

A variational multiscale stabilized finite element formulation for Reissner-Mindlin plates and Timoshenko beams

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Abstract

The theories for thick plates and beams, namely Reissner-Mindlin's and Timoshenko's theories, are well known to suffer numerical locking when approximated using the standard Galerkin finite element method for small thicknesses. This occurs when the same interpolations are used for displacement and rotations, reason for which stabilization becomes necessary. To overcome this problem, a variational multiscale stabilization method is analyzed in this work. In this framework, two different approaches are presented: the Algebraic Sub-Grid Scale formulation and the Orthogonal Sub-Grid Scale formulation. Stability and convergence is proved for both approaches, explaining why the latter performs much better. Although the numerical examples show that the Algebraic Sub-Grid Scale approach is in some cases able to overcome the numerical locking, it is highly sensitive to stabilization parameters and presents difficulties to converge optimally with respect to the element size in the L^2 norm. In this regard, the Orthogonal Sub-Grid Scale approach, which considers the space of the sub-grid scales to be orthogonal to the finite element space, is shown to be stable and optimally convergent independently of the thickness of the solid.

Keywords: Reissner-Mindlin Plate, Timoshenko Beam, Shear Locking, VMS.

1. Introduction

In the context of computational mechanics, plates and beams are structural elements that are able to represent three dimensional objects in which the length of one or two of its dimensions is significantly smaller than the others. This dimensional reduction allows engineers to model complex structures at the cost of introducing rotations as independent variables. The Reissner-Mindlin equations are commonly used to describe the behavior of thin and moderately thick plates under transverse loads. This model differs from the Poisson-Kirchhoff equations for thin plates by considering the shear deformations due to distortion. In a physical sense, it is assumed that a straight line normal to the undeformed middle plane will remain straight but not necessarily perpendicular to the middle plane after deformation. The same analogy applies to the Timoshenko equations for thin-thick beams, with respect to the Euler-Bernoulli equations for thin beams. This leads to two groups of equations: the thin-thick theories, referred to as Reissner-Mindlin plates and Timoshenko beams, and the thin theories, referred to as Poisson-Kirchhoff plates and Euler-Bernoulli beams.

This paper is focused on the Finite Element (FE) stabilization of thin-thick theories of plates and beams, founded on the fact that applications in engineering usually fall into this category, and in the interest of using C^0 interpolations for displacement and rotations that are not possible using thin theories. In this context, both in Reissner-Mindlin plate and Timoshenko beam theories, the thickness of the structure appears explicitly in the partial differential equations, which results in a dominance of the shear term as it tends to the slender limit case. This fact together with the zero shear strain constraint leads to *numerical locking* when the equations are solved using the same interpolation for deflection and rotations, and it is the reason it needs to be stabilized.

A considerable amount of work has been put into the design of *locking-free* elements, namely that the order of convergence does not depend on the thickness of the structure. The existing approaches can be classified depending on whether or not the problem is dealt with in the irreducible form or if the shear strain is interpolated as an additional variable of the problem, which alleviates the zero shear constraint. In the irreducible approach, the equations are presented in terms of deflections and rotations only, and if these equations are discretized using standard polynomial spaces, the numerically simulated structure behaves with a stiffness larger than it should for small thicknesses [1]. This pure numerical response is known as numerical locking, and for this particular case, it is often called *shear-locking* because it occurs due to the inability to reproduce a zero shear strain field. In that regard, the numerical locking will occur for any loading that results in a pure bending state of the structure. On the other hand, the mixed formulation approach consists in interpolating the shear strain from the constitutive equations as an additional unknown. This can be extended, although not necessarily, to the dual-mixed formulation by also interpolating the bending strain. This type of implementations is used to avoid restrictive constraints; however, they are inf-sup deficient and also require stabilization. For a better understanding of the FE approach to plates we refer to [2].

The most common approach used to solve the problem, independently of which variables are interpolated, is to use MITC (Mixed Interpolation of Tensorial Components) elements, that allows one to use different interpolations for each unknown. These approaches have been, and are still being developed for different types of problems [3, 4]. The reader can refer to [5] for a general overview, and also to [6] for a general analysis of the design procedure of MITC elements based on the velocity-pressure pair of the Stokes problem. The main challenge of this approach is to develop a stabilized formulation that converges optimally for every unknown using the lowest possible order of interpolation. A good insight of this in relation to the plate problem can be followed from [7], where it was established that shear stress is convergent if the bending moment converges at a high enough rate. This led to further developments using a variational approach [8] that was valid for any element of high enough order of interpolation for MITC elements, limited to using at least quadratic interpolations for the displacements. This limitation was improved in [9] by adding some consistent terms to the formulation.

A search of the literature shows that, in general, shear-locking is treated by using different types of selective integration. This however can be done from many different approaches such as using Lagrange multipliers [10, 11], or non-conforming method approaches [12, 13] which have also been a topic of discussion. Another approach was taken in [14], where the dual mixed formulation was stabilized using inter-elemental jumps of the unknowns. This approach provides optimal convergence of all the unknowns using linear interpolations only.

A more recent methodology consists in formulating the stabilization from the multiscale perspective. It started from the introduction of the Galerkin-Least Squares method [8] and was further developed in [15]. Another least-squares type formulation was developed in [16] by introducing stabilization terms using the shear stress computed from the equilibrium equations. A rather different approach was used in [17], where a preconditioner was obtained using a multigrid method that was initially proposed in [18], where a conjugate gradient iterative algorithm was used to solve the linear system. The mixed formulation was also solved in [19] by introducing different interpolations for bending and shear effects. Also, in [20] a method was developed using a modified version of the formulation presented in [21], where an additional term was added to the mixed form. A similar approach to one of the stabilization methods to be described in the following was adopted in [22], in particular applied to the Discrete Shear Gap (DSG) formulation, which was already designed to avoid shear locking.

In this work, the Reissner-Mindlin plate and the Timoshenko beam problems are addressed using the Variational Multiscale (VMS) method. The original Galerkin formulation is modified as little as possible using residual-based terms, making it consistent. The stabilized formulations presented in this work share similar characteristics with other formulations presented in the literature [2, 5, 9], however, they are originated from different perspectives, and there are differences that make this work a new contribution to the subject.

The paper is organized as follows: A brief overview of the physical problem is presented in section 2 and the variational form is given in section 3. The general form of the stabilized FE formulation is explained in section 4 and its implementation for the plate and beam problems is presented in section 5. The numerical analysis in which the method is proven to be stable and convergent is presented in section 6 and lastly, numerical results are shown in section 7.

2. Continuous boundary value problem

Consider a general definition of the transverse deflection w and the rotation $\boldsymbol{\theta}$ of a structure. In the case of plates, the rotation is a vector that contains the rotations that make it bend in the x and y directions as $\boldsymbol{\theta} = [\theta_x, \theta_y]$. Since the beam problem is a dimensional reduction of the plate problem, a single rotation in the z direction is considered in this case, as $\boldsymbol{\theta} = \theta = \theta_z$. In the problems presented below, the geometries and coordinate systems are defined under the convention presented in Fig. 1. Only beams in the plane will be considered, the extension to 3D beams being straightforward.

