem, the design of a stabilized FE formulation for the standard form can be found in [2, 3], whereas for the LCR form it can be found in [4]. Some insight on the behavior of these formulations can be obtained analyzing the *linearized* form of the corresponding problems; this was done in [5] for the standard approach and in [6] for the LCR (see also [7] for a review). In all cases, the FE formulation developed and analyzed is based on the Variational-Multiscale (VMS) concept [8, 9], which consists in splitting the unknown in terms of the FE component and the remainder, called sub-grid scale (SGS). The equation for the SGS is then approximated so as to obtain a closed form expression for it in terms of the FE component. Additional approximations can then be performed to obtain a convenient final discrete problem.

In this paper we move our attention to the *transient* viscoelastic flow equations. The time dependent nonlinear problem was analyzed in [10], under classical assumptions to guarantee uniqueness of solution and considering time continuous. Here we study the numerical time integration of the linearized equations. This time integration can be done using standard techniques, and in particular finite difference schemes. For the analysis to be presented here we restrict ourselves to the simplest backward difference scheme of first order (BDF1). This is enough to highlight the importance of one of the main features of our approach, namely, that we allow the SGSs to be *time dependent*, and therefore we need to track them in time. This allows us to avoid the so called small time step instability, which is a well known instability effect which occurs when some stabilized FE methods in space are used in conjunction with time integration schemes and the time step size is small (compared to a reference value) [11]. In particular, this prevents from letting the time step go to zero while keeping fixed the mesh size (or, in general, when anisotropic space-time discretizations are used, see [12]). We also use the BDF1 scheme to integrate in time the SGSs. The design and testing of the formulation whose linarized form is analyzed here can be found in [13].

Even if it might seem a restrictive situation, the problem analyzed allows us to get some insight on the behavior of the formulation considered that the general nonlinear problem does not permit. Let us describe some results known for the nonlinear case and their restrictions. At the continuous level, even well-posedness for general models is not well understood. Global existence in time of solutions has been proved only if the initial conditions are small perturbations of the rest state, and for the steady-state case existence of solutions can be proved only for small perturbations of the Newtonian case (see [14, 15] for comprehensive reviews). The existence of slow steady flows of viscoelastic fluids using differential constitutive equations was proved in [16] for Hilbert spaces. For the time-dependent case, the existence of solutions locally in time, and for small data globally in time, has been proved for Hilbert spaces in [17]. The extension to Banach spaces and a complete review of uniqueness, regularity, well-posedness and stability results can be found in [15]. The existence of global weak solutions for general initial conditions using a co-rotational Oldroyd-B model has been proved in [18] in a simplified setting. In [19] the authors proved global existence of weak solutions in 2D to the Oldroyd-B model regularized with the introduction of a diffusion term in the constitutive equation. An analysis of the effects it has on the numerical approximation can be found in [20].

In the context of the FE approximation, for the steady-state case, one of the first works in which the existence of approximate solutions and error analysis were presented is [21]. The authors used a discontinuous interpolation (Lesaint-Raviart method) to treat the viscoelastic stresses. Later, in [22] it was shown that a certain choice of interpolation of unknowns has a unique solution for which error bounds can be found. In [23] a stationary nonlinear Stokes problem was analyzed. More recently, in [24] the authors presented an error analysis of a very particular Oldroyd-B model with the limiting Weissenberg number going to infinity, but assuming a suitable regularity of the exact solution for FE and finite volume methods.

The extension to the time-dependent case was treated in [25] for a nonlinear Stokes problem, proving global existence in time under the small data assumption. For a Stokes/Oldroyd-B linearized problem, in [26] optimal a priori error estimates using the Interior-Penalty method and adding some artificial diffusion in the constitutive equation were presented. A similar problem was studied in [27] for the steady state case, proving existence and uniqueness of the continuous problem and of a FE approximation under the small data assumption. In [28] the Oldroyd-B time-dependent case was analyzed for small Weissnberg numbers, both in the semi-discrete and in the fully discrete cases, proving existence and deriving a priori error estimates for the numerical approximation. In [29] the authors analyzed the time behavior of the viscoelastic Oldroyd-B model in 2D using a Galerkin formulation in space; in this work, the stress is eliminated through a proper projection operator, resulting in an integro-differential equation in terms of velocity and pressure.

Our objective in this paper is to present the stability analysis of the formulation we propose for the linearized problem, from which convergence follows using more or less classical arguments. More general conditions than those assumed in the references mentioned can be considered, although we will also assume small enough Weissenberg numbers. The analysis to be presented differs significantly from the one corresponding to the stationary problem and presented in [5, 6], which is based on checking an inf-sup condition in appropriate norms. Now we will be able to prove stability directly in a norm of the unknowns involving both space and time, as a result of applying the discrete Gronwall inequality. The path we will follow is essentially the same as in [30] for a much simpler problem, namely, the linearized equations of Newtonian flows. Now, apart from extending the results to viscoelastic flows, using both the standard formulation and the LCR, we consider a slightly different space stabilization, in which different SGSs are introduced for the different terms of the equations (see [13]). Care is now needed in the projections involved in the definition of the SGS spaces.

The outline of the paper is as follows. The problem to be solved and its discretization are presented in Section 2, where both the standard and the LCR forms are considered. The stability results have been split in two sections: in Section 3 they are written in terms of the SGSs, whereas in Section 4 they are formulated in terms of the FE unknowns. Finally, conclusions are drawn in Section 5.

2. Problem statement and discretization

2.1. Initial and boundary value problem

The linearized version of the problem we consider consists in assuming a known velocity in the nonlinear terms of both the momentum and the constitutive equations, i.e., a fixed-point type linearization. Likewise, for the LCR we also need to linearize the exponential of the conformation tensor. Thus, the linearized Navier-Stokes equations can be written as

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho\, \boldsymbol{a} \cdot \nabla \boldsymbol{u} - \nabla \cdot \boldsymbol{T} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega,\ t \in \,]0, t_{\mathrm{f}}[, \tag{2.1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega,\ t \in \,]0, t_{\mathrm{f}}[, \tag{2.2}$$

where Ω is the domain of the problem contained in $\mathbb{R}^d$ ($d = 2$ or 3) whose boundary is denoted by $\partial\Omega$, and $]0, t_{\mathrm{f}}[$ is the time interval of analysis. The constant density is denoted by ρ, $p : \Omega \times \,]0, t_{\mathrm{f}}[\, \rightarrow \mathbb{R}$ is the pressure field, $\boldsymbol{u} : \Omega \times \,]0, t_{\mathrm{f}}[\, \rightarrow \mathbb{R}^d$ is the unknown velocity field, $\boldsymbol{a} : \Omega \times \,]0, t_{\mathrm{f}}[\, \rightarrow \mathbb{R}^d$ is a given velocity field, $\boldsymbol{f} : \Omega \times \,]0, t_{\mathrm{f}}[\, \rightarrow \mathbb{R}^d$ is the force field and $\boldsymbol{T} : \Omega \times \,]0, t_{\mathrm{f}}[\, \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ is the deviatoric stress tensor. In general, $\boldsymbol{T}$ is defined in terms of a viscous and a viscoelastic contribution as $\boldsymbol{T} = 2\eta_s \nabla^s \boldsymbol{u} + \boldsymbol{\sigma}$, where $\nabla^s \boldsymbol{u}$ is the symmetrical part of the velocity gradient and $\boldsymbol{\sigma}$ is the viscoelastic or elastic stress tensor. The effective (or solvent) viscosity η_s and the polymeric viscosity η_p can be written as a function of the total viscosity η_0 as $\eta_s = \beta\eta_0$ and $\eta_p = (1-\beta)\eta_0$, with $\beta \in [0, 1]$.

The constitutive equation for the viscoelastic stress tensor needs now to be defined. We consider the linearized Oldroyd-B model, which reads:

$$\frac{1}{2\eta_p}\boldsymbol{\sigma} - \nabla^s \boldsymbol{u} + \frac{\lambda}{2\eta_p}\left(\frac{\partial \boldsymbol{\sigma}}{\partial t} + \boldsymbol{a} \cdot \nabla \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \nabla \boldsymbol{a} - (\nabla \boldsymbol{a})^T \cdot \boldsymbol{\sigma}\right) = \boldsymbol{0}, \text{ in } \Omega,\ t \in \,]0, t_{\mathrm{f}}[, \tag{2.3}$$

where $\lambda > 0$ is the relaxation time.

As for the boundary conditions, for simplicity $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$ is considered. The elastic stresses do not need to be prescribed. The problem is completed with the initial conditions for velocity and elastic stresses, $\boldsymbol{u} = \boldsymbol{u}^0$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}^0$ at time $t = 0$, being $\boldsymbol{u}^0$ and $\boldsymbol{\sigma}^0$ functions defined in the whole domain Ω.

We will now briefly describe the viscoelastic equations when the LCR is considered. Note that the complete development is extensively explained in [4]. The reformulation is derived from a change of variables, where $\boldsymbol{\sigma}$ is replaced by $\boldsymbol{\sigma} = \frac{\eta_p}{\lambda_0}(\boldsymbol{\tau} - \boldsymbol{I})$, and the conformation tensor $\boldsymbol{\tau}$ is written as $\boldsymbol{\tau} = \exp(\boldsymbol{\psi})$ in (2.1) and (2.3). The parameter λ_0 is defined as $\lambda_0 = \max\{k\lambda, \lambda_{0,\min}\}$, being k a constant and $\lambda_{0,\min}$ a given threshold. Furthermore, $\exp(\boldsymbol{\psi})$ is linearized around a known function $\hat{\boldsymbol{\psi}}$ as $\exp(\boldsymbol{\psi}) \approx \boldsymbol{E}\boldsymbol{\psi} + \boldsymbol{E} - \boldsymbol{E}\hat{\boldsymbol{\psi}}$, where matrix $\boldsymbol{E} = \exp(\hat{\boldsymbol{\psi}})$ is known. With this in mind, the linearized form of the LCR can be written as

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} - \frac{\eta_p}{\lambda_0}\nabla \cdot \boldsymbol{E}\boldsymbol{\psi} - 2\eta_s \nabla \cdot (\nabla^s \boldsymbol{u}) + \rho\, \boldsymbol{a} \cdot \nabla \boldsymbol{u} + \nabla p = \boldsymbol{f}_u, \tag{2.4}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.5}$$

$$\frac{1}{2\lambda_0}\boldsymbol{E}\boldsymbol{\psi} - \nabla^s \boldsymbol{u} + \frac{\lambda}{2\lambda_0}\left(\frac{\partial \boldsymbol{E}\boldsymbol{\psi}}{\partial t} + \boldsymbol{a} \cdot \nabla \boldsymbol{E}\boldsymbol{\psi} - \boldsymbol{E}\boldsymbol{\psi} \cdot \nabla \boldsymbol{a} - (\nabla \boldsymbol{a})^T \cdot \boldsymbol{E}\boldsymbol{\psi} + 2\nabla^s \boldsymbol{u}\right) = \boldsymbol{f}_\psi, \tag{2.6}$$

where $\boldsymbol{f}_u$ and $\boldsymbol{f}_\psi$ involve known terms arising from the linearization.

2.2. **Variational form**

Let us introduce some notation to write the weak form of the problem. Given a set $\omega \subset \Omega$, $L^2(\omega)$ is the space of square integrable functions in ω; $H^m(\omega)$ is the space of functions whose distributional derivatives of order up to $m \geq 0$ (integer) belong to $L^2(\omega)$. The space $H_0^1(\omega)$ comprises functions in $H^1(\omega)$ vanishing on $\partial\omega$, whereas $H^{-1}(\Omega)$ is the topological dual of $H_0^1(\Omega)$, the duality pairing being $\langle\cdot,\cdot\rangle$. The space of essentially bounded functions is $L^\infty(\omega)$, and the space of functions in $L^\infty(\omega)$ with first derivatives in $L^\infty(\omega)$ is $W^{1,\infty}(\omega)$. An analogous notation is used for the time regularity. The L^2 inner product in ω (for scalars, vectors and tensors) is denoted by $(\cdot,\cdot)_\omega$ and the integral over ω of the product of two general functions is written as $\langle\cdot,\cdot\rangle_\omega$, the subscript being omitted when $\omega = \Omega$. The norm in a space X is denoted by $\|\cdot\|_X$, except in the case $X = L^2(\Omega)$, where the subscript is omitted.

Using this notation, the spaces for the continuous standard problem can be taken as: $\boldsymbol{\Upsilon} = H^1(\Omega)^{d\times d}_{\rm sym}$ (symmetric second order tensor with components in $H^1(\Omega)$) for the stress field, $\boldsymbol{\mathcal{V}}_0 = H_0^1(\Omega)^d$ for the velocity field and $\mathcal{Q} = L^2(\Omega)/\mathbb{R}$ for the pressure for each fixed time t (the regularity for the stress space could be relaxed). The weak form of the problem consists then in finding $\boldsymbol{U} = [\boldsymbol{u}, p, \boldsymbol{\sigma}] :]0, t_{\rm f}[\longrightarrow \boldsymbol{\mathcal{X}} := \boldsymbol{\mathcal{V}}_0 \times \mathcal{Q} \times \boldsymbol{\Upsilon}$, such that the initial conditions are satisfied and:

$$
\begin{aligned}
\left(\rho\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v}\right) + (\boldsymbol{\sigma}, \nabla^s \boldsymbol{v}) + 2(\eta_s \nabla^s \boldsymbol{u}, \nabla^s \boldsymbol{v}) + \langle \rho\, \boldsymbol{a}\cdot\nabla\boldsymbol{u}, \boldsymbol{v}\rangle - (p, \nabla\cdot\boldsymbol{v}) &= \langle \boldsymbol{f}, \boldsymbol{v}\rangle,\\
(q, \nabla\cdot\boldsymbol{u}) &= 0,\\
\frac{1}{2\eta_p}(\boldsymbol{\sigma}, \boldsymbol{\chi}) - (\nabla^s \boldsymbol{u}, \boldsymbol{\chi}) + \frac{\lambda}{2\eta_p}\left(\frac{\partial \boldsymbol{\sigma}}{\partial t} + \boldsymbol{a}\cdot\nabla\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot\nabla\boldsymbol{a} - (\nabla\boldsymbol{a})^T\cdot\boldsymbol{\sigma}, \boldsymbol{\chi}\right) &= 0,
\end{aligned}
$$

for all $\boldsymbol{V} = [\boldsymbol{v}, q, \boldsymbol{\chi}] \in \boldsymbol{\mathcal{X}}$, where it is assumed that $\boldsymbol{a} \in L^\infty(]0, t_{\rm f}[; W^{1,\infty}(\Omega)^d)$, with $\nabla\cdot\boldsymbol{a} = 0$, $\boldsymbol{n}\cdot\boldsymbol{a} = 0$ on $\partial\Omega$ (zero normal component), and $\boldsymbol{f}$ is such that $\langle \boldsymbol{f}, \boldsymbol{v}\rangle$ is well defined. In compact form, the problem can be written as:

$$
G_{\rm std}(\boldsymbol{U}, \boldsymbol{V}) + B_{\rm std}(\boldsymbol{U}, \boldsymbol{V}) = L_{\rm std}(\boldsymbol{V}), \tag{2.7}
$$

for all $\boldsymbol{V} \in \boldsymbol{\mathcal{X}}$, where

$$
G_{\rm std}(\boldsymbol{U}, \boldsymbol{V}) = \left(\rho\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v}\right) + \frac{\lambda}{2\eta_p}\left(\frac{\partial \boldsymbol{\sigma}}{\partial t}, \boldsymbol{\chi}\right), \tag{2.8}
$$

$$
\begin{aligned}
B_{\rm std}(\boldsymbol{U}, \boldsymbol{V}) = {}& 2(\eta_s \nabla^s \boldsymbol{u}, \nabla^s \boldsymbol{v}) + \langle \rho\, \boldsymbol{a}\cdot\nabla\boldsymbol{u}, \boldsymbol{v}\rangle + (\boldsymbol{\sigma}, \nabla^s \boldsymbol{v}) - (p, \nabla\cdot\boldsymbol{v}) + (q, \nabla\cdot\boldsymbol{u})\\
&+ \frac{1}{2\eta_p}(\boldsymbol{\sigma}, \boldsymbol{\chi}) - (\nabla^s \boldsymbol{u}, \boldsymbol{\chi}) + \frac{\lambda}{2\eta_p}\left(\boldsymbol{a}\cdot\nabla\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot\nabla\boldsymbol{a} - (\nabla\boldsymbol{a})^T\cdot\boldsymbol{\sigma}, \boldsymbol{\chi}\right),
\end{aligned} \tag{2.9}
$$

$$
L_{\rm std}(\boldsymbol{V}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle. \tag{2.10}
$$

Considering now the LCR, the spaces for the velocity and pressure for the continuous problems are the ones defined above for the standard formulation, and now the space for tensor $\boldsymbol{\psi}$ is denoted by $\bar{\boldsymbol{\Upsilon}}$ for each fixed time t. Assuming that the components of $\boldsymbol{E}$ belong to $L^\infty(]0, t_{\rm f}[; W^{1,\infty}(\Omega))$, we can take $\bar{\boldsymbol{\Upsilon}} = \boldsymbol{\Upsilon}$, but we will keep distinguishing these spaces as $\boldsymbol{\psi}$ and $\boldsymbol{\sigma}$ are different objects.