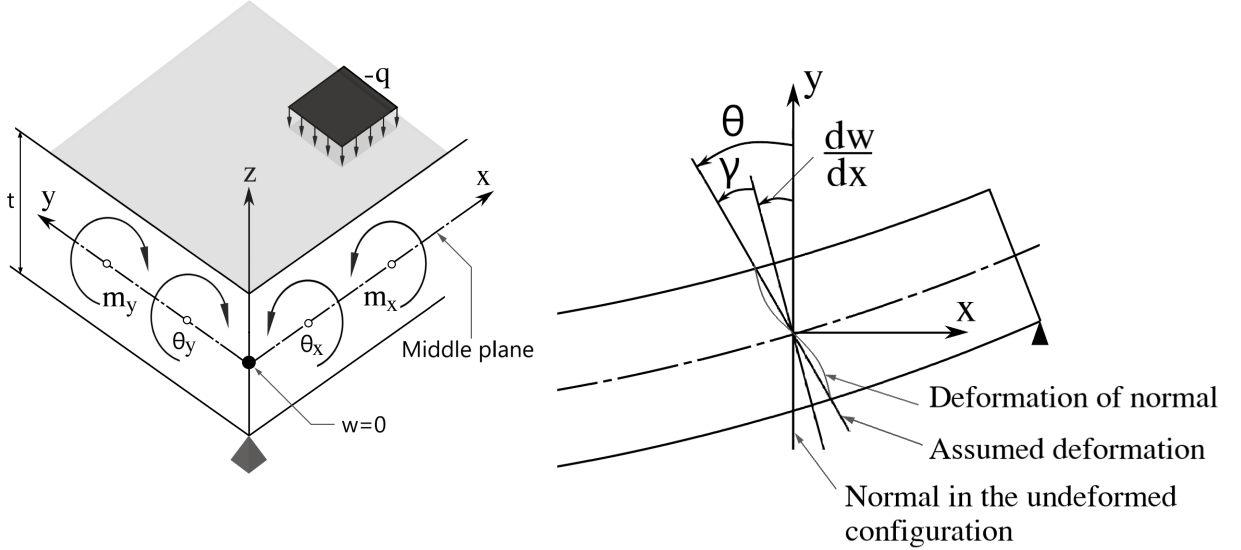


Figure 1: Coordinate system and definition of variables in plates and beams.

Let us define a domain in a general manner as Ω and its boundary $\Gamma = \partial\Omega$. Then Ω can be particularized for plates as $\Omega = \Omega_P$ and for beams as $\Omega = \Omega_B$, with their corresponding boundaries Γ_P and Γ_B . The irreducible form of the Reissner-Mindlin plate problem consists in finding $\mathbf{u} = [\boldsymbol{\theta}, w]$ in the domain Ω_P of \mathbb{R} with boundary Γ_P as the solution to

$$-k_1 \Delta \boldsymbol{\theta} - k_2 \nabla (\nabla \cdot \boldsymbol{\theta}) - \frac{1}{\varepsilon} (\nabla w - \boldsymbol{\theta}) = \mathbf{m} \quad \text{in } \Omega_P, \quad (2.1)$$

$$-\frac{1}{\varepsilon} \nabla \cdot (\nabla w - \boldsymbol{\theta}) = q \quad \text{in } \Omega_P, \quad (2.2)$$

$$w = 0 \quad \text{in } \Gamma_P, \quad (2.3)$$

$$\boldsymbol{\theta} = \mathbf{0} \quad \text{in } \Gamma_P, \quad (2.4)$$

for properly defined external loading moments $\mathbf{m} = [m_x, m_y]$, transverse loads q and

$$k_1 = \frac{Et^3}{24(1+\nu)}, \quad k_2 = \frac{Et^3}{24(1-\nu)}, \quad \varepsilon = \frac{2(1+\nu)}{E\kappa t}, \quad (2.5)$$

where E is the Young modulus, ν the Poisson ratio, t the thickness and κ the shear correction factor. In the development presented below, only homogeneous Dirichlet conditions are considered in (2.3) and (2.4), for simplicity.

Note that $\varepsilon \rightarrow \infty$ as $t \rightarrow 0$. However, k_1 and k_2 are $\mathcal{O}(t^3)$, whereas ε^{-1} is $\mathcal{O}(t)$. We have used the symbol ε^{-1} for the shear stiffness to emphasize that it dominates the bending stiffnesses as $t \rightarrow 0$.

Similarly, the Timoshenko beam problem consists in finding $\mathbf{u} = [\theta, w]$ defined in the domain Ω_B of \mathbb{R} with boundary Γ_B such that

$$-\frac{d}{dx} \left(EI \frac{d\theta}{dx} \right) - GA^* \left(\frac{dw}{dx} - \theta \right) = m \quad \text{in } \Omega_B, \quad (2.6)$$

$$-\frac{d}{dx} \left[GA^* \left(\frac{dw}{dx} - \theta \right) \right] = q \quad \text{in } \Omega_B, \quad (2.7)$$

$$w = 0 \quad \text{in } \Gamma_B, \quad (2.8)$$

$$\theta = 0 \quad \text{in } \Gamma_B, \quad (2.9)$$

where G is the shear modulus, I is the inertia in the bending axis, and A^* is the traverse reduced section area of the beam. The equilibrium equations are completed by the external moment, in this case $\mathbf{m} = m = m_z$, and external loads q .

It is important to mention that thick theories account for the shear strains $\boldsymbol{\gamma} := \nabla w - \boldsymbol{\theta}$, contrary to the $\boldsymbol{\gamma} = \mathbf{0}$ assumption of thin theories. Nevertheless, thick theories should converge to the corresponding thin theory solution when approaching the thin limit $t \rightarrow 0$.

Additional notation is necessary to build the formulations to be described below. Let us define, in a general manner, a linear differential operator \mathcal{L} , the trace operator \mathcal{D} that makes the problem well defined, and an external force vector \mathbf{f} . Either the plate or beam problems can now be written as: find $\mathbf{u} : \Omega \rightarrow \mathbb{R}^n$ such that

$$\mathcal{L}\mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \quad (2.10)$$

$$\mathcal{D}\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \Gamma. \quad (2.11)$$

For the problems we consider, $\mathcal{D}\mathbf{u}$ is just the trace of \mathbf{u} on Γ . As it has been mentioned, we will take $\bar{\mathbf{u}} = \mathbf{0}$ for simplicity.