The weak form of the problem consists in finding $\boldsymbol{U} = [\boldsymbol{u}, p, \boldsymbol{\psi}] :]0, t_{\rm f}[\longrightarrow \bar{\boldsymbol{\mathcal{X}}} := \boldsymbol{\mathcal{V}}_0 \times \mathcal{Q} \times \bar{\boldsymbol{\Upsilon}}$, such that the initial conditions are satisfied and:

$$\left(\rho \frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v}\right) + \frac{\eta_p}{\lambda_0}(\boldsymbol{E}\boldsymbol{\psi}, \nabla^s \boldsymbol{v}) + 2(\eta_s \nabla^s \boldsymbol{u}, \nabla^s \boldsymbol{v}) + \langle \rho\, \boldsymbol{a} \cdot \nabla \boldsymbol{u}, \boldsymbol{v}\rangle - (p, \nabla \cdot \boldsymbol{v}) = \langle \boldsymbol{f}_u, \boldsymbol{v}\rangle, \tag{2.11}$$

$$(q, \nabla \cdot \boldsymbol{u}) = 0, \tag{2.12}$$

$$\frac{1}{2\lambda_0}(\boldsymbol{E}\boldsymbol{\psi}, \boldsymbol{\chi}) - (\nabla^s \boldsymbol{u}, \boldsymbol{\chi}) + \frac{\lambda}{2\lambda_0}\left(\frac{\partial \boldsymbol{E}\boldsymbol{\psi}}{\partial t} + \boldsymbol{a} \cdot \nabla \boldsymbol{E}\boldsymbol{\psi} - \boldsymbol{E}\boldsymbol{\psi} \cdot \nabla \boldsymbol{a} - (\nabla \boldsymbol{a})^T \cdot \boldsymbol{E}\boldsymbol{\psi} + 2\nabla^s \boldsymbol{u}, \boldsymbol{\chi}\right) = \langle \boldsymbol{f}_\psi, \boldsymbol{\chi}\rangle, \tag{2.13}$$

for all $\boldsymbol{V} = [\boldsymbol{v}, q, \boldsymbol{\chi}] \in \boldsymbol{\mathcal{X}}$. Taking into account the new definition of $\boldsymbol{U}$ for this formulation, the problem can be written as:

$$G_{\log}(\boldsymbol{U}, \boldsymbol{V}) + B_{\log}(\boldsymbol{U}, \boldsymbol{V}) = L_{\log}(\boldsymbol{V}), \tag{2.14}$$

where

$$G_{\log}(\boldsymbol{U}, \boldsymbol{V}) = \left(\rho \frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v}\right) + \frac{\lambda}{2\lambda_0}\left(\frac{\partial \boldsymbol{E}\boldsymbol{\psi}}{\partial t}, \boldsymbol{\chi}\right), \tag{2.15}$$

$$B_{\log}(\boldsymbol{U}, \boldsymbol{V}) = \frac{\eta_p}{\lambda_0}(\boldsymbol{E}\boldsymbol{\psi}, \nabla^s \boldsymbol{v}) + 2(\eta_s \nabla^s \boldsymbol{u}, \nabla^s \boldsymbol{v}) + \langle \rho\, \boldsymbol{a} \cdot \nabla \boldsymbol{u}, \boldsymbol{v}\rangle - (p, \nabla \cdot \boldsymbol{v}) + (\nabla \cdot \boldsymbol{u}, q) + \frac{1}{2\lambda_0}(\boldsymbol{E}\boldsymbol{\psi}, \boldsymbol{\chi}) - (\nabla^s \boldsymbol{u}, \boldsymbol{\chi}) + \frac{\lambda}{2\lambda_0}\left(\boldsymbol{a} \cdot \nabla \boldsymbol{E}\boldsymbol{\psi} - \boldsymbol{E}\boldsymbol{\psi} \cdot \nabla \boldsymbol{a} - (\nabla \boldsymbol{a})^T \cdot \boldsymbol{E}\boldsymbol{\psi} + 2\nabla^s \boldsymbol{u}, \boldsymbol{\chi}\right), \tag{2.16}$$

$$L_{\log}(\boldsymbol{V}) = \langle \boldsymbol{f}_u, \boldsymbol{v}\rangle + \langle \boldsymbol{f}_\psi, \boldsymbol{\chi}\rangle. \tag{2.17}$$

Note that we have taken $\boldsymbol{\chi} \in \boldsymbol{\Upsilon}$, not in $\bar{\boldsymbol{\Upsilon}}$, i.e., we take the test function in the space of stresses even if the unknown belongs to the space of logarithms of the conformation tensor. See [4] for further discussion.

2.3. Galerkin finite element discretization and time discretization

Consider a FE partition of the domain Ω, $\mathcal{T}_h = \{K\}$. The diameter of an element $K \in \mathcal{T}_h$ is denoted by h_K and the diameter of the partition is defined as $h = \max\{h_K | K \in \mathcal{T}_h\}$. For simplicity, we will consider quasi-uniform partitions, and thus h can be considered as the representative size of all the elements.

In the particular case of the standard formulation, from $\mathcal{T}_h$ we may construct conforming FE spaces for the velocity, the pressure and the elastic stress, $\boldsymbol{\mathcal{V}}_{h,0} \subset \boldsymbol{\mathcal{V}}_0$, $\mathcal{Q}_h \subset \mathcal{Q}$, $\boldsymbol{\Upsilon}_h \subset \boldsymbol{\Upsilon}$, respectively. So, calling $\boldsymbol{\mathcal{X}}_h := \boldsymbol{\mathcal{V}}_{h,0} \times \mathcal{Q}_h \times \boldsymbol{\Upsilon}_h$, the Galerkin FE approximation of the standard problem consists in finding $\boldsymbol{U}_h :]0, t_{\rm f}[\longrightarrow \boldsymbol{\mathcal{X}}_h$, such that:

$$G_{\rm std}(\boldsymbol{U}_h, \boldsymbol{V}_h) + B_{\rm std}(\boldsymbol{U}_h, \boldsymbol{V}_h) = L_{\rm std}(\boldsymbol{V}_h), \tag{2.18}$$

for all $\boldsymbol{V}_h = [\boldsymbol{v}_h, q_h, \boldsymbol{\chi}_h] \in \boldsymbol{\mathcal{X}}_h$, and satisfying the appropriate initial conditions.

Moving now to the LCR, from $\mathcal{T}_h$ we may construct the FE space for the new variable $\boldsymbol{\psi}$, $\bar{\boldsymbol{\Upsilon}}_h \subset \bar{\boldsymbol{\Upsilon}}$. In the analysis to be presented, we will prove stability for the FE component of $\boldsymbol{E}\boldsymbol{\psi}_h$, $\boldsymbol{\psi}_h \in \bar{\boldsymbol{\Upsilon}}_h$, i.e., for $P_h[\boldsymbol{E}\boldsymbol{\psi}_h]$, where $P_h : L^2(\Omega)^{d\times d}_{\rm sym} \longrightarrow \bar{\boldsymbol{\Upsilon}}_h$ is the $L^2(\Omega)$ projection. Thus, we may take directly as unknown of the problem $\bar{\boldsymbol{\psi}}_h := P_h[\boldsymbol{E}\boldsymbol{\psi}_h]$; note that using $\bar{\boldsymbol{\psi}}_h$ or $\boldsymbol{E}\boldsymbol{\psi}_h$ is equivalent for terms to be tested with FE functions, and in particular for the temporal derivative. If $\bar{\boldsymbol{\mathcal{X}}}_h := \boldsymbol{\mathcal{V}}_{h,0} \times \mathcal{Q}_h \times \bar{\boldsymbol{\Upsilon}}_h$, the Galerkin approximation consists in finding $\boldsymbol{U}_h = [\boldsymbol{u}_h, p_h, \bar{\boldsymbol{\psi}}_h] :]0, t_{\rm f}[\longrightarrow \bar{\boldsymbol{\mathcal{X}}}_h$, such that

$$G_{\log}(\boldsymbol{U}_h, \boldsymbol{V}_h) + B_{\log}(\boldsymbol{U}_h, \boldsymbol{V}_h) = L_{\log}(\boldsymbol{V}_h),$$

for all $\boldsymbol{V}_h = [\boldsymbol{v}_h, q_h, \boldsymbol{\chi}_h] \in \boldsymbol{\mathcal{X}}_h$.

It is well known that the Gakerkin approximation is unstable unless convective terms are not relevant and appropriate compatibility conditions between $\mathcal{Q}_h$ and $\boldsymbol{\mathcal{V}}_{h,0}$, on the one hand, and between $\boldsymbol{\mathcal{V}}_{h,0}$ and $\boldsymbol{\Upsilon}_h$, on the other hand, are met (see for example [2] and references therein). In the next Subsection we will present a stable formulation, able in particular to deal with continuous approximations for all fields, which is the situation we shall consider.

The fully discrete problem is obtained after discretizing in time. To this end, we consider the BDF1 scheme. Let $0 = t^0 < t^1 < \cdots < t^N = t_{\rm f}$ be a uniform partition of the time interval of size $\delta t = t^i - t^{i-1}$, $i = 1, \ldots, N$. We will use a superscript to denote the time step counter, indicating by n the time level up to which the solution is known. Likewise, we employ the notation $\delta f^n := f^{n+1} - f^n$ and $\delta_t f^n := \frac{\delta f^n}{\delta t}$ for any time dependent function f, f^n being an approximation to $f(t^n)$. The fully discrete version of problem (2.18) can be written as follows: given $\boldsymbol{U}_h^n \in \boldsymbol{\mathcal{X}}_h$, find $\boldsymbol{U}_h^{n+1} \in \boldsymbol{\mathcal{X}}_h$ such that

$$(\rho\, \delta_t \boldsymbol{u}_h^n, \boldsymbol{v}_h) + \frac{\lambda}{2\eta_p} (\delta_t \boldsymbol{\sigma}_h^n, \boldsymbol{\chi}_h) + B_{\rm std}(\boldsymbol{U}_h^{n+1}, \boldsymbol{V}_h) = L_{\rm std}(\boldsymbol{V}_h), \tag{2.19}$$

for all $\boldsymbol{V}_h \in \boldsymbol{\mathcal{X}}_h$, with $\boldsymbol{U}_h^0$ determined by the initial conditions. When the forcing terms in $L_{\rm std}(\boldsymbol{V}_h)$ depend on time, it is understood that they are evaluated at t^{n+1}, and likewise for the vector field $\boldsymbol{a}$.

The time discrete version of the LCR problem can be written similarly.

2.4. Stabilized finite element method

Let us describe the stabilized FE method we analyze in the following. We will not derive it here, as it was presented in detail in [13]. Let us only describe the main ingredients. The starting point is the VMS approach, splitting the unknowns into their FE component and their SGS. The SGSs are considered *time dependent*, and *orthogonal* to the FE space in the $L^2(\Omega)$ sense. An important remark is required here: for the velocity SGS, this orthogonality is to the FE space *without boundary conditions*, i.e., without imposing that velocities vanish on the boundary. We will call $\boldsymbol{\mathcal{V}}_h \subset H^1(\Omega)^d$ this space, and $P_h : L^2(\Omega)^d \longrightarrow \boldsymbol{\mathcal{V}}_h$ the $L^2(\Omega)$ projection onto $\boldsymbol{\mathcal{V}}_h$ (we use the same symbol P_h as for the $L^2(\Omega)$ projection onto $\boldsymbol{\Upsilon}_h$ or $\bar{\boldsymbol{\Upsilon}}_h$). Note that $\boldsymbol{\mathcal{V}}_h^\perp \subset \boldsymbol{\mathcal{V}}_{h,0}^\perp$, $\perp$ standing for $L^2(\Omega)$ orthogonality, and also that for any function f defined on Ω the function $P_h^\perp(f) \in \boldsymbol{\mathcal{V}}_h^\perp$ converges to zero as $h \to 0$ in any suitable norm at the optimal rate provided by the FE interpolation. This would *not* happen if P_h is replaced by the $L^2(\Omega)$ projection onto $\boldsymbol{\mathcal{V}}_{h,0}$.

The method to be analyzed has as starting point the decompositions $\boldsymbol{u} \approx \boldsymbol{u}_h + \tilde{\boldsymbol{u}}_1 + \tilde{\boldsymbol{u}}_2 + \tilde{\boldsymbol{u}}_3$, $p \approx p_h$, $\boldsymbol{\sigma} \approx \boldsymbol{\sigma}_h + \tilde{\boldsymbol{\sigma}}$. The velocity SGSs $\tilde{\boldsymbol{u}}_1,\ \tilde{\boldsymbol{u}}_2,\ \tilde{\boldsymbol{u}}_3 \in \boldsymbol{\mathcal{V}}_h^\perp$ are considered to be independent for the design of the method (see [3]). For the pressure, we could also introduce a pressure SGS $\tilde{p} \in \mathcal{Q}_h^\perp$, but this would only introduce a marginal added value in the stability results. Finally, there is only one stress SGS, $\tilde{\boldsymbol{\sigma}} \in \boldsymbol{\Upsilon}_h^\perp$ for the standard formulation, and likewise for the LCR, for the reasons explained in [4].

To simplify the exposition, we shall consider that spaces $\mathcal{Q}_h$ and $\boldsymbol{\Upsilon}_h$ are made of *continuous* functions. Otherwise, terms involving jumps of pressures and stresses need to be introduced in the formulation, as it was done in [5] for the stationary standard problem.

After adequate approximations for the SGSs and neglecting some terms that do not contribute to stability, the method to be analyzed for the standard formulation reads as follows: given $\boldsymbol{u}_h^n \in \boldsymbol{\mathcal{V}}_{h,0}$, $\tilde{\boldsymbol{u}}_i^n \in \boldsymbol{\mathcal{V}}_h^\perp$, $i=1,2,3$, $\boldsymbol{\sigma}_h^n \in \boldsymbol{\Upsilon}_h$ and $\tilde{\boldsymbol{\sigma}}^n \in \boldsymbol{\Upsilon}_h^\perp$, find $\boldsymbol{u}_h^{n+1} \in \boldsymbol{\mathcal{V}}_{h,0}$, $\tilde{\boldsymbol{u}}_i^{n+1} \in \boldsymbol{\mathcal{V}}_h^\perp$, $i=1,2,3$, $\boldsymbol{\sigma}_h^{n+1} \in \boldsymbol{\Upsilon}_h$, $p_h^{n+1} \in \mathcal{Q}_h$ and $\tilde{\boldsymbol{\sigma}}^{n+1} \in \boldsymbol{\Upsilon}_h^\perp$ such that

$$
\begin{aligned}
&(\rho\,\delta_t \boldsymbol{u}_h^n, \boldsymbol{v}_h) + \frac{\lambda}{2\eta_p}(\delta_t \boldsymbol{\sigma}_h^n, \boldsymbol{\chi}_h) + B_{\rm std}(\boldsymbol{U}_h^{n+1}, \boldsymbol{V}_h) - L_{\rm std}(\boldsymbol{V}_h) \\
&\quad - \left(\tilde{\boldsymbol{u}}_1^{n+1}, \rho\,\boldsymbol{a}\cdot\nabla \boldsymbol{v}_h\right) - \left(\tilde{\boldsymbol{u}}_2^{n+1}, \nabla q_h\right) + \left(\tilde{\boldsymbol{u}}_3^{n+1}, \nabla\cdot \boldsymbol{\chi}_h\right) \\
&\quad + \left(\tilde{\boldsymbol{\sigma}}^{n+1}, \nabla^s \boldsymbol{v}_h - \frac{\lambda}{2\eta_p}\left(\boldsymbol{a}\cdot\nabla\boldsymbol{\chi}_h + \boldsymbol{\chi}_h\cdot(\nabla\boldsymbol{a})^T + \nabla\boldsymbol{a}\cdot\boldsymbol{\chi}_h\right)\right) = 0, && (2.20)\\
&\rho\,\delta_t \tilde{\boldsymbol{u}}_1^n + \alpha_1^{-1}\tilde{\boldsymbol{u}}_1^{n+1} = -P_h^\perp[\rho\,\boldsymbol{a}\cdot\nabla \boldsymbol{u}_h^{n+1}], && (2.21)\\
&\rho\,\delta_t \tilde{\boldsymbol{u}}_2^n + \alpha_1^{-1}\tilde{\boldsymbol{u}}_2^{n+1} = -P_h^\perp[\nabla p_h^{n+1}], && (2.22)\\
&\rho\,\delta_t \tilde{\boldsymbol{u}}_3^n + \alpha_1^{-1}\tilde{\boldsymbol{u}}_3^{n+1} = P_h^\perp[\nabla\cdot\boldsymbol{\sigma}_h^{n+1}], && (2.23)\\
&\frac{\lambda}{2\eta_p}\delta_t\tilde{\boldsymbol{\sigma}}^n + \alpha_3^{-1}\tilde{\boldsymbol{\sigma}}^{n+1} = P_h^\perp\left[\nabla^s \boldsymbol{u}_h^{n+1} - \frac{\lambda}{2\eta_p}\left(\boldsymbol{a}\cdot\nabla\boldsymbol{\sigma}_h^{n+1} - \boldsymbol{\sigma}_h^{n+1}\cdot\nabla\boldsymbol{a} - (\nabla\boldsymbol{a})^T\cdot\boldsymbol{\sigma}_h^{n+1}\right)\right], && (2.24)
\end{aligned}
$$

where α_1 and α_3 are the stabilization parameters (the notation of [31] has been kept), computed within each element K and at each time level $n+1$ as

$$
\begin{aligned}
\alpha_1^{-1} &= c_1\frac{\eta_0}{h^2} + \frac{\|\boldsymbol{a}\|_{L^\infty(K)}}{h}, && (2.25)\\
\alpha_3^{-1} &= c_3\frac{1}{2\eta_p} + c_4\left(\frac{\lambda}{2\eta_p}\frac{\|\boldsymbol{a}\|_{L^\infty(K)}}{h} + \frac{\lambda}{\eta_p}\|\nabla\boldsymbol{a}\|_{L^\infty(K)}\right), && (2.26)
\end{aligned}
$$

where c_i, $i=1,2,3,4$, are algorithmic constants (see [7, 9] for an overview of the motivation to obtain α_1, α_3 and the choice of the algorithmic constants). Observe that equations (2.21)-(2.24) result from the time evolution of the SGSs; these are in fact ordinary differential equations that need to be integrated in time at the numerical integration points of the FE mesh.

Let us move now to the LCR. Even though it is not crucial, from the implementation point of view it is easier to compute the SGSs in terms of stresses rather than in terms of the variable $\boldsymbol{\psi}$. Thus, the stabilized FE problem we consider in this case can be written as follows: given $\boldsymbol{u}_h^n \in \boldsymbol{\mathcal{V}}_{h,0}$, $\tilde{\boldsymbol{u}}_i^n \in \boldsymbol{\mathcal{V}}_h^\perp$, $i=1,2,3$, $\bar{\boldsymbol{\psi}}_h^n \in \bar{\boldsymbol{\Upsilon}}_h$ and $\tilde{\boldsymbol{\sigma}}^n \in \boldsymbol{\Upsilon}_h^\perp$, find $\boldsymbol{U}_h^{n+1} = [\boldsymbol{u}_h^{n+1}, p_h^{n+1}, \bar{\boldsymbol{\psi}}_h^{n+1}] \in \boldsymbol{\mathcal{V}}_{h,0}\times\mathcal{Q}_h\times\bar{\boldsymbol{\Upsilon}}_h$, $\tilde{\boldsymbol{u}}_i^{n+1} \in \boldsymbol{\mathcal{V}}_h^\perp$, $i=1,2,3$, and $\tilde{\boldsymbol{\sigma}}^{n+1} \in \boldsymbol{\Upsilon}_h^\perp$ such that

$$
\begin{aligned}
&(\rho\,\delta_t \boldsymbol{u}_h^n, \boldsymbol{v}_h) + \left(\frac{\lambda}{2\lambda_0}\delta_t\bar{\boldsymbol{\psi}}_h^n, \boldsymbol{\chi}_h\right) + B_{\log}(\boldsymbol{U}_h^{n+1}, \boldsymbol{V}_h) - L_{\log}(\boldsymbol{V}_h) \\
&\quad - \left(\tilde{\boldsymbol{u}}_1^{n+1}, \rho\,\boldsymbol{a}\cdot\nabla \boldsymbol{v}_h\right) - \left(\tilde{\boldsymbol{u}}_2^{n+1}, \nabla q_h\right) + \left(\tilde{\boldsymbol{u}}_3^{n+1}, \nabla\cdot \boldsymbol{\chi}_h\right), \\
&\quad + \left(\tilde{\boldsymbol{\sigma}}^{n+1}, \frac{1}{2\eta_p}\boldsymbol{\chi}_h + \nabla^s \boldsymbol{v}_h - \frac{\lambda}{2\eta_p}\left(\boldsymbol{a}\cdot\nabla\boldsymbol{\chi}_h + \boldsymbol{\chi}_h\cdot(\nabla\boldsymbol{a})^T + \nabla\boldsymbol{a}\cdot\boldsymbol{\chi}_h\right)\right) = 0 && (2.27)\\
&\rho\,\delta_t \tilde{\boldsymbol{u}}_1^n + \alpha_1^{-1}\tilde{\boldsymbol{u}}_1^{n+1} = -P_h^\perp\left[\rho\,\boldsymbol{a}\cdot\nabla \boldsymbol{u}_h^{n+1}\right], && (2.28)
\end{aligned}
$$

$$\rho\,\delta_t\tilde{\boldsymbol{u}}_2^n + \alpha_1^{-1}\tilde{\boldsymbol{u}}_2^{n+1} = -P_h^{\perp}\left[\nabla p_h^{n+1}\right], \tag{2.29}$$

$$\rho\,\delta_t\tilde{\boldsymbol{u}}_3^n + \alpha_1^{-1}\tilde{\boldsymbol{u}}_3^{n+1} = P_h^{\perp}\left[\frac{\eta_p}{\lambda_0}\nabla\cdot\bar{\boldsymbol{\psi}}_h^{n+1}\right], \tag{2.30}$$

$$\frac{\lambda}{2\eta_p}\delta_t\tilde{\boldsymbol{\sigma}}^n + \alpha_3^{-1}\tilde{\boldsymbol{\sigma}}^{n+1} = P_h^{\perp}\left[\nabla\boldsymbol{u}_h^{n+1} - \frac{\lambda}{2\lambda_0}\left(\boldsymbol{a}\cdot\nabla\bar{\boldsymbol{\psi}}_h^{n+1} - \bar{\boldsymbol{\psi}}_h^{n+1}\cdot\nabla\boldsymbol{a} - (\nabla\boldsymbol{a})^T\cdot\bar{\boldsymbol{\psi}}_h^{n+1} + 2\nabla^s\boldsymbol{u}_h\right)\right]. \tag{2.31}$$

The parameters α_1 and α_3 are still given by (2.25) and (2.26), respectively.

3. Stability analysis in terms of the sub-grid scales

The analysis to be presented follows the same lines as the one presented in [30] for the stabilized FE approximation of the incompressible flow of Newtonian fluids, where the SGSs are also considered time-dependent. As in that reference, the analysis developed in this section aims at emphasizing the effect of tracking the SGSs in time from the analytical perspective. Here we extend it to the two versions we are considering of the viscoelastic flow problem and also using the three velocity SGSs introduced rather than only one.

In this section we will prove stability considering that the SGSs are independent unknowns, deferring to the following section stability results in terms of the FE unknowns only. Let us remark here that the stability results to be obtained are *independent* of the expression of the stabilization parameters α_1 and α_3.

3.1. **Preliminaries**

Let us introduce some additional notation and tools to be used in the following:

- Consider a sequence of functions defined on Ω, $F = \{f^n\}$, with index n ranging from 1 to N, the number of time intervals of the partition in time. Given a Banach space X of functions in Ω, for $1 \leq p < \infty$ we say that $F \in \ell^p(X)$ if $\sum_{n=1}^N \delta t\,\|f^n\|_X^p \leq C < \infty$, and $F \in \ell^\infty(X)$ if $\max_{n=1,\dots,N}\|f^n\|_X \leq C < \infty$. Here and below, C denotes a generic positive constant.
- Given two sequences of functions defined in Ω, $F = \{f^n\}$ and $G = \{g^n\}$, with f^0 and g^0 also given, we will make use of the following discrete version of the integration-by-parts formula:

$$\sum_{n=0}^{N-1}\left\langle \delta f^n, g^{n+1}\right\rangle = -\sum_{n=0}^{N-1}\left\langle f^n, \delta g^n\right\rangle + \left\langle f^N, g^N\right\rangle - \left\langle f^0, g^0\right\rangle.$$

- For FE functions f_h in Ω, we will make use of the classical inverse estimate

$$\|\nabla f_h\| \leq \frac{C_{\text{inv}}}{h}\|f_h\|, \tag{3.1}$$

  which holds for quasi-uniform FE partitions as those we are considering, C_{inv} being a constant.
- Regarding the initial conditions, we will assume that they all have components in $L^2(\Omega)$, both the FE components and the SGSs. The latter, in fact, can be taken as zero, as considering the initial conditions in the FE spaces keeps the convergence order of the formulation (which will not be analyzed here). Furthermore, for the LCR we assume that $\left\|P_h\left(\boldsymbol{E}\boldsymbol{\psi}_h^0\right)\right\|$ is bounded uniformly in h.

- Concerning the force terms, the assumption of $\boldsymbol{f} \in L^2\left(]0,t_{\mathrm{f}}[;L^2(\Omega)^d\right)$ for the standard formulation leads to stability. We consider t_{f} fixed and bounded (its exponential appears when applying the discrete Gronwall inequality). For the time discrete problem, the counterpart of $\boldsymbol{f} \in L^2\left(]0,t_{\mathrm{f}}[;L^2(\Omega)^d\right)$ is $\{\boldsymbol{f}^n\} \in \ell^2\left(L^2(\Omega)^d\right)$. Similar assumptions apply to the LCR for $\boldsymbol{f}_u$ and $\boldsymbol{f}_\psi$.
- We shall use the notation $\lesssim$ for $\leq$ up to positive constants.

3.2. Stability bounds for the standard formulation

The first result we will prove is a classical stability estimate, in which only velocity and stresses are implied, but not the pressure. In particular, for the standard formulation, the terms involved are the FE components of the velocity and stresses and the SGSs of the velocity and the stresses (Theorem 1). In the case of the LCR, the analogous result is proved in Theorem 2.

Theorem 1 (Stability bounds for the standard formulation)**.** *Let $\boldsymbol{u}_h^{n+1}$, p_h^{n+1} and $\boldsymbol{\sigma}_h^{n+1}$ be the solution of* (2.20) *and $\tilde{\boldsymbol{u}}_1^{n+1}$, $\tilde{\boldsymbol{u}}_2^{n+1}$, $\tilde{\boldsymbol{u}}_3^{n+1}$, $\tilde{\boldsymbol{\sigma}}^{n+1}$ solutions of* (2.21)*,* (2.22)*,* (2.23) *and* (2.24)*, respectively. Under the assumptions stated for $\boldsymbol{a}$, the following stability bounds hold for all $\delta t > 0$ and λ small enough:*

$$\begin{aligned}
&\max_{n=0,\ldots,N-1}\left\{\rho\left\|\boldsymbol{u}_h^{n+1}\right\|^2+\frac{\lambda}{2\eta_p}\left\|\boldsymbol{\sigma}_h^{n+1}\right\|^2+\rho\left\|\tilde{\boldsymbol{u}}_1^{n+1}\right\|^2+\rho\left\|\tilde{\boldsymbol{u}}_2^{n+1}\right\|^2+\rho\left\|\tilde{\boldsymbol{u}}_3^{n+1}\right\|^2+\frac{\lambda}{2\eta_p}\left\|\tilde{\boldsymbol{\sigma}}^{n+1}\right\|^2\right\}\\
&\quad+\sum_{n=0}^{N-1}\delta t 2\eta_s\left\|\nabla\boldsymbol{u}_h^{n+1}\right\|^2+\sum_{n=0}^{N-1}\delta t\frac{1}{2\eta_p}\left\|\boldsymbol{\sigma}_h^{n+1}\right\|^2+\sum_{n=0}^{N-1}\delta t\left\|\alpha_3^{-1/2}\tilde{\boldsymbol{\sigma}}^{n+1}\right\|^2\\
&\quad+\sum_{n=0}^{N-1}\delta t\left(\left\|\alpha_1^{-1/2}\tilde{\boldsymbol{u}}_1^{n+1}\right\|^2+\left\|\alpha_1^{-1/2}\tilde{\boldsymbol{u}}_2^{n+1}\right\|^2+\left\|\alpha_1^{-1/2}\tilde{\boldsymbol{u}}_3^{n+1}\right\|^2\right)\\
&\quad\lesssim\sum_{n=0}^{N-1}\delta t\frac{\lambda}{\rho}\left\|\boldsymbol{f}^{n+1}\right\|^2+\rho\left\|\boldsymbol{u}_h^0\right\|^2+\frac{\lambda}{2\eta_p}\left\|\boldsymbol{\sigma}_h^0\right\|^2.
\end{aligned}$$

Therefore, if $\{\boldsymbol{f}^n\} \in \ell^2(L^2(\Omega)^d)$, $\boldsymbol{u}_h^0 \in L^2(\Omega)^d$, and $\boldsymbol{\sigma}_h^0 \in L^2(\Omega)^{d\times d}_{\mathrm{sym}}$ we have that

$$\begin{aligned}
&\{\boldsymbol{u}_h^n\} \in \ell^\infty\left(L^2(\Omega)^d\right)\cap\ell^2\left(H^1(\Omega)^d\right);\{\tilde{\boldsymbol{u}}_1^n\},\{\tilde{\boldsymbol{u}}_2^n\},\{\tilde{\boldsymbol{u}}_3^n\} \in \ell^\infty\left(L^2(\Omega)^d\right);\\
&\{\alpha_1^{-1/2}\tilde{\boldsymbol{u}}_1^n\},\{\alpha_1^{-1/2}\tilde{\boldsymbol{u}}_2^n\},\{\alpha_1^{-1/2}\tilde{\boldsymbol{u}}_3^n\} \in \ell^2\left(L^2(\Omega)^d\right);\\
&\{\boldsymbol{\sigma}_h^n\},\{\tilde{\boldsymbol{\sigma}}^n\} \in \ell^\infty\left(L^2(\Omega)^{d\times d}_{\mathrm{sym}}\right);\{\alpha_3^{-1/2}\tilde{\boldsymbol{\sigma}}^n\} \in \ell^2\left(L^2(\Omega)^{d\times d}_{\mathrm{sym}}\right).
\end{aligned}$$

Proof. In order to obtain stability bounds for the FE solution, first of all we test (2.20) by $\boldsymbol{v}_h = \boldsymbol{u}_h^{n+1}$, $q_h = p_h^{n+1}$ and $\boldsymbol{\chi}_h = \boldsymbol{\sigma}_h^{n+1}$. Under the assumptions of the Theorem, we have that $\left(\boldsymbol{a}\cdot\nabla\boldsymbol{u}_h^{n+1},\boldsymbol{u}_h^{n+1}\right) = 0$ and $\left(\boldsymbol{a}\cdot\nabla\boldsymbol{\sigma}_h^{n+1},\boldsymbol{\sigma}_h^{n+1}\right) = 0$. Thus, we are left with:

$$\begin{aligned}
&\rho\underbrace{\left(\delta\boldsymbol{u}_h^n,\boldsymbol{u}_h^{n+1}\right)}_{(1)}+\underbrace{\frac{\lambda}{2\eta_p}\left(\delta\boldsymbol{\sigma}_h^n,\boldsymbol{\sigma}_h^{n+1}\right)}_{(2)}+\delta t 2\eta_s\left(\nabla^s\boldsymbol{u}_h^{n+1},\nabla^s\boldsymbol{u}_h^{n+1}\right)\\
&\quad+\delta t\frac{1}{2\eta_p}\left(\boldsymbol{\sigma}_h^{n+1},\boldsymbol{\sigma}_h^{n+1}\right)+\underbrace{\delta t\frac{\lambda}{2\eta_p}\left(-\boldsymbol{\sigma}_h^{n+1}\cdot\nabla\boldsymbol{a}-(\nabla\boldsymbol{a})^T\cdot\boldsymbol{\sigma}_h^{n+1},\boldsymbol{\sigma}_h^{n+1}\right)}_{(3)}
\end{aligned}$$

$$
\begin{aligned}
&- \delta t \left(\tilde{\boldsymbol{u}}_1^{n+1}, \rho \, \boldsymbol{a} \cdot \nabla \boldsymbol{u}_h^{n+1} \right) - \delta t \left(\tilde{\boldsymbol{u}}_2^{n+1}, \nabla p_h^{n+1} \right) \\
&+ \delta t \left(\tilde{\boldsymbol{u}}_3^{n+1}, \nabla \cdot \boldsymbol{\sigma}_h^{n+1} \right) + \delta t \left(\tilde{\boldsymbol{\sigma}}^{n+1}, \nabla^s \boldsymbol{u}_h^{n+1} \right) \\
&- \delta t \left(\tilde{\boldsymbol{\sigma}}^{n+1}, \frac{\lambda}{2\eta_p} \left(\boldsymbol{a} \cdot \nabla \boldsymbol{\sigma}_h^{n+1} + \boldsymbol{\sigma}_h^{n+1} \cdot (\nabla \boldsymbol{a})^T + \nabla \boldsymbol{a} \cdot \boldsymbol{\sigma}_h^{n+1} \right) \right) \\
&= \underbrace{\delta t \left(\boldsymbol{f}^{n+1}, \boldsymbol{u}_h^{n+1} \right)}_{(4)} .
\end{aligned}
$$