We can refer to the Reissner-Mindlin plate problem when $n = 3$, $\mathcal{L} = \mathcal{L}_{RM}$ and $\mathbf{f} = \mathbf{f}_{RM}$, given by

$$\mathcal{L}_{RM}\mathbf{u} = \begin{bmatrix} -k_1 \Delta \boldsymbol{\theta} - k_2 \nabla (\nabla \cdot \boldsymbol{\theta}) - \frac{1}{\varepsilon} (\nabla w - \boldsymbol{\theta}) \\ -\frac{1}{\varepsilon} \nabla \cdot (\nabla w - \boldsymbol{\theta}) \end{bmatrix}, \quad \mathbf{f}_{RM} = \begin{bmatrix} \mathbf{m} \\ q \end{bmatrix}, \quad (2.12)$$

and to the Timoshenko beam problem when $n = 2$, $\mathcal{L} = \mathcal{L}_T$ and $\mathbf{f} = \mathbf{f}_T$, given by

$$\mathcal{L}_T\mathbf{u} = \begin{bmatrix} -\frac{d}{dx} \left(EI \frac{d\theta}{dx} \right) - GA^* \left(\frac{dw}{dx} - \theta \right) \\ -\frac{d}{dx} \left(GA^* \left(\frac{dw}{dx} - \theta \right) \right) \end{bmatrix}, \quad \mathbf{f}_T = \begin{bmatrix} m \\ q \end{bmatrix}. \quad (2.13)$$

To build the variational method below, it also is necessary to define the flux operator \mathcal{F} . Following the same notation as before, the flux operator corresponds to the Reissner-Mindlin problem when $\mathcal{F} = \mathcal{F}_{RM}$, defined as

$$\mathcal{F}_{RM}(\mathbf{u}) = \begin{bmatrix} k_1 \mathbf{n} \cdot \nabla \boldsymbol{\theta} + k_2 \mathbf{n} \nabla \cdot \boldsymbol{\theta} \\ \mathbf{n} \cdot \frac{1}{\varepsilon} (\nabla w - \boldsymbol{\theta}) \end{bmatrix}, \quad (2.14)$$

and to the Timoshenko beam problem when $\mathcal{F} = \mathcal{F}_T$, given by

$$\mathcal{F}_T(\mathbf{u}) = \begin{bmatrix} EI \frac{d\theta}{dx} \\ GA^* \left(\frac{dw}{dx} - \theta \right) \end{bmatrix}. \quad (2.15)$$

3. Variational Form

Consider $H^1(\Omega)$ as the space of functions in $L^2(\Omega)$ whose derivatives belong to $L^2(\Omega)$ and $H_0^1(\Omega)$ the subspace of $H^1(\Omega)$ of functions vanishing on Γ . In both specific problems, we denote the space of the deflection as $\mathcal{V} = H_0^1(\Omega)$; the space of the rotations \mathcal{Q} will correspond to $\mathcal{Q} = H_0^1(\Omega)^2$ when referring to plates and to $\mathcal{Q} = H_0^1(\Omega)$ when referring to beams.

Let us denote by $(\cdot, \cdot)_\Omega$ the L^2 -inner product and define $\langle \cdot, \cdot \rangle_\Omega$ as the integral of the product of two functions in Ω . We also denote as $\mathcal{X} = \mathcal{V} \times \mathcal{Q}$ the spaces where the weak problem is defined, whose test functions are $\mathbf{v} = [\phi, v] \in \mathcal{X}$. Let us introduce the bilinear form of the problem B and the linear form L as

$$B(\mathbf{u}, \mathbf{v}) = \langle \mathcal{L}\mathbf{u}, \mathbf{v} \rangle_\Omega + \langle \mathcal{F}\mathbf{u}, \mathcal{D}\mathbf{v} \rangle_{\partial\Omega}, \quad (3.1)$$

$$L(\mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_\Omega. \quad (3.2)$$

With all the above, the general problem in (2.10) and (2.11) is equivalent to the weak form of the problem, which consists in finding $\mathbf{u} \in \mathcal{X}$ such that

$$B(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}), \quad (3.3)$$

for all $\mathbf{v} \in \mathcal{X}$.

4. Stabilized Finite Element Formulation

The standard FE discretization consists in taking a finite partition $\{K\}$ of the domain Ω . Let the functional space where the continuous problem is posed be denoted by \mathcal{X} from which the constructed conforming FE space \mathcal{X}_h is a subset. Then the Galerkin FE approximation consists in finding $\mathbf{u}_h \in \mathcal{X}$ such that

$$B(\mathbf{u}_h, \mathbf{v}_h) = L(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathcal{X}_h. \quad (4.1)$$

The stabilized formulation analyzed in this paper is constructed in the Variational Multiscale (VMS) framework. The idea is to add additional terms to the Galerkin formulation of the problem that enhance stability without upsetting accuracy. This is achieved by splitting the space of unknowns as $\mathcal{X} = \mathcal{X}_h \oplus \mathcal{X}'$, where \mathcal{X}_h is the part that can be solved in the FE space and \mathcal{X}' is the remainder, or sub-grid scale, part. This leads to the splitting of the unknowns $\mathbf{u} = \mathbf{u}_h + \mathbf{u}'$ and test functions $\mathbf{v} = \mathbf{v}_h + \mathbf{v}'$. This splitting modifies the original formulation shown in (4.1) and turns the problem into: find $\mathbf{u}_h \in \mathcal{X}_h$ and $\mathbf{u}' \in \mathcal{X}'$ such that

$$B(\mathbf{u}_h, \mathbf{v}_h) + B(\mathbf{u}', \mathbf{v}_h) = L(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathcal{X}_h, \quad (4.2)$$

$$B(\mathbf{u}_h, \mathbf{v}') + B(\mathbf{u}', \mathbf{v}') = L(\mathbf{v}') \quad \forall \mathbf{v}' \in \mathcal{X}'. \quad (4.3)$$

The way this formulation is constructed requires an approximation of the sub-grid scales to be complete, which will be computed as a function of the FE part, as shown below. Note that choosing $\mathcal{X}' = \{0\}$ yields the Galerkin method, making this method consistent by construction.

4.1. Sub-grid scales in the element interiors

At this point, the approximation requires to solve more variables than the initial problem. This is dealt with by modifying (4.2) by using the additivity of the integral and the identity of the bilinear operator:

$$B(\mathbf{u}, \mathbf{v}) = \sum_K \langle \mathcal{L}\mathbf{u}, \mathbf{v} \rangle_K + \sum_K \langle \mathcal{F}\mathbf{u}, \mathcal{D}\mathbf{v} \rangle_{\partial K} = \sum_K \langle \mathbf{u}, \mathcal{L}^* \mathbf{v} \rangle_K + \sum_K \langle \mathcal{D}\mathbf{u}, \mathcal{F}^* \mathbf{v} \rangle_{\partial K}, \quad (4.4)$$

where the superscript $*$ denotes the adjoint of an operator. In the current problems, the operators involved are self-adjoint, which means that $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{F}^* = \mathcal{F}$; nevertheless, the superscript is left as a reference. In this

manner, equation (4.2) can be written as

$$B(\mathbf{u}_h, \mathbf{v}_h) + \sum_K [\langle \mathbf{u}', \mathcal{L}^* \mathbf{v}_h \rangle_K + \langle \mathcal{D}\mathbf{u}', \mathcal{F}^* \mathbf{v}_h \rangle_{\partial K}] = L(\mathbf{v}_h). \quad (4.5)$$

Then the problem reduces to find a proper approximation for \mathbf{u}' . This can be achieved in (4.3) by approximating \mathbf{u}' in terms of \mathbf{u}_h . We have that

$$\begin{aligned} B(\mathbf{u}', \mathbf{v}') &= L(\mathbf{v}') - B(\mathbf{u}_h, \mathbf{v}'), \\ &= L(\mathbf{v}') - \sum_K [\langle \mathcal{L}\mathbf{u}_h, \mathbf{v}' \rangle_K + \langle \mathcal{F}\mathbf{u}_h, \mathcal{D}\mathbf{v}' \rangle_{\partial K}], \\ &= \sum_K [\langle \mathcal{R}\mathbf{u}_h, \mathbf{v}' \rangle_K + \langle \mathcal{F}\mathbf{u}_h, \mathcal{D}\mathbf{v}' \rangle_{\partial K}], \end{aligned} \quad (4.6)$$

where $\mathcal{R}\mathbf{u}_h = \mathbf{f} - \mathcal{L}\mathbf{u}_h$ is the FE residual. For now let us impose that $\mathcal{D}\mathbf{u}' = \mathbf{0}$ on ∂K in an essential way, making the sub-grid scale test function satisfy $\mathcal{D}\mathbf{v}' = \mathbf{0}$ on ∂K . This yields the following approximation

$$\mathbf{u}' = \mathcal{L}^{-1}(\mathcal{R}\mathbf{u}_h + \mathbf{v}'^\perp), \quad (4.7)$$

where \mathbf{v}'^\perp guarantees that $\mathbf{u}' \in \mathcal{X}'$. In this expression, \mathcal{L}^{-1} cannot be computed directly, so it is approximated element by element as

$$\mathbf{u}'|_K \approx \boldsymbol{\tau}_K (\mathcal{R}\mathbf{u}_h + \mathbf{v}'^\perp)|_K, \quad (4.8)$$

where $\boldsymbol{\tau}_K$ is a matrix that approximates \mathcal{L}^{-1} on each element K . The function \mathbf{v}'^\perp depends on the space of the sub-grid scales, so it can be written in a general form as

$$\mathbf{u}'|_K \approx \boldsymbol{\tau}_K P'(\mathcal{R}\mathbf{u}_h)|_K, \quad (4.9)$$

where P' is the L^2 projection onto the sub-grid scale space \mathcal{X}' and $P' = I - P'^\perp$, with I being the identity in \mathcal{X} . This approximation allows us to rewrite the modified version of the problem in (4.5) as a function of the FE variables, making the number of unknowns to be the same as the original problem, as follows

$$B(\mathbf{u}_h, \mathbf{v}_h) + \sum_K \langle \boldsymbol{\tau}_K P'(\mathcal{R}\mathbf{u}_h), \mathcal{L}^* \mathbf{v}_h \rangle_K = L(\mathbf{v}_h). \quad (4.10)$$

From this point onwards, it only remains to choose a proper projection of the residual. A typical choice of the sub-grid scale space is the identity $P' = I$, leading to the Algebraic Sub-Grid Scale (ASGS) formulation, which means $\mathbf{v}'^\perp = \mathbf{0}$. Choosing $P' = I - P_h := P_h^\perp$, where P_h is the L^2 projection onto the FE space (including boundary conditions) yields the Orthogonal Sub-Grid Scale (OSGS) formulation, because it corresponds to taking \mathcal{X}' as the orthogonal complement of \mathcal{X}_h .

4.2. Sub-grid scales in the element edges

The common procedure for the VMS formulation is to neglect the contribution of the sub-grid scales to the inter-element boundaries; however, enhanced stability properties can be achieved by considering them [23]. With the approach we follow, it is important to note that the sub-grid scales in the element interiors do not account for the boundary values, and the sub-grid scales in the inter-element boundaries have to be computed afterwards.

Let $\mathcal{E}_h = \{E\}$ the collection of interior edges of the FE partition, and \mathbf{u}'_E the sub-grid scale on edge E , assumed to be uniquely valued. The sub-grid scales on the edges of the boundary of Ω are taken as zero. Then, we may write equation (4.5) as:

$$B(\mathbf{u}_h, \mathbf{v}_h) + \sum_K \langle \mathbf{u}', \mathcal{L}^* \mathbf{v}_h \rangle_K + \sum_E \langle \mathbf{u}'_E, \llbracket \mathcal{F}^* \mathbf{v}_h \rrbracket \rangle_E = L(\mathbf{v}_h). \quad (4.11)$$

where $[[\mathcal{F}^*\mathbf{v}_h]]$ denotes the jump of $\mathcal{F}^*\mathbf{v}_h$ across E , i.e., the sum of the values of $\mathcal{F}^*\mathbf{v}_h$ computed with the normals exterior to the elements that share edge E .

Imposing that the total fluxes are continuous across inter-element boundaries, it is argued in [24] that the sub-grid scales on the edges of the FE partition can be approximated by

$$\mathbf{u}'_E \approx -\frac{\delta}{2}[[\mathcal{F}\mathbf{u}_h]],$$

where $\delta = \delta_0 h$ and δ_0 is a dimensionless algorithmic parameter. Combining this with (4.9) we obtain the stabilized FE formulation:

$$B(\mathbf{u}_h, \mathbf{v}_h) + \sum_K \langle \boldsymbol{\tau}_K P'(\mathcal{R}\mathbf{u}_h), \mathcal{L}^*\mathbf{v}_h \rangle_K - \sum_E \frac{\delta}{2} \langle [[\mathcal{F}\mathbf{u}_h]], [[\mathcal{F}^*\mathbf{v}_h]] \rangle_E = L(\mathbf{v}_h). \quad (4.12)$$

5. Implementation

In this subsection, the stabilized formulation of the Reissner-Mindlin plate and the Timoshenko beam theories are presented. To simplify the exposition, in this section we consider only linear elements, and thus higher order derivatives in the element interiors are not considered. Nevertheless, the case of arbitrary order of interpolation will be analyzed in the following section.