Now, we can add up the resulting equalities from $n = 0$ to an arbitrary time level M and consider the equality

$$
(a, a - b) = \frac{1}{2} a^{\mathbf{2}} - \frac{1}{2} b^2 + \frac{1}{2} (a - b)^2 .
$$

In particular, if a and b are consecutive terms of a series, the term $\dfrac{1}{2}a^{\mathbf{2}} - \dfrac{1}{2}b^2$ would be a telescoping series. We then have:

$$
\begin{aligned}
(1) &= \sum_{n=0}^{M} \rho \left(\delta \boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1} \right) = \frac{1}{2} \sum_{n=0}^{M} \rho \left\| \delta \boldsymbol{u}_h^n \right\|^2 + \frac{1}{2} \rho \left\| \boldsymbol{u}_h^{M+1} \right\|^2 - \frac{1}{2} \rho \left\| \boldsymbol{u}_h^0 \right\|^2 , \\
(2) &= \sum_{n=0}^{M} \frac{\lambda}{2\eta_p} \left(\delta \boldsymbol{\sigma}_h^n, \boldsymbol{\sigma}_h^{n+1} \right) = \frac{1}{2} \sum_{n=0}^{M} \frac{\lambda}{2\eta_p} \left\| \delta \boldsymbol{\sigma}_h^n \right\|^2 + \frac{1}{2} \frac{\lambda}{2\eta_p} \left\| \boldsymbol{\sigma}_h^{M+1} \right\|^2 - \frac{1}{2} \frac{\lambda}{2\eta_p} \left\| \boldsymbol{\sigma}_h^0 \right\|^2 .
\end{aligned}
$$

We can bound (3) as:

$$
(3) = - \sum_{n=0}^{M} \delta t \frac{\lambda}{2\eta_p} \left(\boldsymbol{\sigma}_h^{n+1} \cdot \nabla \boldsymbol{a} + (\nabla \boldsymbol{a})^T \cdot \boldsymbol{\sigma}_h^{n+1}, \boldsymbol{\sigma}_h^{n+1} \right) \geq - \sum_{n=0}^{M} \delta t \frac{\lambda}{\eta_p} \left\| \nabla \boldsymbol{a} \right\|_{L^\infty(\Omega)} \left\| \boldsymbol{\sigma}_h^{n+1} \right\|^2 .
$$

Also considering the inequality $\left(\boldsymbol{f}^{n+1}, \boldsymbol{u}_h^{n+1} \right) \leq \frac{\gamma}{2} \left\| \boldsymbol{f}^{n+1} \right\|^2 + \frac{1}{2\gamma} \left\| \boldsymbol{u}_h^{n+1} \right\|^2 \ \forall \gamma > 0$, and taking the constant γ as $\gamma = \dfrac{\lambda \varepsilon_0}{\rho}$, with $\varepsilon_0 > 0$ dimensionless:

$$
(4) = \sum_{n=0}^{M} \delta t \left(\boldsymbol{f}^{n+1}, \boldsymbol{u}_h^{n+1} \right) \leq \sum_{n=0}^{M} \delta t \frac{\lambda \varepsilon_0}{2\rho} \left\| \boldsymbol{f}^{n+1} \right\|^2 + \sum_{n=0}^{M} \delta t \frac{\rho}{2\lambda \varepsilon_0} \left\| \boldsymbol{u}_h^{n+1} \right\|^2 .
$$

The last term is absorbed by the left-hand-side (LHS) using the discrete Gronwall inequality (see for example [32]) for ε_0 large enough. Thus, we obtain the following expression:

$$
\begin{aligned}
&\rho \left\| \boldsymbol{u}_h^{M+1} \right\|^2 + \sum_{n=0}^{M} \rho \left\| \delta \boldsymbol{u}_h^n \right\|^2 + \frac{\lambda}{2\eta_p} \left\| \boldsymbol{\sigma}_h^{M+1} \right\|^2 + \sum_{n=0}^{M} \frac{\lambda}{2\eta_p} \left\| \delta \boldsymbol{\sigma}_h^n \right\|^2 \\
&\quad + \sum_{n=0}^{M} \delta t 2\eta_s \left\| \nabla^s \boldsymbol{u}_h^{n+1} \right\|^2 + \sum_{n=0}^{M} \delta t \left(\frac{1}{2\eta_p} - \frac{\lambda}{\eta_p} \left\| \nabla \boldsymbol{a} \right\|_{L^\infty(\Omega)} \right) \left\| \boldsymbol{\sigma}_h^{n+1} \right\|^2 \\
&\quad - \sum_{n=0}^{M} \delta t \left(\tilde{\boldsymbol{u}}_1^{n+1}, \rho \, \boldsymbol{a} \cdot \nabla \boldsymbol{u}_h^{n+1} \right) - \sum_{n=0}^{M} \delta t \left(\tilde{\boldsymbol{u}}_2^{n+1}, \nabla p_h^{n+1} \right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n=0}^{M}\delta t\left(\tilde{\boldsymbol{u}}_3^{n+1},\nabla\cdot\boldsymbol{\sigma}_h^{n+1}\right)+\sum_{n=0}^{M}\delta t\left(\tilde{\boldsymbol{\sigma}}^{n+1},\nabla^s\boldsymbol{u}_h^{n+1}\right)\\
&-\sum_{n=0}^{M}\delta t\left(\tilde{\boldsymbol{\sigma}}^{n+1},\frac{\lambda}{2\eta_p}\left(\boldsymbol{a}\cdot\nabla\boldsymbol{\sigma}_h^{n+1}+\boldsymbol{\sigma}_h^{n+1}\cdot(\nabla\boldsymbol{a})^T+\nabla\boldsymbol{a}\cdot\boldsymbol{\sigma}_h^{n+1}\right)\right)\\
&\lesssim\sum_{n=0}^{M}\delta t\frac{\lambda}{2\rho}\left\|\boldsymbol{f}^{n+1}\right\|^2+\rho\left\|\boldsymbol{u}_h^0\right\|^2+\frac{\lambda}{2\eta_p}\left\|\boldsymbol{\sigma}_h^0\right\|^2.
\end{aligned}\tag{3.2}
$$

Now, multiplying (2.21) by $\tilde{\boldsymbol{u}}_1^{n+1}$, integrating over the whole domain and adding up the result from $n=0$ to $n=M$, we get

$$
\rho\left\|\tilde{\boldsymbol{u}}_1^{M+1}\right\|^2+\sum_{n=0}^{M}\rho\left\|\delta\tilde{\boldsymbol{u}}_1^{n}\right\|^2+\sum_{n=0}^{M}\delta t\left\|\alpha_1^{-1/2}\tilde{\boldsymbol{u}}_1^{n+1}\right\|^2\lesssim-\sum_{n=0}^{M}\delta t\left(\tilde{\boldsymbol{u}}_1^{n+1},P_h^{\perp}\left[\rho\,\boldsymbol{a}\cdot\nabla\boldsymbol{u}_h^{n+1}\right]\right)+\rho\left\|\tilde{\boldsymbol{u}}_1^{0}\right\|^2.\tag{3.3}
$$

Proceeding analogously for the remaining SGS expressions, (2.22) is multiplied by $\tilde{\boldsymbol{u}}_2^{n+1}$, (2.23) by $\tilde{\boldsymbol{u}}_3^{n+1}$ and (2.24) by $\tilde{\boldsymbol{\sigma}}^{n+1}$:

$$
\rho\left\|\tilde{\boldsymbol{u}}_2^{M+1}\right\|^2+\sum_{n=0}^{M}\rho\left\|\delta\tilde{\boldsymbol{u}}_2^{n}\right\|^2+\sum_{n=0}^{M}\delta t\left\|\alpha_1^{-1/2}\tilde{\boldsymbol{u}}_2^{n+1}\right\|^2\lesssim-\sum_{n=0}^{M}\delta t\left(\tilde{\boldsymbol{u}}_2^{n+1},P_h^{\perp}\left[\nabla p_h^{n+1}\right]\right)+\rho\left\|\tilde{\boldsymbol{u}}_2^{0}\right\|^2,\tag{3.4}
$$

$$
\rho\left\|\tilde{\boldsymbol{u}}_3^{M+1}\right\|^2+\sum_{n=0}^{M}\rho\left\|\delta\tilde{\boldsymbol{u}}_3^{n}\right\|^2+\sum_{n=0}^{M}\delta t\left\|\alpha_1^{-1/2}\tilde{\boldsymbol{u}}_3^{n+1}\right\|^2\lesssim\sum_{n=0}^{M}\delta t\left(\tilde{\boldsymbol{u}}_3^{n+1},P_h^{\perp}\left[\nabla\cdot\boldsymbol{\sigma}_h^{n+1}\right]\right)+\rho\left\|\tilde{\boldsymbol{u}}_3^{0}\right\|^2,\tag{3.5}
$$

$$
\begin{aligned}
\frac{\lambda}{2\eta_p}\left\|\tilde{\boldsymbol{\sigma}}^{M+1}\right\|^2+\sum_{n=0}^{M}\frac{\lambda}{2\eta_p}\left\|\delta\tilde{\boldsymbol{\sigma}}^{n}\right\|^2+\sum_{n=0}^{M}\delta t\left\|\alpha_3^{-1/2}\tilde{\boldsymbol{\sigma}}^{n+1}\right\|^2\lesssim\sum_{n=0}^{M}\delta t\Big(\tilde{\boldsymbol{\sigma}}^{n+1},P_h^{\perp}\Big[\nabla^s\boldsymbol{u}_h^{n+1}\\
-\frac{\lambda}{2\eta_p}\Big(\boldsymbol{a}\cdot\nabla\boldsymbol{\sigma}_h^{n+1}-\boldsymbol{\sigma}_h^{n+1}\cdot\nabla\boldsymbol{a}-(\nabla\boldsymbol{a})^T\cdot\boldsymbol{\sigma}_h^{n+1}\Big)\Big]\Big)+\frac{\lambda}{2\eta_p}\left\|\tilde{\boldsymbol{\sigma}}^{0}\right\|^2.
\end{aligned}\tag{3.6}
$$

Finally, adding up equations (3.2)-(3.6) some terms are cancelled. Note that for any $L^2(\Omega)$ vector function $\boldsymbol{v}$ we have $\left(\tilde{\boldsymbol{u}}^{n+1},\boldsymbol{v}\right)=\left(\tilde{\boldsymbol{u}}^{n+1},P_h^{\perp}[\boldsymbol{v}]\right)$ and for any $L^2(\Omega)^{d\times d}$ tensor $\left(\tilde{\boldsymbol{\sigma}}^{n+1},\boldsymbol{\chi}\right)=\left(\tilde{\boldsymbol{\sigma}}^{n+1},P_h^{\perp}[\boldsymbol{\chi}]\right)$. The only term that remains to be bounded in the right-hand-side (RHS) of (3.6) is

$$
\sum_{n=0}^{M}\delta t\left(\tilde{\boldsymbol{\sigma}}^{n+1},\frac{\lambda}{\eta_p}\left(\boldsymbol{\sigma}_h^{n+1}\cdot\nabla\boldsymbol{a}+(\nabla\boldsymbol{a})^T\cdot\boldsymbol{\sigma}_h^{n+1}\right)\right)\leq\sum_{n=0}^{M}\delta t\frac{\lambda}{\eta_p}\left\|\nabla\boldsymbol{a}\right\|_{L^\infty(\Omega)}\left[\frac{\varepsilon_1}{2}\left\|\tilde{\boldsymbol{\sigma}}^{n+1}\right\|^2+\frac{1}{2\varepsilon_1}\left\|\boldsymbol{\sigma}_h^{n+1}\right\|^2\right],
$$

for any $\varepsilon_1>0$. For this parameter small enough, the first term can be absorbed by $\frac{\lambda}{2\eta_p}\left\|\tilde{\boldsymbol{\sigma}}^{M+1}\right\|^2$ applying the discrete Gronwall inequality. Therefore, the final inequality is as follows:

$$
\begin{aligned}
&\rho\left\|\boldsymbol{u}_h^{M+1}\right\|^2+\frac{\lambda}{2\eta_p}\left\|\boldsymbol{\sigma}_h^{M+1}\right\|^2+\rho\left\|\tilde{\boldsymbol{u}}_1^{M+1}\right\|^2+\rho\left\|\tilde{\boldsymbol{u}}_2^{M+1}\right\|^2+\rho\left\|\tilde{\boldsymbol{u}}_3^{M+1}\right\|^2+\frac{\lambda}{2\eta_p}\left\|\tilde{\boldsymbol{\sigma}}^{M+1}\right\|^2\\
&+\sum_{n=0}^{M}\delta t2\eta_s\left\|\nabla^s\boldsymbol{u}_h^{n+1}\right\|^2+\sum_{n=0}^{M}\delta t\frac{1}{2\eta_p}\left(1-\lambda\left\|\nabla\boldsymbol{a}\right\|_{L^\infty(\Omega)}-\frac{\lambda}{2\varepsilon_1}\left\|\nabla\boldsymbol{a}\right\|_{L^\infty(\Omega)}\right)\left\|\boldsymbol{\sigma}_h^{n+1}\right\|^2\\
&+\sum_{n=0}^{M}\delta t\left\|\alpha_1^{-1/2}\tilde{\boldsymbol{u}}_1^{n+1}\right\|^2+\sum_{n=0}^{M}\delta t\left\|\alpha_1^{-1/2}\tilde{\boldsymbol{u}}_2^{n+1}\right\|^2+\sum_{n=0}^{M}\delta t\left\|\alpha_1^{-1/2}\tilde{\boldsymbol{u}}_3^{n+1}\right\|^2+\sum_{n=0}^{M}\delta t\left\|\alpha_3^{-1/2}\tilde{\boldsymbol{\sigma}}^{n+1}\right\|^2
\end{aligned}
$$

$$\lesssim \sum_{n=0}^{M} \delta t \frac{\lambda}{2\rho} \left\| \boldsymbol{f}^{n+1} \right\|^2 + \rho \left\| \boldsymbol{u}_h^0 \right\|^2 + \rho \left(\left\| \tilde{\boldsymbol{u}}_1^0 \right\|^2 + \left\| \tilde{\boldsymbol{u}}_2^0 \right\|^2 + \left\| \tilde{\boldsymbol{u}}_3^0 \right\|^2 \right) + \frac{\lambda}{2\eta_p} \left\| \boldsymbol{\sigma}_h^0 \right\|^2 + \frac{\lambda}{2\eta_p} \left\| \tilde{\boldsymbol{\sigma}}^0 \right\|^2,$$

from where the theorem follows for λ small enough. □

3.3. Stability bounds for the log-conformation reformulation

We can prove now a similar stability estimate for the LCR:

Theorem 2 (Stability bounds for the LCR)**.** *Let $\boldsymbol{u}_h^{n+1}$, p_h^{n+1} and $\bar{\boldsymbol{\psi}}_h^{n+1}$ be the solution of (2.27) and $\tilde{\boldsymbol{u}}_1^{n+1}$, $\tilde{\boldsymbol{u}}_2^{n+1}$, $\tilde{\boldsymbol{u}}_3^{n+1}$, $\tilde{\boldsymbol{\sigma}}^{n+1}$ solutions of (2.28), (2.29), (2.30) and (2.31), respectively. Under the assumptions of Theorem 1, the following stability bounds hold for all $\delta t > 0$:*

$$\begin{aligned}
&\max_{n=0,\dots,N-1} \left\{ \rho \left\| \boldsymbol{u}_h^{n+1} \right\|^2 + \frac{\lambda}{2\lambda_0} \frac{\eta_p}{\lambda_0} \left\| \bar{\boldsymbol{\psi}}_h^{n+1} \right\|^2 + \rho \left\| \tilde{\boldsymbol{u}}_1^{n+1} \right\|^2 + \rho \left\| \tilde{\boldsymbol{u}}_2^{n+1} \right\|^2 + \rho \left\| \tilde{\boldsymbol{u}}_3^{n+1} \right\|^2 + \frac{\lambda}{2\eta_p} \left\| \tilde{\boldsymbol{\sigma}}^{n+1} \right\|^2 \right\} \\
&\quad + \sum_{n=0}^{N-1} \delta t 2\eta_s \left\| \nabla \boldsymbol{u}_h^{n+1} \right\|^2 + \sum_{n=0}^{N-1} \delta t \frac{\eta_p}{\lambda_0^2} \left\| \bar{\boldsymbol{\psi}}_h^{n+1} \right\|^2 + \sum_{n=0}^{N-1} \delta t \left\| \alpha_3^{-1/2} \tilde{\boldsymbol{\sigma}}^{n+1} \right\|^2 \\
&\quad + \sum_{n=0}^{N-1} \delta t \left(\left\| \alpha_1^{-1/2} \tilde{\boldsymbol{u}}_1^{n+1} \right\|^2 + \left\| \alpha_1^{-1/2} \tilde{\boldsymbol{u}}_2^{n+1} \right\|^2 + \left\| \alpha_1^{-1/2} \tilde{\boldsymbol{u}}_3^{n+1} \right\|^2 \right) \\
&\quad \lesssim \sum_{n=0}^{M} \frac{\lambda}{2\rho} \delta t \left\| \boldsymbol{f}_u^{n+1} \right\|^2 + \sum_{n=0}^{M} \frac{\lambda}{2} \frac{\eta_p}{\lambda_0} \delta t \left\| \boldsymbol{f}_\psi^{n+1} \right\|^2 + \rho \left\| \boldsymbol{u}_h^0 \right\|^2 + \frac{\lambda}{2\lambda_0} \frac{\eta_p}{\lambda_0} \left\| \bar{\boldsymbol{\psi}}_h^0 \right\|^2 .
\end{aligned}$$

Therefore, if $\{\boldsymbol{f}_u^n\} \in \ell^2(L^2(\Omega)^d)$, $\{\boldsymbol{f}_\psi^n\} \in L^2(\Omega)_{\mathrm{sym}}^{d\times d}$, $\boldsymbol{u}_h^0 \in L^2(\Omega)^d$ and $\bar{\boldsymbol{\psi}}_h^0 \in L^2(\Omega)_{\mathrm{sym}}^{d\times d}$ we have that

$$\begin{aligned}
&\{\boldsymbol{u}_h^n\} \in \ell^\infty \left(L^2(\Omega)^d \right) \cap \ell^2 \left(H^1(\Omega)^d \right); \{\tilde{\boldsymbol{u}}_1^n\}, \{\tilde{\boldsymbol{u}}_2^n\}, \{\tilde{\boldsymbol{u}}_3^n\} \in \ell^\infty \left(L^2(\Omega)^d \right); \\
&\{\alpha_1^{-1/2} \tilde{\boldsymbol{u}}_1^n\}, \{\alpha_1^{-1/2} \tilde{\boldsymbol{u}}_2^n\}, \{\alpha_1^{-1/2} \tilde{\boldsymbol{u}}_3^n\} \in \ell^2 \left(L^2(\Omega)^d \right); \\
&\{\tilde{\boldsymbol{\sigma}}^n\}, \{\bar{\boldsymbol{\psi}}_h^n\} \in \ell^\infty \left(L^2(\Omega)_{\mathrm{sym}}^{d\times d} \right); \{\alpha_3^{-1/2} \tilde{\boldsymbol{\sigma}}^n\} \in \ell^2 \left(L^2(\Omega)_{\mathrm{sym}}^{d\times d} \right).
\end{aligned}$$

Proof. In order to obtain stability bounds for the FE solution, first of all we test (2.27) by $\boldsymbol{v}_h = \boldsymbol{u}_h^{n+1}$, $q_h = p_h^{n+1}$ and $\boldsymbol{\chi}_h = \frac{\eta_p}{\lambda_0} \bar{\boldsymbol{\psi}}_h^{n+1}$. Using the assumptions of Theorem 1, we have that $\left(\boldsymbol{a} \cdot \nabla \boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1} \right) = 0$ and $\left(\boldsymbol{a} \cdot \nabla \bar{\boldsymbol{\psi}}_h^{n+1}, \bar{\boldsymbol{\psi}}_h^{n+1} \right) = 0$. We thus obtain:

$$\begin{aligned}
&\rho \underbrace{\left(\delta \boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1} \right)}_{(1)} + \underbrace{\frac{\lambda}{2\lambda_0} \frac{\eta_p}{\lambda_0} \left(\delta \bar{\boldsymbol{\psi}}_h^n, \bar{\boldsymbol{\psi}}_h^{n+1} \right)}_{(2)} + \delta t 2\eta_s \left(\nabla^s \boldsymbol{u}_h^{n+1}, \nabla^s \boldsymbol{u}_h^{n+1} \right) \\
&\quad + \delta t \frac{1}{2\lambda_0} \frac{\eta_p}{\lambda_0} \left(\bar{\boldsymbol{\psi}}_h^{n+1}, \bar{\boldsymbol{\psi}}_h^{n+1} \right) + \underbrace{\frac{\lambda}{2\lambda_0} \frac{\eta_p}{\lambda_0} \delta t \left(2\nabla^s \boldsymbol{u}_h^{n+1}, \bar{\boldsymbol{\psi}}_h^{n+1} \right)}_{(3)} \\
&\quad \underbrace{- \delta t \frac{\lambda}{2\lambda_0} \frac{\eta_p}{\lambda_0} \left(\bar{\boldsymbol{\psi}}_h^{n+1} \cdot \nabla \boldsymbol{a} + (\nabla \boldsymbol{a})^T \cdot \bar{\boldsymbol{\psi}}_h^{n+1}, \bar{\boldsymbol{\psi}}_h^{n+1} \right)}_{(4)} \\
&\quad - \delta t \left(\tilde{\boldsymbol{u}}_1^{n+1}, \rho\, \boldsymbol{a} \cdot \nabla \boldsymbol{u}_h^{n+1} \right) - \delta t \left(\tilde{\boldsymbol{u}}_2^{n+1}, \nabla p_h^{n+1} \right) + \frac{\eta_p}{\lambda_0} \delta t \left(\tilde{\boldsymbol{u}}_3^{n+1}, \nabla \cdot \bar{\boldsymbol{\psi}}_h^{n+1} \right)
\end{aligned}$$

$$
+\,\delta t\left(\tilde{\boldsymbol{\sigma}}^{n+1},\nabla^s\boldsymbol{u}_h^{n+1}-\frac{\lambda}{2\lambda_0}\left(\boldsymbol{a}\cdot\nabla\bar{\boldsymbol{\psi}}_h^{n+1}+\bar{\boldsymbol{\psi}}_h^{n+1}\cdot(\nabla\boldsymbol{a})^T+\nabla\boldsymbol{a}\cdot\bar{\boldsymbol{\psi}}_h^{n+1}\right)\right)
$$

$$
=\underbrace{\delta t\left(\boldsymbol{f}_u^{n+1},\boldsymbol{u}_h^{n+1}\right)}_{(5)}+\underbrace{\delta t\left(\boldsymbol{f}_\psi^{n+1},\frac{\eta_p}{\lambda_0}\bar{\boldsymbol{\psi}}_h^{n+1}\right)}_{(6)}.
$$

Following the same strategy as in the proof of Theorem 1, we have:

$$
\begin{aligned}
(1)&=\sum_{n=0}^{M}\rho\left(\delta\boldsymbol{u}_h^n,\boldsymbol{u}_h^{n+1}\right)=\frac{1}{2}\sum_{n=0}^{M}\rho\left\|\delta\boldsymbol{u}_h^n\right\|^2+\frac{1}{2}\rho\left\|\boldsymbol{u}_h^{M+1}\right\|^2-\frac{1}{2}\rho\left\|\boldsymbol{u}_h^0\right\|^2,\\
(2)&=\sum_{n=0}^{M}\frac{\lambda}{2\lambda_0}\frac{\eta_p}{\lambda_0}\left(\delta\bar{\boldsymbol{\psi}}_h^n,\bar{\boldsymbol{\psi}}_h^{n+1}\right)=\frac{1}{2}\sum_{n=0}^{M}\frac{\lambda}{2\lambda_0}\frac{\eta_p}{\lambda_0}\left\|\delta\bar{\boldsymbol{\psi}}_h^n\right\|^2+\frac{1}{2}\frac{\lambda}{2\lambda_0}\frac{\eta_p}{\lambda_0}\left\|\bar{\boldsymbol{\psi}}_h^{M+1}\right\|^2-\frac{1}{2}\frac{\lambda}{2\lambda_0}\frac{\eta_p}{\lambda_0}\left\|\bar{\boldsymbol{\psi}}_h^0\right\|^2,\\
(3)&=\sum_{n=0}^{M}\frac{\lambda}{2\lambda_0}\frac{\eta_p}{\lambda_0}\delta t\left(\nabla^s\boldsymbol{u}_h^{n+1},\bar{\boldsymbol{\psi}}_h^{n+1}\right)\geq\sum_{n=0}^{M}\eta_p\frac{\lambda}{2\lambda_0}\delta t\left[\frac{1}{2\varepsilon_0\lambda_0^2}\left\|\bar{\boldsymbol{\psi}}_h^{n+1}\right\|^2+\frac{\varepsilon_0}{2}\left\|\nabla^s\boldsymbol{u}_h^{n+1}\right\|^2\right],\\
(4)&=-\sum_{n=0}^{M}\delta t\frac{\lambda}{2\lambda_0}\frac{\eta_p}{\lambda_0}\left(\bar{\boldsymbol{\psi}}_h^{n+1}\cdot\nabla\boldsymbol{a}+(\nabla\boldsymbol{a})^T\cdot\bar{\boldsymbol{\psi}}_h^{n+1},\bar{\boldsymbol{\psi}}_h^{n+1}\right)\geq-\sum_{n=0}^{M}\delta t\frac{\lambda}{\lambda_0}\frac{\eta_p}{\lambda_0}\left\|\nabla\boldsymbol{a}\right\|_{L^\infty(\Omega)}\left\|\bar{\boldsymbol{\psi}}_h^{n+1}\right\|^2,\\
(5)&=\sum_{n=0}^{M}\delta t\left(\boldsymbol{f}_u^{n+1},\boldsymbol{u}_h^{n+1}\right)\leq\sum_{n=0}^{M}\frac{\lambda\varepsilon_1}{2\rho}\delta t\left\|\boldsymbol{f}_u^{n+1}\right\|^2+\sum_{n=0}^{M}\frac{\rho}{2\lambda\varepsilon_1}\delta t\left\|\boldsymbol{u}_h^{n+1}\right\|^2,\\
(6)&=\sum_{n=0}^{M}\delta t\left(\boldsymbol{f}_\psi^{n+1},\frac{\eta_p}{\lambda_0}\bar{\boldsymbol{\psi}}_h^{n+1}\right)\leq\sum_{n=0}^{M}\delta t\frac{\eta_p}{\lambda_0}\frac{\lambda\varepsilon_2}{2}\left\|\boldsymbol{f}_\psi^{n+1}\right\|^2+\sum_{n=0}^{M}\delta t\frac{\eta_p}{\lambda_0}\frac{1}{2\lambda\varepsilon_2}\left\|\bar{\boldsymbol{\psi}}_h^{n+1}\right\|^2.
\end{aligned}
$$

The last terms in (5) and (6) can be absorbed by the LHS for ε_1 and ε_2 large enough using the discrete Gronwall inequality. To sum up we obtain the following expression:

$$
\begin{aligned}
&\rho\left\|\boldsymbol{u}_h^{M+1}\right\|^2+\sum_{n=0}^{M}\rho\left\|\delta\boldsymbol{u}_h^n\right\|+\frac{\lambda}{2\lambda_0}\frac{\eta_p}{\lambda_0}\left\|\bar{\boldsymbol{\psi}}_h^{M+1}\right\|^2\\
&\quad+\sum_{n=0}^{M}\frac{\lambda}{2\lambda_0}\frac{\eta_p}{\lambda_0}\left\|\delta\bar{\boldsymbol{\psi}}_h^n\right\|^2+\sum_{n=0}^{M}\delta t\left(2\eta_s-\eta_p\frac{\varepsilon_0}{2}\frac{\lambda}{\lambda_0}\right)\left\|\nabla^s\boldsymbol{u}_h^{n+1}\right\|^2\\
&\quad+\sum_{n=0}^{M}\delta t\frac{\eta_p}{2\lambda_0}\left(\frac{1}{\lambda_0}-\frac{\lambda}{2\varepsilon_0\lambda_0^2}-\frac{2\lambda}{\lambda_0}\left\|\nabla\boldsymbol{a}\right\|_{L^\infty(\Omega)}\right)\left\|\bar{\boldsymbol{\psi}}_h^{n+1}\right\|^2\\
&\quad-\sum_{n=0}^{M}\delta t\left(\tilde{\boldsymbol{u}}_1^{n+1},\rho\,\boldsymbol{a}\cdot\nabla\boldsymbol{u}_h^{n+1}\right)-\sum_{n=0}^{M}\delta t\left(\tilde{\boldsymbol{u}}_2^{n+1},\nabla p_h^{n+1}\right)+\sum_{n=0}^{M}\delta t\frac{\eta_p}{\lambda_0}\left(\tilde{\boldsymbol{u}}_3^{n+1},\nabla\cdot\bar{\boldsymbol{\psi}}_h^{n+1}\right)\\
&\quad+\sum_{n=0}^{M}\delta t\left(\tilde{\boldsymbol{\sigma}}^{n+1},\nabla^s\boldsymbol{u}_h^{n+1}-\frac{\lambda}{2\lambda_0}\left(\boldsymbol{a}\cdot\nabla\bar{\boldsymbol{\psi}}_h^{n+1}+\bar{\boldsymbol{\psi}}_h^{n+1}\cdot(\nabla\boldsymbol{a})^T+\nabla\boldsymbol{a}\cdot\bar{\boldsymbol{\psi}}_h^{n+1}\right)\right)\\
&\quad\lesssim\sum_{n=0}^{M}\frac{\lambda}{2\rho}\delta t\left\|\boldsymbol{f}_u^{n+1}\right\|^2+\sum_{n=0}^{M}\frac{\lambda}{2}\frac{\eta_p}{\lambda_0}\delta t\left\|\boldsymbol{f}_\psi^{n+1}\right\|^2+\rho\left\|\boldsymbol{u}_h^0\right\|^2+\frac{\lambda}{2\lambda_0}\frac{\eta_p}{\lambda_0}\left\|\bar{\boldsymbol{\psi}}_h^0\right\|^2.
\end{aligned}
\tag{3.7}
$$

Now, we multiply (2.28) by $\tilde{\boldsymbol{u}}_1^{n+1}$, integrate over the whole domain and add up the result from $n=0$ to $n=M$:

$$\rho\left\|\tilde{\boldsymbol{u}}_1^{M+1}\right\|^2+\sum_{n=0}^{M}\rho\left\|\delta\tilde{\boldsymbol{u}}_1^{n}\right\|^2+\sum_{n=0}^{M}\delta t\left\|\alpha_1^{-1/2}\tilde{\boldsymbol{u}}_1^{n+1}\right\|^2\lesssim-\sum_{n=0}^{M}\delta t\left(\tilde{\boldsymbol{u}}_1^{n+1},P_h^{\perp}\left[\rho\,\boldsymbol{a}\cdot\nabla\boldsymbol{u}_h^{n+1}\right]\right)+\rho\left\|\tilde{\boldsymbol{u}}_1^{0}\right\|^2, \quad (3.8)$$