Let us first write the Galerkin FE form of each problem:

- Reissner-Mindlin plate

$$k_1 (\nabla \boldsymbol{\theta}_h, \nabla \boldsymbol{\phi}_h) + k_2 (\nabla \cdot \boldsymbol{\theta}_h, \nabla \cdot \boldsymbol{\phi}_h) + \frac{1}{\varepsilon} (\nabla w_h - \boldsymbol{\theta}_h, \nabla v_h - \boldsymbol{\phi}_h) = \langle \mathbf{m}, \boldsymbol{\phi}_h \rangle + \langle q, v_h \rangle. \quad (5.1)$$

- Timoshenko beam

$$EI \left(\frac{d\theta_h}{dx}, \frac{d\phi_h}{dx} \right) + GA^* \left(\frac{dw_h}{dx} - \theta_h, \frac{dv_h}{dx} - \phi_h \right) = \langle m, \phi_h \rangle + \langle q, v_h \rangle. \quad (5.2)$$

Consider α and β as the constants that multiply the shear and bending terms, respectively, namely $\alpha = GA^*$ and $\beta = EI$ for beams and $\alpha = \varepsilon^{-1}$, $\beta = Et^3/24$ for plates. In a general manner, the formulation yields a system of equations of the form

$$\begin{bmatrix} \alpha S_1 & \alpha S_2 \\ \alpha S_3 & \alpha S_4 + \beta B \end{bmatrix} \begin{bmatrix} W \\ \Theta \end{bmatrix} = \begin{bmatrix} F \\ M \end{bmatrix}, \quad (5.3)$$

where W and Θ are the displacements and rotations arrays, respectively, F is the array coming from the transverse loads, M from the bending moments, B and S_i ($i = 1, 2, 3, 4$) are the components arising from the bending and shear terms, respectively (with $S_3 = S_2^T$). These notations can refer to matrix arrays depending on the problem. The standard formulations gives a symmetric system that locks when $t \rightarrow 0$ and $S_i \gg B$ (the inequality being understood component-wise), while in the stabilized formulations presented below, the stabilization terms allow the system to avoid the dominance of one term over the other.

In the following, the stabilization matrices are taken to be diagonal, of the form $\boldsymbol{\tau}_K = \text{diag}(\tau_\theta, \tau_\theta, \tau_w)$ for plates and $\boldsymbol{\tau}_K = \text{diag}(\tau_\theta, \tau_w)$ for beams, where τ_θ and τ_w are still to be defined. It is understood that these parameters are evaluated element by element.

We will start writing the stabilized formulations without sub-grid scales on the inter-element boundaries. These will be introduced later, as they are the same for both the ASGS and the OSGS formulations.

5.1. Algebraic Sub-Grid Scales

We first consider the ASGS formulation, which corresponds to $\mathbf{v}'^\perp = \mathbf{0}$. Then P' is the identity on the element residuals, and thus $\mathbf{u}' = \boldsymbol{\tau}_K[\mathbf{f} - \mathcal{L}\mathbf{u}_h]$ in each element K . The formulation for linear elements and assuming constant

stabilization parameters (i.e., constant element sizes) reads:

- Reissner-Mindlin plates:

$$k_1 (\nabla \boldsymbol{\theta}_h, \nabla \boldsymbol{\phi}_h) + k_2 (\nabla \cdot \boldsymbol{\theta}_h, \nabla \cdot \boldsymbol{\phi}_h) + \frac{1}{\varepsilon} (\nabla w_h - \boldsymbol{\theta}_h, \nabla v_h - \boldsymbol{\phi}_h) - \frac{\tau_w}{\varepsilon^2} \sum_K \langle \nabla \cdot \boldsymbol{\theta}_h, \nabla \cdot \boldsymbol{\phi}_h \rangle_K - \frac{\tau_\theta}{\varepsilon^2} \sum_K \langle \nabla w_h - \boldsymbol{\theta}_h, \nabla v_h - \boldsymbol{\phi}_h \rangle_K = \langle \mathbf{m}, \boldsymbol{\phi}_h \rangle + \langle q, v_h \rangle. \quad (5.4)$$

- Timoshenko beams:

$$EI \left(\frac{d\theta_h}{dx}, \frac{d\phi_h}{dx} \right) + GA^* \left(\frac{dw_h}{dx} - \theta_h, \frac{dv_h}{dx} - \phi_h \right) - \tau_w (GA^*)^2 \sum_K \left\langle \frac{d\theta_h}{dx}, \frac{d\phi_h}{dx} \right\rangle_K - \tau_\theta (GA^*)^2 \sum_K \left\langle \frac{dw_h}{dx} - \theta_h, \frac{dv_h}{dx} - \phi_h \right\rangle_K = \langle m, \phi_h \rangle + \langle q, v_h \rangle. \quad (5.5)$$

In a general manner, the matrix version of these equations is

$$\begin{bmatrix} (\alpha - \tau_\theta \alpha^2) S_1 & (\alpha - \tau_\theta \alpha^2) S_2 \\ (\alpha - \tau_\theta \alpha^2) S_3 & (\alpha - \tau_\theta \alpha^2) S_4 + (\beta - \tau_w \alpha^2) B \end{bmatrix} \begin{bmatrix} W \\ \Theta \end{bmatrix} = \begin{bmatrix} F \\ M \end{bmatrix}. \quad (5.6)$$

This allows the system not to fall into shear dominance when τ_K is properly designed. This part is crucial to obtain a locking-free formulation, and it will be explained below in more detail.

5.2. Orthogonal Sub-Grid Scales

For this approach, the sub-grid scales are computed considering the orthogonal component of the residual as $\mathbf{u}' = \tau_K P^\perp [\mathbf{f} - \mathcal{L}\mathbf{u}_h]$ in each element K , i.e., $P' = P^\perp = I - P_h$. If we denote, in a general manner, $\boldsymbol{\xi}_h = [\boldsymbol{\xi}_\theta, \xi_w]$ to be the FE projections of the residual onto \mathcal{X}_h , and $\boldsymbol{\eta}_h = [\boldsymbol{\eta}_\theta, \eta_w]$ to be the respective test functions in \mathcal{X}_h , the formulations for plates and beams can be written as:

- Reissner-Mindlin plates:

$$k_1 (\nabla \boldsymbol{\theta}_h, \nabla \boldsymbol{\phi}_h) + k_2 (\nabla \cdot \boldsymbol{\theta}_h, \nabla \cdot \boldsymbol{\phi}_h) + \frac{1}{\varepsilon} (\nabla w_h - \boldsymbol{\theta}_h, \nabla v_h - \boldsymbol{\phi}_h) - \frac{\tau_w}{\varepsilon^2} \sum_K \langle \nabla \cdot \boldsymbol{\theta}_h, \nabla \cdot \boldsymbol{\phi}_h \rangle_K - \frac{\tau_\theta}{\varepsilon^2} \sum_K \langle \nabla w_h - \boldsymbol{\theta}_h, \nabla v_h - \boldsymbol{\phi}_h \rangle_K + \frac{\tau_w}{\varepsilon} \sum_K \langle \xi_w, \nabla \cdot \boldsymbol{\phi}_h \rangle_K + \frac{\tau_\theta}{\varepsilon} \sum_K \langle \xi_\theta, \nabla v_h - \boldsymbol{\phi}_h \rangle_K = \langle \mathbf{m}, \boldsymbol{\phi}_h \rangle + \langle q, v_h \rangle, \quad (5.7)$$

$$\frac{1}{\varepsilon} (\nabla \cdot \boldsymbol{\theta}_h, \boldsymbol{\eta}_w) - (\xi_w, \boldsymbol{\eta}_w) = 0, \quad (5.8)$$

$$\frac{1}{\varepsilon} (\nabla w_h - \boldsymbol{\theta}_h, \boldsymbol{\eta}_\theta) - (\xi_\theta, \boldsymbol{\eta}_\theta) = 0, \quad (5.9)$$

for all $[\boldsymbol{\phi}_h, v_h] \in \mathcal{X}_h$ and $[\boldsymbol{\eta}_\theta, \eta_w] \in \mathcal{X}_h$.

- Timoshenko beams:

$$EI \left(\frac{d\theta_h}{dx}, \frac{d\phi_h}{dx} \right) + GA^* \left(\frac{dw_h}{dx} - \theta_h, \frac{dv_h}{dx} - \phi_h \right) - \tau_w (GA^*)^2 \sum_K \left\langle \frac{d\theta_h}{dx}, \frac{d\phi_h}{dx} \right\rangle_K - \tau_\theta (GA^*)^2 \sum_K \left\langle \frac{dw_h}{dx} - \theta_h, \frac{dv_h}{dx} - \phi_h \right\rangle_K + \tau_w GA^* \sum_K \left\langle \xi_w, \frac{d\phi_h}{dx} \right\rangle_K + \tau_\theta GA^* \sum_K \left\langle \xi_\theta, \frac{dv_h}{dx} - \phi_h \right\rangle_K = \langle m, \phi_h \rangle + \langle q, v_h \rangle, \quad (5.10)$$

$$GA^* \left(\frac{dw_h}{dx} - \theta_h, \boldsymbol{\eta}_\theta \right) - (\xi_\theta, \boldsymbol{\eta}_\theta) = 0, \quad (5.11)$$

$$GA^* \left(\frac{d\theta_h}{dx}, \eta_w \right) - \left(\xi_w, \eta_w \right) = 0, \quad (5.12)$$

for all $[\phi_h, v_h] \in \mathcal{X}_h$ and $[\eta_\theta, \eta_w] \in \mathcal{X}_h$.

For this formulation, the matrix version of the equations is:

$$\begin{bmatrix} (\alpha - \tau_\theta \alpha^2)S_1 & (\alpha - \tau_\theta \alpha^2)S_2 & \tau_\theta \alpha P_1^\theta & 0 \\ (\alpha - \tau_\theta \alpha^2)S_3 & (\alpha - \tau_\theta \alpha^2)S_4 + (\beta - \tau_w \alpha^2)B & \tau_\theta \alpha P_2^\theta & \tau_w \alpha P^w \\ \alpha Q_1^\theta & \alpha Q_2^\theta & N & 0 \\ 0 & \alpha Q^w & 0 & N \end{bmatrix} \begin{bmatrix} W \\ \Theta \\ \Xi_\theta \\ \Xi_w \end{bmatrix} = \begin{bmatrix} F \\ M \\ 0 \\ 0 \end{bmatrix}, \quad (5.13)$$

where N is the Gram matrix, P^w and P_i^θ ($i = 1, 2$) are the matrices containing ξ_w and ξ_θ , respectively, with the corresponding test function associated to the index. In the same manner, Q^w and Q_i^θ ($i = 1, 2$) are the matrices containing deflection and rotations terms, depending on the index, with the test functions of the projection equations. Note that the system can be symmetrized by multiplying the last two rows by τ_θ and τ_w , respectively, since $(Q_i^\theta)^T = P_i^\theta$, $i = 1, 2$.

The OSGS implementation yields a system that shares some similarities with the mixed form of the problem. The difference is that the later yields a saddle point problem that is usually addressed using a mixed interpolation of the unknowns [8]. Note that similar formulations have been obtained in [2, 5, 9], where the shear force is computed using the L^2 projection of the rotation onto the FE space in order to soothe the zero shear strain constraint of the problem, which differs from the sub-grid scale approach of the present work.

5.3. Inter-element edge stabilization

The stabilization using the sub-grid scales in the element edges can be implemented independently of the sub-grid scales in the element interiors, as stated in section 4. For the implementation, consider the terms from the sub-grid scale in the inter-element edges added to the Galerkin form of the problem. The resulting discrete problem is:

- Reissner-Mindlin plate

$$\begin{aligned} & k_1 (\nabla \theta_h, \nabla \phi_h) + k_2 (\nabla \cdot \theta_h, \nabla \cdot \phi_h) + \frac{1}{\varepsilon} (\nabla w_h - \theta_h, \nabla v_h - \phi_h) \\ & - \frac{\delta k_1}{2} \sum_E \langle [\mathbf{n} \cdot \nabla \theta_h], [\mathbf{n} \cdot \nabla \phi_h] \rangle_E - \frac{\delta k_2}{2} \sum_E \langle [\mathbf{n} \nabla \cdot \theta_h], [\mathbf{n} \nabla \cdot \phi_h] \rangle_E \\ & - \frac{\delta}{2\varepsilon} \sum_E \langle [\mathbf{n} \cdot \nabla w_h], [\mathbf{n} \cdot \nabla v_h] \rangle_E = \langle \mathbf{m}, \phi_h \rangle + \langle q, v_h \rangle. \end{aligned} \quad (5.14)$$

- Timoshenko beam

$$\begin{aligned} & EI \left(\frac{d\theta_h}{dx}, \frac{d\phi_h}{dx} \right) + GA^* \left(\frac{dw_h}{dx} - \theta_h, \frac{dv_h}{dx} - \phi_h \right) \\ & - \frac{\delta EI}{2} \sum_E \left\langle \left[\frac{d\theta_h}{dx} \right], \left[\frac{d\phi_h}{dx} \right] \right\rangle_E - \frac{\delta GA^*}{2} \sum_E \left\langle \left[\frac{dw_h}{dx} \right], \left[\frac{dv_h}{dx} \right] \right\rangle_E = \langle m, \phi_h \rangle + \langle q, v_h \rangle. \end{aligned} \quad (5.15)$$

Recall that, in both problems, δ is a parameter of the order of the element size.

5.4. Stabilization parameters

The design of τ_K is based on the definition proposed in [25] for the Reissner-Mindlin case, with some modifications. According to that work, shear dominance can be dealt with by just introducing the shear stabilization parameter τ_θ , and it was successful in that regard. However, convergence ratios were not tested. Taking this into consideration, the stabilization parameters are defined as

- Reissner-Mindlin plate:

$$\boldsymbol{\tau}_K = \text{diag}(\tau_\theta, \tau_\theta, \tau_w), \quad \tau_\theta = \left(c_1 \frac{k}{h^2} + c_2 \varepsilon^{-1} \right)^{-1}, \quad \tau_w = \left(c_3 \frac{\varepsilon^{-1}}{h^2} + c_4 \frac{\varepsilon^{-2}}{k} \right)^{-1}, \quad (5.16)$$

- Timoshenko beam:

$$\boldsymbol{\tau}_K = \text{diag}(\tau_\theta, \tau_w), \quad \tau_\theta = \left(c_1 \frac{EI}{h^2} + c_2 GA^* \right)^{-1}, \quad \tau_w = \left(c_3 \frac{GA^*}{h^2} + c_4 \frac{(GA^*)^2}{EI} \right)^{-1}, \quad (5.17)$$

where h is the element size, $k = k_1 + k_2$, and c_i , $i = 1, 2$ are constants to be defined. Note that in [25] the constant c_2 must be taken as $c_2 = 1$ to eliminate shear dominance; this is confirmed in the stability analysis in section 6. In [22] the ASGS formulation is used together with the DSG approach taking $\tau_w = 0$ and a similar expression of τ_θ to the one we propose, but taking also into account the possible anisotropy of the elements.