Analogously, we multiply (2.29) by $\tilde{\boldsymbol{u}}_2^{n+1}$, (2.30) by $\tilde{\boldsymbol{u}}_3^{n+1}$ and (2.31) by $\tilde{\boldsymbol{\sigma}}^{n+1}$, to obtain:

$$\rho\left\|\tilde{\boldsymbol{u}}_2^{M+1}\right\|^2+\sum_{n=0}^{M}\rho\left\|\delta\tilde{\boldsymbol{u}}_2^{n}\right\|^2+\sum_{n=0}^{M}\delta t\left\|\alpha_1^{-1/2}\tilde{\boldsymbol{u}}_2^{n+1}\right\|^2\lesssim-\sum_{n=0}^{M}\delta t\left(\tilde{\boldsymbol{u}}_2^{n+1},P_h^{\perp}\left[\nabla p_h^{n+1}\right]\right)+\rho\left\|\tilde{\boldsymbol{u}}_2^{0}\right\|^2, \quad (3.9)$$

$$\rho\left\|\tilde{\boldsymbol{u}}_3^{M+1}\right\|^2+\sum_{n=0}^{M}\rho\left\|\delta\tilde{\boldsymbol{u}}_3^{n}\right\|^2+\sum_{n=0}^{M}\delta t\left\|\alpha_1^{-1/2}\tilde{\boldsymbol{u}}_3^{n+1}\right\|^2\lesssim\sum_{n=0}^{M}\delta t\frac{\eta_p}{\lambda_0}\left(\tilde{\boldsymbol{u}}_3^{n+1},P_h^{\perp}\left[\nabla\cdot\bar{\boldsymbol{\psi}}_h^{n+1}\right]\right)+\rho\left\|\tilde{\boldsymbol{u}}_3^{0}\right\|^2, \quad (3.10)$$

$$\begin{aligned}&\frac{\lambda}{2\eta_p}\left\|\tilde{\boldsymbol{\sigma}}^{M+1}\right\|^2+\sum_{n=0}^{M}\frac{\lambda}{2\eta_p}\left\|\delta\tilde{\boldsymbol{\sigma}}^{n}\right\|^2+\sum_{n=0}^{M}\delta t\left\|\alpha_3^{-1/2}\tilde{\boldsymbol{\sigma}}^{n+1}\right\|^2\\&\lesssim\sum_{n=0}^{M}\delta t\Big(\tilde{\boldsymbol{\sigma}}^{n+1},P_h^{\perp}\Big[\nabla^s\boldsymbol{u}_h^{n+1}-\frac{\lambda}{\lambda_0}\nabla^s\boldsymbol{u}_h^{n+1}\\&\quad-\frac{\lambda}{2\lambda_0}\Big(\boldsymbol{a}\cdot\nabla\bar{\boldsymbol{\psi}}_h^{n+1}-\bar{\boldsymbol{\psi}}_h^{n+1}\cdot(\nabla\boldsymbol{a})^T-\nabla\boldsymbol{a}\cdot\bar{\boldsymbol{\psi}}_h^{n+1}\Big)\Big]\Big)+\frac{\lambda}{2\eta_p}\left\|\tilde{\boldsymbol{\sigma}}^{0}\right\|^2. \quad (3.11)\end{aligned}$$

Adding equations (3.7)-(3.11), some terms are cancelled. The term that remains to be bounded is:

$$\begin{aligned}&\frac{\lambda}{2\lambda_0}\sum_{n=0}^{M}\delta t\left(\tilde{\boldsymbol{\sigma}}^{n+1},\bar{\boldsymbol{\psi}}_h^{n+1}\cdot(\nabla\boldsymbol{a})^T+\nabla\boldsymbol{a}\cdot\bar{\boldsymbol{\psi}}_h^{n+1}\right)\\&\quad\geq-\frac{\lambda}{2\lambda_0}\sum_{n=0}^{M}\delta t\left\|\nabla\boldsymbol{a}\right\|_{L^\infty(\Omega)}\left[\frac{\lambda_0}{2\eta_p\varepsilon_3}\left\|\tilde{\boldsymbol{\sigma}}^{n+1}\right\|^2+\frac{\varepsilon_3}{2}\frac{\eta_p}{\lambda_0}\left\|\bar{\boldsymbol{\psi}}_h^{n+1}\right\|^2\right].\end{aligned}$$

The first term in the RHS can be controlled by the first term in the LHS of (3.11) using the discrete Gronwall inequality for ε_3 large enough. Altogether, we are left with:

$$\begin{aligned}&\rho\left\|\boldsymbol{u}_h^{M+1}\right\|^2+\frac{\lambda}{2\lambda_0}\frac{\eta_p}{\lambda_0}\left\|\bar{\boldsymbol{\psi}}_h^{M+1}\right\|^2+\rho\left\|\tilde{\boldsymbol{u}}_1^{M+1}\right\|^2+\rho\left\|\tilde{\boldsymbol{u}}_2^{M+1}\right\|^2+\rho\left\|\tilde{\boldsymbol{u}}_3^{M+1}\right\|^2+\frac{\lambda}{2\eta_p}\left\|\tilde{\boldsymbol{\sigma}}^{M+1}\right\|^2\\&+\sum_{n=0}^{M}\delta t\left(2\eta_s-\eta_p\frac{\varepsilon_0}{2}\frac{\lambda}{\lambda_0}\right)\left\|\nabla^s\boldsymbol{u}_h^{n+1}\right\|^2\\&+\sum_{n=0}^{M}\delta t\frac{\eta_p}{\lambda_0}\left(\frac{1}{2\lambda_0}-\frac{\lambda}{2\varepsilon_0\lambda_0^2}-\frac{\lambda}{\lambda_0}\left(2+\frac{\varepsilon_3}{2}\right)\left\|\nabla\boldsymbol{a}\right\|_{L^\infty(\Omega)}\right)\left\|\bar{\boldsymbol{\psi}}_h^{n+1}\right\|^2\\&+\sum_{n=0}^{M}\delta t\left\|\alpha_1^{-1/2}\tilde{\boldsymbol{u}}_1^{n+1}\right\|^2+\sum_{n=0}^{M}\delta t\left\|\alpha_1^{-1/2}\tilde{\boldsymbol{u}}_2^{n+1}\right\|^2+\sum_{n=0}^{M}\delta t\left\|\alpha_1^{-1/2}\tilde{\boldsymbol{u}}_3^{n+1}\right\|^2+\sum_{n=0}^{M}\delta t\left\|\alpha_3^{-1/2}\tilde{\boldsymbol{\sigma}}^{n+1}\right\|^2\\&\lesssim\sum_{n=0}^{M}\frac{\lambda\varepsilon_1}{2\rho}\delta t\left\|\boldsymbol{f}_u^{n+1}\right\|^2+\sum_{n=0}^{M}\frac{\lambda\varepsilon_2}{2}\frac{\eta_p}{\lambda_0}\delta t\left\|\boldsymbol{f}_\psi^{n+1}\right\|^2+\rho\left\|\boldsymbol{u}_h^{0}\right\|^2\end{aligned}$$

$$+\frac{\lambda}{2\lambda_0}\frac{\eta_p}{\lambda_0}\left\|\bar{\boldsymbol{\psi}}_h^0\right\|^2+\rho\left\|\tilde{\boldsymbol{u}}_1^0\right\|^2+\rho\left\|\tilde{\boldsymbol{u}}_2^0\right\|^2+\rho\left\|\tilde{\boldsymbol{u}}_3^0\right\|^2+\frac{\lambda}{2\eta_p}\left\|\tilde{\boldsymbol{\sigma}}^0\right\|^2$$

from where the theorem follows for λ small enough. □

4. Stability analysis in terms of the finite element unknowns

4.1. Preliminaries

The stability obtained for $\tilde{\boldsymbol{u}}^{n+1}$ and $\tilde{\boldsymbol{\sigma}}^{n+1}$ will be translated now into stability in terms of $\boldsymbol{u}_h^{n+1}$, $\boldsymbol{\sigma}_h^{n+1}$ and p_h^{n+1} (and in terms of $\boldsymbol{u}_h^{n+1}$, $\boldsymbol{\psi}_h^{n+1}$ and p_h^{n+1} in the case of the LCR) using first dual norms (Theorems 3, 4, 5). Later, these results will be extended to a natural norm if a certain relationship between the time step size and the stabilization parameters α_1 and α_3 holds (Theorem 6). If this relationship does not hold, the stability results to be obtained are rather weak. For the case of the transient Stokes problem, they can be improved by obtaining bounds for the discrete time derivative as done in [12]; in fact, this technique also allows one to obtain explicit estimates for the pressure. However, we will not pursue this path in this work, as for the problem we consider that would require stringent and unrealistic conditions on the Reynolds and Weissenberg numbers.

The result obtained in Theorem 1 gives stability for $\{\boldsymbol{u}_h^n\}$, $\{\tilde{\boldsymbol{u}}^n\}$, $\{\boldsymbol{\sigma}_h^n\}$ and $\{\tilde{\boldsymbol{\sigma}}^n\}$, and Theorem 2 for $\{\bar{\boldsymbol{\psi}}_h^n\}$ instead of $\{\boldsymbol{\sigma}_h^n\}$. In the following we shall see that from these results we can get stability on the following terms:

$$\boldsymbol{m}_1^n := \rho\,\boldsymbol{a}\cdot\nabla\boldsymbol{u}_h^n, \tag{4.1}$$
$$\boldsymbol{m}_2^n := \nabla p_h^n, \tag{4.2}$$
$$\boldsymbol{m}_3^n := -\nabla\cdot\boldsymbol{\sigma}_h^n, \tag{4.3}$$
$$\boldsymbol{m}_4^n := \frac{1}{2\eta_p}\boldsymbol{\sigma}_h^n-\nabla^s\boldsymbol{u}_h^n+\frac{\lambda}{2\eta_p}\left(\boldsymbol{a}\cdot\nabla\boldsymbol{\sigma}_h^n-\boldsymbol{\sigma}_h^n\cdot\nabla\boldsymbol{a}-(\nabla\boldsymbol{a})^T\cdot\boldsymbol{\sigma}_h^n\right), \tag{4.4}$$

for $n=1,...,N$. In case of the LCR $\boldsymbol{m}_1^n$ and $\boldsymbol{m}_2^n$ are defined as in the standard formulation, but $\boldsymbol{m}_3^n$ and $\boldsymbol{m}_4^n$ must be redefined:

$$\boldsymbol{m}_3^n := -\frac{\eta_p}{\lambda_0}\nabla\cdot\bar{\boldsymbol{\psi}}_h^n, \tag{4.5}$$
$$\boldsymbol{m}_4^n := \frac{1}{2\lambda_0}\bar{\boldsymbol{\psi}}_h^n-\nabla^s\boldsymbol{u}_h^n+\frac{\lambda}{2\lambda_0}\left(2\nabla^s\boldsymbol{u}_h^n+\boldsymbol{a}\cdot\nabla\bar{\boldsymbol{\psi}}_h^n-\bar{\boldsymbol{\psi}}_h^n\cdot\nabla\boldsymbol{a}-(\nabla\boldsymbol{a})^T\cdot\bar{\boldsymbol{\psi}}_h^n\right). \tag{4.6}$$

The term $\boldsymbol{m}_1^n$ provides control on the convective derivative, $\boldsymbol{m}_2^n$ over the gradient of the pressure, $\boldsymbol{m}_3^n$ on the divergence of the stress tensor and $\boldsymbol{m}_4^n$ on some terms of the constitutive equation. We shall see that for terms $\boldsymbol{m}_1^n$, $\boldsymbol{m}_2^n$ and $\boldsymbol{m}_3^n$ it is possible to obtain separate stability of their components in $\boldsymbol{\mathcal{V}}_h^{\perp}$, but not of their component in $\boldsymbol{\mathcal{V}}_{h,0}$. Furthermore, an additional condition on the FE mesh will be needed to obtain full stability of the sum of these terms. For deeper discussion in the case of the Oseen problem, see [33].

We will follow the same procedure as in [30]. As indicated there, in the general situation without imposing any condition on δt and α_1, α_3, we will prove stability in a rather weak dual norm. However, if we assume a condition between α_1, α_3 and δt it is possible to improve this stability bound.

Let us describe the dual norm in which stability will be first proved. Given a sequence $F = \{f^n\}$, of scalar, vector or tensor functions on Ω, we define the following norms:

$$\|F\|_{X_i} := \left(\sum_{n=0}^{N} \delta t \,\|f^n\|^2\right)^{1/2} + \sum_{n=0}^{N-1} \delta t \left\|\alpha_i^{1/2}\delta_t f^n\right\| + \max_{n=0,\dots,N} \left\{\alpha_i^{1/2}\,\|f^n\|\right\}, \tag{4.7}$$

with $i = 1, 3$. These norms endow the spaces of sequences

$$X_i = \left\{F = \{f^n\} \mid F \in \ell^2\left(L^2(\Omega)\right), \left\{\alpha_i^{1/2}\delta_t f^n\right\} \in \ell^1\left(L^2(\Omega)\right), \left\{\alpha_i^{1/2} f^n\right\} \in \ell^\infty\left(L^2(\Omega)\right)\right\}$$

of a Banach structure. Note that $\{\delta_t f^n\} \in \ell^1\left(L^2(\Omega)\right)$ and $F \in \ell^\infty\left(L^2(\Omega)\right)$ certainly define strong topologies, but the factors $\alpha_i^{1/2}$, $i = 1, 3$, prevent from any comparison between the different terms. Let now X_i' be the dual space of X_i, the duality pairing being

$$\langle F, G\rangle_{X_i \times X_i'} := \sum_{n=1}^{N} \delta t \,\langle f^n, g^n\rangle\,,$$

with $F = \{f^n\} \in X_i$, $G = \{g^n\} \in X_i'$. The norm in X_i' is given by

$$\|G\|_{X_i'} := \sup_{F \in X_i, F \neq 0} \frac{\langle F, G\rangle_{X_i \times X_i'}}{\|F\|_{X_i}}. \tag{4.8}$$

4.2. **Stability in dual norms**

Let us start proving some stability for $\{\boldsymbol{m}_i^n\}$, $i = 1, 2, 3$, in the norm of X_1'. Recall that the $L^2(\Omega)$ projection onto $\boldsymbol{\mathcal{V}}_h$ is P_h, whereas $P_h^\perp$ is the $L^2(\Omega)$ projection onto $\boldsymbol{\mathcal{V}}_h^\perp$. The $L^2(\Omega)$ projection onto the velocity FE space $\boldsymbol{\mathcal{V}}_{h,0}$ will be denoted by $P_{h,0}$.

Theorem 3 (Stability in the dual norm X_1' for the momentum equation terms)**.** *Under the assumptions of Theorem 1, suppose also that the constants c_1 and c_3 in the definitions* (2.25)-(2.26) *of α_1 and α_3, respectively, satisfy $2c_3 \leq c_1$. Then, there is a constant C such that*

$$\sum_{i=1}^{3} \left\|\left\{\alpha_1^{1/2} P_h^\perp\left[\boldsymbol{m}_i^n\right]\right\}\right\|_{X_1'} + \left\|\left\{\alpha_1^{1/2} P_{h,0}\left[\boldsymbol{m}_1^n + \boldsymbol{m}_2^n + \boldsymbol{m}_3^n\right]\right\}\right\|_{X_1'} \leq C.$$

This result holds for both the standard formulation and the LCR.