To find which values of the stabilization constants can be used in the stabilization parameters, the matrix form of the equations using the ASGS formulation of (5.6) is compared to the exact solution of the elastic equations for the bending of an unloaded beam using Timoshenko's theory. Consider a two noded beam of length L , with nodal deflections w_i and rotations θ_i , for nodes $i = 1, 2$, and the corresponding nodal loads P_i and M_i . For $\mu = \frac{12EI}{GA^*L^2}$, the following system of equations is obtained:

$$\begin{bmatrix} \frac{12EI}{(1+\mu)L^3} & \frac{6EI}{(1+\mu)L^2} & -\frac{12EI}{(1+\mu)L^3} & \frac{6EI}{(1+\mu)L^2} \\ \frac{6EI}{(1+\mu)L^2} & \frac{(4+\mu)EI}{(1+\mu)L} & -\frac{12EI}{(1+\mu)L^2} & \frac{(2-\mu)EI}{(1+\mu)L} \\ -\frac{12EI}{(1+\mu)L^2} & -\frac{6EI}{(1+\mu)L^2} & \frac{12EI}{(1+\mu)L^2} & -\frac{6EI}{(1+\mu)L^2} \\ \frac{6EI}{(1+\mu)L^2} & \frac{(2-\mu)EI}{(1+\mu)L} & -\frac{12EI}{(1+\mu)L^2} & \frac{(4+\mu)EI}{(1+\mu)L} \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ M_1 \\ P_2 \\ M_2 \end{bmatrix}. \quad (5.18)$$

Comparing this system to the one obtained using the stabilized FE approximation we propose, we obtain that, in the ASGS formulation of beams, the stabilization constants can be taken as $c_1 = c_3 = 12$ and $c_2 = c_4 = 1$. In this case, the stiffness matrix of an element of length L is exactly the same as that of the elastic equations of a Timoshenko beam. The behavior of the FE formulation in response to this selection of constants as well as the constants used in the OSGS formulation will be discussed in section 7.

6. Numerical Analysis

In this section, the numerical analysis of the stabilized formulation for plates is analysed. The results are inherited by the beam problem since it is equivalent to the dimensional reduction of the plate problem.

Consider $\|\cdot\|$ to be the $L^2(\Omega)$ norm. Let us define some inequalities that will allow us to obtain a stability estimate. For simplicity, we will assume the FE partition to be quasi-uniform, of size h . We may thus assume that there is a constant C_{inv} , independent of the mesh size h , such that the following inverse estimate holds:

$$\|\nabla v_h\|_K \leq \frac{C_{\text{inv}}}{h} \|v_h\|_K, \quad (6.1)$$

for all FE functions v_h defined on the partition $\{K\}$. Similarly, the following trace inequality holds: there exists a constant C_{trace} independent of h such that

$$\|v\|_{\partial K}^2 \leq C_{\text{trace}} \left(\frac{1}{h} \|v\|_K^2 + h \|\nabla v\|_K^2 \right), \quad (6.2)$$

for functions $v \in H^1(K)$. In this expression, the last term is dropped if v is a polynomial on the element domain K .

Let now $\mathcal{E}_h = \{E\}$ be the edges of the FE partition. For piecewise discontinuous polynomials φ_h and continuous

polynomials ψ_h there holds:

$$\sum_E \|\llbracket \mathbf{n}\varphi_h \rrbracket\|_E^2 \leq 2 \frac{C_{\text{trace}}}{h} \sum_K \|\varphi_h\|_K^2, \quad \sum_E \|\psi_h\|_E^2 \leq \frac{C_{\text{trace}}}{2h} \sum_K \|\psi_h\|_K^2. \quad (6.3)$$

In the following, C will denote a generic positive constant, not necessarily the same at different occurrences. The stabilized FE formulation for elements of arbitrary polynomial degree can be written as:

$$B_{\text{stab}}(\mathbf{u}_h, \mathbf{v}_h) = L_{\text{stab}}(\mathbf{v}_h), \quad (6.4)$$

where

$$\begin{aligned} B_{\text{stab}}(\mathbf{u}_h, \mathbf{v}_h) &= k_1 (\nabla \boldsymbol{\theta}_h, \nabla \boldsymbol{\phi}_h) + k_2 (\nabla \cdot \boldsymbol{\theta}_h, \nabla \cdot \boldsymbol{\phi}_h) + \frac{1}{\varepsilon} (\nabla w_h - \boldsymbol{\theta}_h, \nabla v_h - \boldsymbol{\phi}_h) \\ &+ \tau_\theta \sum_K \left\langle P' \left[k_1 \Delta \boldsymbol{\theta}_h + k_2 \nabla (\nabla \cdot \boldsymbol{\theta}_h) + \frac{1}{\varepsilon} (\nabla w_h - \boldsymbol{\theta}_h) \right], -k_1 \Delta \boldsymbol{\phi}_h - k_2 \nabla (\nabla \cdot \boldsymbol{\phi}_h) - \frac{1}{\varepsilon} (\nabla v_h - \boldsymbol{\phi}_h) \right\rangle_K \\ &+ \tau_w \sum_K \left\langle P' \left[\frac{1}{\varepsilon} \nabla \cdot (\nabla w_h - \boldsymbol{\theta}_h) \right], -\frac{1}{\varepsilon} \nabla \cdot (\nabla v_h - \boldsymbol{\phi}_h) \right\rangle_K, \end{aligned} \quad (6.5)$$

and

$$\begin{aligned} L_{\text{stab}}(\mathbf{v}_h) &= \langle \mathbf{m}, \boldsymbol{\phi}_h \rangle + \langle q, v_h \rangle - \tau_\theta \sum_K \left\langle P' [\mathbf{m}], -k_1 \Delta \boldsymbol{\phi}_h - k_2 \nabla (\nabla \cdot \boldsymbol{\phi}_h) - \frac{1}{\varepsilon} (\nabla v_h - \boldsymbol{\phi}_h) \right\rangle_K \\ &- \tau_w \sum_K \left\langle P' [q], -\frac{1}{\varepsilon} \nabla \cdot (\nabla v_h - \boldsymbol{\phi}_h) \right\rangle_K, \end{aligned} \quad (6.6)$$

where $P' = I$ for the ASGS formulation and $P' = P^\perp$ for the OSGS method.