Proof. Let $\{\boldsymbol{v}^n\} \in X_1$ be an arbitrary sequence, which can be split as $\boldsymbol{v}^n = \boldsymbol{v}_h^n + \boldsymbol{v}_b^n + \tilde{\boldsymbol{v}}^n$ with $\boldsymbol{v}_h^n = P_{h,0}[\boldsymbol{v}^n]$ and $\tilde{\boldsymbol{v}}^n = P_h^\perp[\boldsymbol{v}^n]$. We can write equations (2.21)-(2.23) as

$$\rho\,\delta\tilde{\boldsymbol{u}}_i^n + \delta t \alpha_1^{-1}\tilde{\boldsymbol{u}}_i^{n+1} = -\delta t P_h^\perp\left[\boldsymbol{m}_i^{n+1}\right],\ i = 1, 2, 3.$$

Multiplying each equation by $\alpha_1^{1/2}\tilde{\boldsymbol{v}}^{n+1}$, integrating over Ω, and adding up the result from $n = 0$ to $N-1$, integrating by parts using the discrete integration formula and using the Cauchy-Schwartz inequality, we have:

$$\begin{aligned}
\sum_{n=0}^{N-1} \delta t \alpha_1^{1/2} \left(P_h^{\perp}\left[\boldsymbol{m}_i^{n+1}\right], \tilde{\boldsymbol{v}}^{n+1}\right) &= -\sum_{n=0}^{N-1} \rho\, \alpha_1^{1/2} \left(\delta \tilde{\boldsymbol{u}}_i^n, \tilde{\boldsymbol{v}}^{n+1}\right) - \sum_{n=0}^{N-1} \delta t \alpha_1^{-1/2} \left(\tilde{\boldsymbol{u}}_i^{n+1}, \tilde{\boldsymbol{v}}^{n+1}\right) \\
&= \sum_{n=0}^{N-1} \rho\, \alpha_1^{1/2} \left(\tilde{\boldsymbol{u}}_i^n, \delta \tilde{\boldsymbol{v}}^n\right) - \rho\, \alpha_1^{1/2} \left(\tilde{\boldsymbol{u}}_i^N, \tilde{\boldsymbol{v}}^N\right) + \rho\, \alpha_1^{1/2} \left(\tilde{\boldsymbol{u}}_i^0, \tilde{\boldsymbol{v}}^0\right) - \sum_{n=0}^{N-1} \delta t \alpha_1^{-1/2} \left(\tilde{\boldsymbol{u}}_i^{n+1}, \tilde{\boldsymbol{v}}^{n+1}\right) \\
&\lesssim \rho \max_{n=0,\dots,N-1} \left\{\|\tilde{\boldsymbol{u}}_i^n\|\right\} \sum_{n=0}^{N-1} \delta t \alpha_1^{1/2} \|\delta_t \tilde{\boldsymbol{v}}^n\| + 2\rho \max_{n=0,\dots,N-1} \left\{\|\tilde{\boldsymbol{u}}_i^n\|\right\} \alpha_1^{1/2} \max_{n=0,..,N} \left\{\|\tilde{\boldsymbol{v}}^n\|\right\} \\
&\quad + \left(\sum_{n=0}^{N-1} \delta t \left\|\alpha_1^{-1/2} \tilde{\boldsymbol{u}}_i^{n+1}\right\|^2\right)^{1/2} \left(\sum_{n=0}^{N-1} \delta t \left\|\tilde{\boldsymbol{v}}^{n+1}\right\|^2\right)^{1/2}. \qquad (4.9)
\end{aligned}$$

Here and in what follows we have considered α_1 constant to simplify the writing. Now, using Theorem 1 and the definition of norm X_1 (4.7), we can conclude that

$$\left\langle \{\boldsymbol{v}^n\}, \alpha_1^{1/2} \left\{P_h^{\perp}\left[\boldsymbol{m}_i^n\right]\right\} \right\rangle_{X_1 \times X_1'} = \sum_{n=0}^{N-1} \delta t \alpha_1^{1/2} \left(P_h^{\perp}\left[\boldsymbol{m}_i^{n+1}\right], \tilde{\boldsymbol{v}}^{n+1}\right) \lesssim \|\{\boldsymbol{v}^n\}\|_{X_1}.$$

Consequently, adding the three inequalities for $i = 1, 2, 3$ we have bounded

$$\left\|\left\{\alpha_1^{1/2} P_h^{\perp}\left[\boldsymbol{m}_1^n\right]\right\}\right\|_{X_1'} + \left\|\left\{\alpha_1^{1/2} P_h^{\perp}\left[\boldsymbol{m}_2^n\right]\right\}\right\|_{X_1'} + \left\|\left\{\alpha_1^{1/2} P_h^{\perp}\left[\boldsymbol{m}_3^n\right]\right\}\right\|_{X_1'} \le C. \qquad (4.10)$$

The next step is to control the FE component, that is, to say, the term $P_{h,0}\left[\boldsymbol{m}_1^n + \boldsymbol{m}_2^n + \boldsymbol{m}_3^n\right]$. Let us define, for simplicity, the sequence $\boldsymbol{m}_s^n = \sum_{i=1}^3 \boldsymbol{m}_i^n$. For this, we take equation (2.20), considering $q_h = 0$ and $\boldsymbol{\chi}_h = \boldsymbol{0}$, and $\boldsymbol{v}_h^n$ as defined at the beginning of the proof. Therefore, employing arguments similar to those of the previous development, we have that:

$$\begin{aligned}
\sum_{n=0}^{N-1} \delta t \alpha_1^{1/2} \left(P_{h,0}\left[\boldsymbol{m}_s^{n+1}\right], \boldsymbol{v}_h^{n+1}\right) &= \sum_{n=0}^{N-1} \delta t \alpha_1^{1/2} \left(\boldsymbol{a} \cdot \nabla \boldsymbol{u}_h^{n+1} + \nabla p_h^{n+1} - \nabla \cdot \boldsymbol{\sigma}_h^{n+1}, \boldsymbol{v}_h^{n+1}\right) \\
&= \sum_{n=0}^{N-1} \delta t \alpha_1^{1/2} \left\langle \boldsymbol{f}^{n+1}, \boldsymbol{v}_h^{n+1}\right\rangle - \sum_{n=0}^{N-1} \alpha_1^{1/2} \left(\delta \boldsymbol{u}_h^n, \boldsymbol{v}_h^{n+1}\right) - \sum_{n=0}^{N-1} 2\eta_s \delta t \alpha_1^{1/2} \left(\nabla^s \boldsymbol{u}_h^{n+1}, \nabla^s \boldsymbol{v}_h^{n+1}\right) \\
&\quad + \sum_{n=0}^{N-1} \delta t \alpha_1^{1/2} \left(\tilde{\boldsymbol{u}}_1^{n+1}, \boldsymbol{a} \cdot \nabla \boldsymbol{v}_h^{n+1}\right) - \sum_{n=0}^{N-1} \delta t \alpha_1^{1/2} \left(\tilde{\boldsymbol{\sigma}}^{n+1}, \nabla^s \boldsymbol{v}_h^{n+1}\right) \\
&\lesssim \alpha_1^{1/2} \max_{n=1,\dots,N} \left\{\|\boldsymbol{v}_h^n\|\right\} \sum_{n=0}^{N-1} \delta t \left\|\boldsymbol{f}^{n+1}\right\| + \max_{n=1,\dots,N} \left\{\|\boldsymbol{u}_h^n\|\right\} \sum_{n=0}^{N-1} \delta t \alpha_1^{1/2} \|\delta_t \boldsymbol{v}_h^n\| \\
&\quad + \max_{n=0,\dots,N} \left\{\|\boldsymbol{u}_h^n\|\right\} \alpha_1^{1/2} \max_{n=1,\dots,N} \left\{\|\boldsymbol{v}_h^n\|\right\} + \sum_{n=0}^{N-1} \delta t \eta_s^{1/2} \left\|\nabla^s \boldsymbol{u}_h^{n+1}\right\| \left\|\boldsymbol{v}_h^{n+1}\right\| \\
&\quad + \sum_{n=0}^{N-1} \delta t {\alpha_1}^{-1/2} \left\|\tilde{\boldsymbol{u}}_1^{n+1}\right\| \left\|\boldsymbol{v}_h^{n+1}\right\| + \sum_{n=0}^{N-1} \delta t \alpha_3^{-1/2} \left\|\tilde{\boldsymbol{\sigma}}^{n+1}\right\| \left\|\boldsymbol{v}_h^{n+1}\right\|. \qquad (4.11)
\end{aligned}$$

In particular, the three last terms have been obtained using the inverse estimate (3.1) together with the definitions of α_1 and α_3, which in particular imply:

$$\frac{\eta_0}{h}\alpha_1^{1/2} \le C\eta_0^{1/2}, \quad \frac{\|\boldsymbol{a}\|_{L^\infty(K)}}{h} \le \alpha_1^{-1}, \quad \frac{\alpha_1^{1/2}}{h} \le \alpha_3^{-1/2}.$$

The last inequality holds under the condition $2c_3 \le c_1$, being c_1 and c_3 the constants of the stabilization parameters α_1 and α_3. Using now the assumptions on the data, Cauchy's inequality for the last three terms of (4.11) and finally using Theorem 1, it follows that

$$\left\langle \{\boldsymbol{v}^n\}, \alpha_1^{1/2}\{P_{h,0}[\boldsymbol{m}_s^n]\}\right\rangle_{X_1\times X_1'} = \sum_{n=0}^{N-1}\delta t\alpha_1^{1/2}\left(P_{h,0}\left[\boldsymbol{m}_s^{n+1}\right], \boldsymbol{v}_h^{n+1}\right) \lesssim \|\{\boldsymbol{v}^n\}\|_{X_1}.$$

Therefore

$$\left\|\left\{\alpha_1^{1/2}P_{h,0}\left[\boldsymbol{m}_1^n + \boldsymbol{m}_2^n + \boldsymbol{m}_3^n\right]\right\}\right\|_{X_1'} \le C. \tag{4.12}$$

The theorem follows from the addition of (4.10) and (4.12).

The proof is analogous for the LCR case, considering (4.5) and replacing $\nabla\cdot\boldsymbol{\sigma}_h^{n+1}$ by $\dfrac{\eta_p}{\lambda_0}\nabla\cdot\bar{\boldsymbol{\psi}}_h^{n+1}$ in (4.11). Details are omitted. □

Now we can proceed to prove a similar result for the constitutive equation. Since the stress or the logarithm of the conformation tensor have no boundary conditions, we will be able to prove stability for $\{P_h[\alpha_3^{1/2}\boldsymbol{m}_4^n]\}$ and $\{P_h^\perp[\alpha_3^{1/2}\boldsymbol{m}_4^n]\}$, that is to say, for the whole sequence $\{\alpha_3^{1/2}\boldsymbol{m}_4^n\}$.

Theorem 4 (Stability in the dual norm X_3' for the constitutive equation)**.** *Under the assumptions of Theorem 3, there is a constant C such that*

$$\left\|\left\{\alpha_3^{1/2}\boldsymbol{m}_4^n\right\}\right\|_{X_3'} \le C,$$

both for the standard formulation and for the LCR.

Proof. Let $\{\boldsymbol{\chi}^n\} \in X_3$ be an arbitrary sequence, which can be split as $\boldsymbol{\chi}^n = \boldsymbol{\chi}_h^n + \tilde{\boldsymbol{\chi}}^n$, with $\boldsymbol{\chi}_h^n = P_h[\boldsymbol{\chi}^n]$. We can write (2.24) as

$$\frac{\lambda}{2\eta_p}\delta\tilde{\boldsymbol{\sigma}}^n + \delta t\alpha_3^{-1}\tilde{\boldsymbol{\sigma}}^{n+1} = -\delta t P_h^\perp\left[\boldsymbol{m}_4^{n+1}\right].$$

Now, we follow the same procedure as in Theorem 3: multiplying by $\alpha_3^{1/2}\tilde{\boldsymbol{\chi}}^{n+1}$, integrating over Ω and adding up the result from $n = 0$ to $N-1$, integrating by parts using the discrete integration formula and using the Cauchy-Schwartz inequality, we have:

$$\begin{aligned}
\sum_{n=0}^{N-1}\delta t\alpha_3^{1/2}\left(P_h^\perp\left[\boldsymbol{m}_4^{n+1}\right], \tilde{\boldsymbol{\chi}}^{n+1}\right) &= -\sum_{n=0}^{N-1}\frac{\lambda}{2\eta_p}\alpha_3^{1/2}\left(\delta\tilde{\boldsymbol{\sigma}}^n, \tilde{\boldsymbol{\chi}}^{n+1}\right) - \sum_{n=0}^{N-1}\delta t\alpha_3^{-1/2}\left(\tilde{\boldsymbol{\sigma}}^{n+1}, \tilde{\boldsymbol{\chi}}^{n+1}\right)\\
&= \sum_{n=0}^{N-1}\frac{\lambda}{2\eta_p}\alpha_3^{1/2}\left(\tilde{\boldsymbol{\sigma}}^n, \delta\tilde{\boldsymbol{\chi}}^n\right) - \frac{\lambda}{2\eta_p}\alpha_3^{1/2}\left(\tilde{\boldsymbol{\sigma}}^N, \tilde{\boldsymbol{\chi}}^N\right) + \frac{\lambda}{2\eta_p}\alpha_3^{1/2}\left(\tilde{\boldsymbol{\sigma}}^0, \tilde{\boldsymbol{\chi}}^0\right) - \sum_{n=0}^{N-1}\delta t\alpha_3^{-1/2}\left(\tilde{\boldsymbol{\sigma}}^{n+1}, \tilde{\boldsymbol{\chi}}^{n+1}\right)\\
&\lesssim \frac{\lambda}{2\eta_p}\max_{n=0,\dots,N-1}\{\|\tilde{\boldsymbol{\sigma}}^n\|\}\sum_{n=0}^{N-1}\delta t\alpha_3^{1/2}\|\delta_t\tilde{\boldsymbol{\chi}}^n\| + \frac{\lambda}{\eta_p}\max_{n=0,\dots,N-1}\{\|\tilde{\boldsymbol{\sigma}}^n\|\}\alpha_3^{1/2}\max_{n=0,..,N}\{\|\tilde{\boldsymbol{\chi}}^n\|\}
\end{aligned}$$

$$+\left(\sum_{n=0}^{N-1}\delta t\left\|\alpha_3^{-1/2}\tilde{\boldsymbol{\sigma}}^{n+1}\right\|^2\right)^{1/2}\left(\sum_{n=0}^{N-1}\delta t\left\|\tilde{\boldsymbol{\chi}}^{n+1}\right\|^2\right)^{1/2}. \tag{4.13}$$

Therefore, using Theorem 1 and the definition of the norm in X_3 (4.7), we can conclude that:

$$\left\langle\{\boldsymbol{\chi}^n\},\alpha_3^{1/2}\left\{P_h^{\perp}\left[\boldsymbol{m}_4^n\right]\right\}\right\rangle_{X_3\times X_3'}=\sum_{n=0}^{N-1}\delta t\alpha_3^{1/2}\left(P_h^{\perp}\left[\boldsymbol{m}_4^{n+1}\right],\tilde{\boldsymbol{\chi}}^{n+1}\right)\lesssim\|\{\boldsymbol{\chi}_h^n\}\|_{X_3}. \tag{4.14}$$

The next step is to control the FE component, that is, $P_h\left[\boldsymbol{m}_4^n\right]$. Now, we use equation (2.20) considering $q_h=0$ and $\boldsymbol{v}_h=\boldsymbol{0}$, and $\boldsymbol{\chi}_h$ as defined at the beginning of the proof. Therefore, employing arguments similar to those of the previous development, we have that:

$$\begin{aligned}
&\sum_{n=0}^{N-1}\delta t\alpha_3^{1/2}\left(P_h\left[\boldsymbol{m}_4^{n+1}\right],\boldsymbol{\chi}_h^{n+1}\right)\\
&=\sum_{n=0}^{N-1}\delta t\alpha_3^{1/2}\left(\frac{1}{2\eta_p}\boldsymbol{\sigma}_h^{n+1}-\nabla^s\boldsymbol{u}_h^{n+1},\boldsymbol{\chi}_h^{n+1}\right)\\
&\quad+\sum_{n=0}^{N-1}\delta t\alpha_3^{1/2}\frac{\lambda}{2\eta_p}\left(\boldsymbol{a}\cdot\nabla\boldsymbol{\sigma}_h^{n+1}-\boldsymbol{\sigma}_h^{n+1}\cdot\nabla\boldsymbol{a}-(\nabla\boldsymbol{a})^T\cdot\boldsymbol{\sigma}_h^{n+1},\boldsymbol{\chi}_h^{n+1}\right)\\
&=-\sum_{n=0}^{N-1}\alpha_3^{1/2}\frac{\lambda}{2\eta_p}\left(\delta\boldsymbol{\sigma}_h^n,\boldsymbol{\chi}_h^{n+1}\right)-\sum_{n=0}^{N-1}\delta t\alpha_3^{1/2}\left(\tilde{\boldsymbol{u}}_3^{n+1},\nabla\cdot\boldsymbol{\chi}_h^{n+1}\right)\\
&\quad-\sum_{n=0}^{N-1}\delta t\alpha_3^{1/2}\left(\tilde{\boldsymbol{\sigma}}^{n+1},\frac{1}{2\eta_p}\boldsymbol{\chi}_h-\frac{\lambda}{2\eta_p}\left(\boldsymbol{a}\cdot\nabla\boldsymbol{\chi}_h+\boldsymbol{\chi}_h\cdot(\nabla\boldsymbol{a})^T+\nabla\boldsymbol{a}\cdot\boldsymbol{\chi}_h\right)\right)\\
&\lesssim\frac{\lambda}{2\eta_p}\max_{n=1,\ldots,N}\{\|\boldsymbol{\sigma}_h^n\|\}\sum_{n=0}^{N-1}\delta t\alpha_3^{1/2}\|\delta\boldsymbol{\chi}_h^n\|+2\frac{\lambda}{2\eta_p}\max_{n=0,\ldots,N}\{\|\boldsymbol{\sigma}_h^n\|\}\alpha_3^{1/2}\max_{n=1,\ldots,N}\{\|\boldsymbol{\chi}_h^n\|\}\\
&\quad+\sum_{n=0}^{N-1}\delta t\alpha_3^{-1/2}\left\|\tilde{\boldsymbol{\sigma}}^{n+1}\right\|\left\|\boldsymbol{\chi}_h^{n+1}\right\|+\sum_{n=0}^{N-1}\delta t\alpha_1^{-1/2}\left\|\tilde{\boldsymbol{u}}_3^{n+1}\right\|\left\|\boldsymbol{\chi}_h^{n+1}\right\|.
\end{aligned} \tag{4.15}$$

In particular, the two last terms have been obtained using the inverse estimate (3.1) together with the definitions of α_1 and α_3, which imply:

$$\frac{C}{2\eta_0}\leq\alpha_3^{-1},\quad\frac{\lambda}{2\eta_0}\frac{\|\boldsymbol{a}\|_{L^\infty(K)}}{h}\leq\alpha_3^{-1},\quad\frac{\lambda}{2\eta_0}\|\nabla\boldsymbol{a}\|_{L^\infty(K)}\leq\alpha_3^{-1},\quad\frac{\alpha_3^{1/2}}{h}\leq\alpha_1^{-1/2}.$$

As in Theorem 3, the last inequality holds under the assumption $2c_3\leq c_1$, being c_1 and c_3 the constants of the stabilization parameters α_1 and α_3.

Using the assumptions on the data, Cauchy's inequality for the last two terms of (4.15) and finally using Theorem 1, it follows that

$$\left\langle\{\boldsymbol{\chi}^n\},\alpha_3^{1/2}\left\{P_h\left[\boldsymbol{m}_4^n\right]\right\}\right\rangle_{X_3\times X_3'}=\sum_{n=0}^{N-1}\delta t\alpha_3^{1/2}\left(P_h\left[\boldsymbol{m}_4^{n+1}\right],\boldsymbol{\chi}_h^{n+1}\right)\lesssim\|\{\boldsymbol{\chi}_h^n\}\|_{X_3}. \tag{4.16}$$

For the standard formulation, the theorem follows from (4.14) and (4.16).

For the LCR, there are slight differences in the proof. In this case we have to consider expression (4.6) of $\boldsymbol{m}_4^n$. In the step where the FE component is controlled, we use equation (2.27), considering $q_h = 0$ and $\boldsymbol{v}_h = \boldsymbol{0}$, and $\boldsymbol{\chi}_h$ as defined above. Therefore, we have that:

$$\begin{aligned}
&\sum_{n=0}^{N-1} \delta t \alpha_3^{1/2} \left(P_h \left[\boldsymbol{m}_4^{n+1} \right], \boldsymbol{\chi}_h^{n+1} \right) \\
&\quad = \sum_{n=0}^{N-1} \delta t \alpha_3^{1/2} \left(\frac{1}{2\lambda_0} \bar{\boldsymbol{\psi}}_h^{n+1} - \nabla^s \boldsymbol{u}_h^{n+1}, \boldsymbol{\chi}_h^{n+1} \right) + \sum_{n=0}^{N-1} \delta t \alpha_3^{1/2} \frac{\lambda}{2\lambda_0} \Big(2\nabla^s \boldsymbol{u}_h^{n+1} + \boldsymbol{a} \cdot \nabla \bar{\boldsymbol{\psi}}_h^{n+1} \\
&\qquad - \bar{\boldsymbol{\psi}}_h^{n+1} \cdot \nabla \boldsymbol{a} - (\nabla \boldsymbol{a})^T \cdot \bar{\boldsymbol{\psi}}_h^{n+1}, \boldsymbol{\chi}_h^{n+1} \Big) \\
&\quad = - \sum_{n=0}^{N-1} \alpha_3^{1/2} \frac{\lambda}{2\lambda_0} \left(\delta \bar{\boldsymbol{\psi}}_h^n, \boldsymbol{\chi}_h^{n+1} \right) - \sum_{n=0}^{N-1} \delta t \alpha_3^{1/2} \left(\tilde{\boldsymbol{u}}_3^{n+1}, \nabla \cdot \boldsymbol{\chi}_h^{n+1} \right) \\
&\qquad - \sum_{n=0}^{N-1} \delta t \alpha_3^{1/2} \left(\tilde{\boldsymbol{\sigma}}^{n+1}, -\frac{\lambda}{2\eta_p} \left(\boldsymbol{a} \cdot \nabla \boldsymbol{\chi}_h + \boldsymbol{\chi}_h \cdot (\nabla \boldsymbol{a})^T + \nabla \boldsymbol{a} \cdot \boldsymbol{\chi}_h \right) \right) \\
&\quad \lesssim \frac{\lambda}{2\lambda_0} \max_{n=1,\ldots,N} \left\{ \left\| \bar{\boldsymbol{\psi}}_h^n \right\| \right\} \sum_{n=0}^{N-1} \delta t \alpha_3^{1/2} \left\| \delta_t \boldsymbol{\chi}_h^n \right\| \\
&\qquad + 2 \frac{\lambda}{2\lambda_0} \max_{n=0,\ldots,N} \left\{ \left\| \bar{\boldsymbol{\psi}}_h^n \right\| \right\} \alpha_3^{1/2} \max_{n=1,\ldots,N} \left\{ \left\| \boldsymbol{\chi}_h^n \right\| \right\} \\
&\qquad + \sum_{n=0}^{N-1} \delta t \alpha_3{}^{-1/2} \left\| \tilde{\boldsymbol{\sigma}}^{n+1} \right\| \left\| \boldsymbol{\chi}_h^{n+1} \right\| + \sum_{n=0}^{N-1} \delta t \alpha_1^{-1/2} \left\| \tilde{\boldsymbol{u}}_3^{n+1} \right\| \left\| \boldsymbol{\chi}_h^{n+1} \right\|,
\end{aligned}$$

and this result is analogous to the one obtained for the standard formulation. □

As done in [30], we can obtain a sharper stability estimate using the fact the BDF1 time integration scheme also provides control on the increments of the unknown between time steps. While the results in Theorems 3 and 4 could be extended to other time integration schemes with a more or less tedious algebra, the following result is specific of the BDF1 scheme.

Although disregarded later, along the proofs of Theorems 1 and 2, we have obtained the following estimates:

$$\rho \sum_{n=0}^{M} \left\| \delta \boldsymbol{u}_h^n \right\|^2 + \rho \left(\sum_{n=0}^{M} \left\| \delta \tilde{\boldsymbol{u}}_1^n \right\|^2 + \sum_{n=0}^{M} \left\| \delta \tilde{\boldsymbol{u}}_2^n \right\|^2 + \sum_{n=0}^{M} \left\| \delta \tilde{\boldsymbol{u}}_3^n \right\|^2 \right) \leq C, \tag{4.17}$$

$$\sum_{n=0}^{M} \frac{\lambda}{2\eta_p} \left\| \delta \boldsymbol{\sigma}_h^n \right\|^2 + \sum_{n=0}^{M} \frac{\lambda}{2\eta_p} \left\| \delta \tilde{\boldsymbol{\sigma}}^n \right\|^2 \leq C. \tag{4.18}$$

For the LCR, the inequality to consider is:

$$\sum_{n=0}^{M} \frac{\lambda}{2\lambda_0} \left\| \delta \bar{\boldsymbol{\psi}}_h^n \right\|^2 + \sum_{n=0}^{M} \frac{\lambda}{2\eta_p} \left\| \delta \tilde{\boldsymbol{\sigma}}^n \right\|^2 \leq C. \tag{4.19}$$

Now we will define the following norm, that will replace norm $\|\cdot\|_{X_i}$ in (4.7):

$$\|F\|_{Y_i} := \left(\sum_{n=0}^{N} \max\{\delta t, \alpha_i\} \|f^n\|^2 \right)^{1/2}, \quad i = 1, 3. \tag{4.20}$$

Theorem 5 (Stability in the dual norm Y_i'). *Under the assumptions of Theorem 3, there holds:*

$$a) \qquad \sum_{i=1}^{3} \left\| \left\{ \alpha_1^{1/2} P_h^{\perp}(\boldsymbol{m}_i^n) \right\} \right\|_{Y_1'} + \left\| \left\{ \alpha_1^{1/2} P_{h,0}(\boldsymbol{m}_1^n + \boldsymbol{m}_2^n + \boldsymbol{m}_3^n) \right\} \right\|_{Y_1'} \leq C,$$

$$b) \qquad \left\| \left\{ \alpha_3^{1/2} \boldsymbol{m}_4^n \right\} \right\|_{Y_3'} \leq C.$$

Proof. For the terms of the momentum equation, a), the only difference with Theorem 3 is that we have to deal with the terms:

$$\sum_{n=0}^{N-1} \rho\, \alpha_1^{1/2} \left(\delta\tilde{\boldsymbol{u}}_i^n, \tilde{\boldsymbol{v}}^{n+1} \right) \leq \left(\sum_{n=0}^{N-1} \rho \,\|\delta\tilde{\boldsymbol{u}}_i^n\|^2 \right)^{1/2} \left(\sum_{n=0}^{N-1} \rho\, \alpha_1 \|\tilde{\boldsymbol{v}}^n\|^2 \right)^{1/2}, \tag{4.21}$$

$$\sum_{n=0}^{N-1} \rho\, \alpha_1^{1/2} \left(\delta\boldsymbol{u}_h^n, \boldsymbol{v}_h^{n+1} \right) \leq \left(\sum_{n=0}^{N-1} \rho \,\|\delta\boldsymbol{u}_h^n\|^2 \right)^{1/2} \left(\sum_{n=0}^{N-1} \rho\, \alpha_1 \|\boldsymbol{v}_h^n\|^2 \right)^{1/2}. \tag{4.22}$$

The first expression would be employed in step (4.9), where the SGSs are bounded, and the second expression would be used in step (4.11). In this last case the fields in the FE space are bounded. Using the properties defined previously, (4.21) and (4.22) will be finally bounded by $C\|\boldsymbol{v}^n\|_{Y_1}$, from where result a) follows as in Theorem 3.

Now in order to bound the terms corresponding to the constitutive equation, b), the inequality used is:

$$\sum_{n=0}^{N-1} \frac{\lambda}{2\eta_p} \alpha_3^{1/2} \left(\delta\tilde{\boldsymbol{\sigma}}^n, \tilde{\boldsymbol{\chi}}^{n+1} \right) \leq \left(\sum_{n=0}^{N-1} \frac{\lambda}{2\eta_p} \|\delta\tilde{\boldsymbol{\sigma}}^n\|^2 \right)^{1/2} \left(\sum_{n=0}^{N-1} \frac{\lambda}{2\eta_p} \alpha_3 \|\tilde{\boldsymbol{\chi}}^n\|^2 \right)^{1/2}. \tag{4.23}$$

Employing again the properties defined previously, (4.18), we can bound (4.23) by $C\|\boldsymbol{\chi}^n\|_{Y_3}$, from where result b) follows analogously as in Theorem 4.

For the LCR the arguments would be analogous for a) and b). The only difference is that in b) the inequality for bounding the FE space component is the next one:

$$\sum_{n=0}^{N-1} \frac{\lambda}{2\lambda_0} \alpha_3^{1/2} \left(\delta\bar{\boldsymbol{\psi}}_h^n, \boldsymbol{\chi}_h^{n+1} \right) \leq \left(\sum_{n=0}^{N-1} \frac{\lambda}{2\lambda_0} \|\delta\bar{\boldsymbol{\psi}}_h^n\|^2 \right)^{1/2} \left(\sum_{n=0}^{N-1} \frac{\lambda}{2\lambda_0} \alpha_3 \|\boldsymbol{\chi}_h^n\|^2 \right)^{1/2}. \tag{4.24}$$

Finally, using (4.19), the expression above would be bounded by $C\|\boldsymbol{\chi}_h^n\|_{Y_3}$. □

4.3. **Stability in natural norms**

From Theorem 5 the next result is straightforward to prove, which holds if the inequality $\delta t \geq C\alpha_i$ for $i = 1, 3$ is satisfied.

Theorem 6 (Stability in the natural norm)**.** *Under the assumptions of Theorem 5, assume also that the stabilization parameters satisfy* $\alpha_1 \leq C\delta t$ *and* $\alpha_3 \leq C\delta t$ *as* $h \to 0$ *and* $\delta t \to 0$*. Then, there holds:*

$$a) \qquad \sum_{i=1}^{3}\sum_{n=0}^{N} \delta t \left\| \left\{ \alpha_1^{1/2} P_h^{\perp} \left[\boldsymbol{m}_i^n \right] \right\} \right\|^2 + \sum_{n=0}^{N} \delta t \left\| \left\{ \alpha_1^{1/2} P_{h,0} \left[\boldsymbol{m}_1^n + \boldsymbol{m}_2^n + \boldsymbol{m}_3^n \right] \right\} \right\|^2 \leq C,$$

$$b) \qquad \sum_{n=0}^{N} \delta t \left\| \left\{ \alpha_3^{1/2} \boldsymbol{m}_4^n \right\} \right\|^2 \leq C.$$

Proof. If inequalities $\alpha_1 \leq C\delta t$ and $\alpha_3 \leq C\delta t$ hold, it is immediately seen from definition (4.20) that $Y = \ell^2\left(L^2(\Omega)\right)$. This space satisfies $Y = Y'$. Therefore the theorem is directly a consequence of Theorem 5. □

According to Theorem 6, we have control on the component of $\boldsymbol{m}_1 + \boldsymbol{m}_2 + \boldsymbol{m}_3$ in $\boldsymbol{\mathcal{V}}_{h,0}$ and $\boldsymbol{\mathcal{V}}_h^{\perp}$, but it remains to prove stability on the component in $\boldsymbol{\mathcal{V}}_h$ associated to functions defined on $\partial\Omega$. As discussed in previous works [6, 33], this is done by prescribing a (mild) condition on the FE mesh.

Corollary 1. *Under the assumptions of Theorem 6, suppose also that the FE mesh is such that given* $\boldsymbol{a}, \boldsymbol{v}_h \in \boldsymbol{\mathcal{V}}_h$*,* $q_h \in \mathcal{Q}_h$*,* $\boldsymbol{\chi}_h \in \boldsymbol{\Upsilon}_h$ *and* $\boldsymbol{z}_h := \rho \boldsymbol{a} \cdot \nabla \boldsymbol{v}_h + \nabla q_h - \nabla \cdot \boldsymbol{\chi}_h$*, there holds*

$$\|\boldsymbol{z}_h\| \leq c_{\mathrm{m}} \left(\|P_{h,0}\left[\boldsymbol{z}_h\right]\| + \left\|P_h^{\perp}\left[\boldsymbol{z}_h\right]\right\| \right), \text{ for a constant } c_{\mathrm{m}} > 0.$$

Then:

$$\sum_{n=0}^{N} \delta t \left\| \alpha_1^{1/2} \left[\boldsymbol{m}_1^n + \boldsymbol{m}_2^n + \boldsymbol{m}_3^n \right] \right\|^2 \leq C. \tag{4.25}$$

Proof. Let $\boldsymbol{z}_h^n = \alpha_1^{1/2}\left[\boldsymbol{m}_1^n + \boldsymbol{m}_2^n + \boldsymbol{m}_3^n\right]$. From the assumption made we have that

$$\begin{aligned} \|\{\boldsymbol{z}_h^n\}\|_{\ell^2(L^2(\Omega)^d)} &\lesssim \|\{P_{h,0}[\boldsymbol{z}_h^n]\}\|_{\ell^2(L^2(\Omega)^d)} + \|\{P_h^{\perp}[\boldsymbol{z}_h^n]\}\|_{\ell^2(L^2(\Omega)^d)} \\ &\lesssim \|\{P_{h,0}[\boldsymbol{z}_h^n]\}\|_{\ell^2(L^2(\Omega)^d)} + \sum_{i=1}^{3} \left\| \left\{ \alpha_1^{1/2} P_h^{\perp} \left[\boldsymbol{m}_i^n \right] \right\} \right\|_{\ell^2(L^2(\Omega)^d)}^2, \end{aligned}$$

from where the result follows using Theorem 6. □

5. Conclusions

In this paper we have presented the stability analysis of a VMS-based stabilized FE method for a linearized viscoelastic flow problem discretized in time using the BDF1 scheme, using both the standard formulation and the LCR. The main features of the formulation are to consider the SGSs as dynamic, orthogonal to the FE space and assigning one velocity SGS for each of the terms of the momentum equation that need to be stabilized. First, we have obtained stability for the FE component and the SGSs in standard norms; the results obtained are the expected ones (Theorems 1 and 2). Of particular relevance is the $\ell^\infty(L^2(\Omega)^d)$ stability obtained for the SGSs, which is a direct consequence of considering them dynamic. The stability bounds for the SGSs can be translated into additional stability for the FE components of the unknowns. This is done first in rather weak dual norms (Theorems 3 and 4), with a certain improvement if the dissipative nature of the BDF1 scheme is employed (Theorem 5). Furthermore, if the classical condition relating the time step size and the stabilization parameters is assumed, a classical stability estimate can be derived (Theorem 6). In all results proven, the constants in the stability estimates do not depend on the physical properties of the problem being solved. All these results serve to assess that the formulation proposed is stable in the terms that could be expected. From this stability one could prove convergence using more or less standard arguments.

R. Codina gratefully acknowledges the support received from the ICREA Acadèmia Program, from the Catalan Government. L. Moreno acknowledges the support received from the Spanish Government through a predoctoral FPI Grant.

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