6.1. Stability analysis of the Algebraic Sub-Grid Scale formulation

Let us first recall the stability estimate presented in [25] for the plate problem, where the ASGS approach was considered. In that work the terms multiplied by τ_w were neglected because they were not needed to get rid of the numerical locking. This fact however, does not account for the convergence rate of the solution, and τ_w is indeed necessary for it to be optimal, as will be explained and numerically proven below. The formulation analyzed in [25] will now be extended in the same manner: for the stability estimate, take $\mathbf{v}_h = \mathbf{u}_h$ and consider $\boldsymbol{\gamma}_h = \nabla w_h - \boldsymbol{\theta}_h$. Using Schwarz's inequality leads to:

$$\begin{aligned} B_{\text{stab}}(\mathbf{u}_h, \mathbf{u}_h) &\geq \sum_K \left[k_1 \|\nabla \boldsymbol{\theta}_h\|_K^2 + k_2 \|\nabla \cdot \boldsymbol{\theta}_h\|_K^2 + \frac{1}{\varepsilon} \|\boldsymbol{\gamma}_h\|_K^2 - \tau_\theta k_1^2 \|\Delta \boldsymbol{\theta}_h\|_K^2 - \tau_\theta k_2^2 \|\nabla (\nabla \cdot \boldsymbol{\theta}_h)\|_K^2 - \tau_\theta \frac{1}{\varepsilon^2} \|\boldsymbol{\gamma}_h\|_K^2 \right. \\ &\left. - 2\tau_\theta \frac{k_1}{\varepsilon} \|\Delta \boldsymbol{\theta}_h\|_K \|\boldsymbol{\gamma}_h\|_K - 2\tau_\theta \frac{k_2}{\varepsilon} \|\nabla (\nabla \cdot \boldsymbol{\theta}_h)\|_K \|\boldsymbol{\gamma}_h\|_K - 2\tau_\theta k_1 k_2 \|\Delta \boldsymbol{\theta}_h\|_K \|\nabla (\nabla \cdot \boldsymbol{\theta}_h)\|_K - \tau_w \frac{1}{\varepsilon^2} \|\nabla \cdot \boldsymbol{\gamma}_h\|_K^2 \right]. \end{aligned} \quad (6.7)$$

Using the inverse estimate (6.1) and Young's inequality, we obtain:

$$\begin{aligned} B_{\text{stab}}(\mathbf{u}_h, \mathbf{u}_h) &\geq \sum_K \left[k_1 \|\nabla \boldsymbol{\theta}_h\|_K^2 + k_2 \|\nabla \cdot \boldsymbol{\theta}_h\|_K^2 + \frac{1}{\varepsilon} \|\boldsymbol{\gamma}_h\|_K^2 - \tau_\theta k_1^2 \frac{C_{\text{inv}}^2}{h^2} \|\nabla \boldsymbol{\theta}_h\|_K^2 - \tau_\theta k_2^2 \frac{C_{\text{inv}}^2}{h^2} \|\nabla \cdot \boldsymbol{\theta}_h\|_K^2 - \tau_\theta \frac{1}{\varepsilon^2} \|\boldsymbol{\gamma}_h\|_K^2 \right. \\ &- \tau_\theta \frac{k_1}{\varepsilon} \left(\|\nabla \boldsymbol{\theta}_h\|_K^2 + \frac{C_{\text{inv}}^2}{h^2} \|\boldsymbol{\gamma}_h\|_K^2 \right) - \tau_\theta \frac{k_2}{\varepsilon} \left(\|\nabla \cdot \boldsymbol{\theta}_h\|_K^2 + \frac{C_{\text{inv}}^2}{h^2} \|\boldsymbol{\gamma}_h\|_K^2 \right) \\ &- \tau_\theta \left(k_1^2 \frac{C_{\text{inv}}^2}{h^2} \|\nabla \boldsymbol{\theta}_h\|_K^2 + k_2^2 \frac{C_{\text{inv}}^2}{h^2} \|\nabla \cdot \boldsymbol{\theta}_h\|_K^2 \right) - \tau_w \frac{1}{\varepsilon^2} \frac{C_{\text{inv}}^2}{h^2} \|\boldsymbol{\gamma}_h\|_K^2 \left. \right] \\ &= \sum_K \left[\beta_1 \|\nabla \boldsymbol{\theta}_h\|_K^2 + \beta_2 \|\nabla \cdot \boldsymbol{\theta}_h\|_K^2 + \beta_3 \|\boldsymbol{\gamma}_h\|_K^2 \right], \end{aligned} \quad (6.8)$$

where

$$\beta_1 = \tau_\theta \left[c_1 \frac{k_1^2}{h^2} + c_1 \frac{k_1 k_2}{h^2} + c_2 \frac{k_1}{\varepsilon} - 2 \frac{C_{\text{inv}}^2}{h^2} k_1^2 - \frac{k_1}{\varepsilon} \right], \quad (6.9)$$

$$\beta_2 = \tau_\theta \left[c_1 \frac{k_1 k_2}{h^2} + c_1 \frac{k_2^2}{h^2} + c_2 \frac{k_2}{\varepsilon} - 2 \frac{C_{\text{inv}}^2}{h^2} k_2^2 - \frac{k_2}{\varepsilon} \right], \quad (6.10)$$

$$\beta_3 = \tau_\theta \left[c_1 \frac{k}{h^2 \varepsilon} + c_2 \frac{1}{\varepsilon^2} - \frac{1}{\varepsilon^2} - \frac{k}{\varepsilon} \frac{C_{\text{inv}}^2}{h^2} - \frac{k}{\varepsilon} \frac{C_{\text{inv}}^2}{h^2} \left(\frac{c_1 k \varepsilon + c_2 h^2}{c_3 k \varepsilon + c_4 h^2} \right) \right]. \quad (6.11)$$

Except for the values of the constants $c_1 = c_3 = 12$, $c_2 = c_4 = 1$ that make the stiffness matrix of an element equal to the matrix coming from the elastic equations, estimate (6.8) is not satisfactory. To understand the reason, suppose that the physical properties and the mesh size h are fixed and let us analyze the scaling of the parameters β_i , $i = 1, 2, 3$, with respect to the thickness t . Assuming $c_3 \geq c_1$, $c_4 \geq c_2$ and $c_1 > 2C_{\text{inv}}^2$ and noting that k_1, k_2 scale as t^3 and ε scales as t^{-1} , we have that

$$\beta_1, \beta_2 \sim \tau_\theta t^4 \left[A_1 \frac{t^2}{h^2} + A_2 (c_2 - 1) \right], \quad \beta_3 \sim \tau_\theta t^4 \left[B_1 \frac{1}{h^2} + \frac{B_2}{t^2} (c_2 - 1) \right], \quad (6.12)$$

where \sim stands for scaling and A_1, A_2, B_1, B_2 are independent of t and h . From this we observe that in order to avoid shear locking we need to take $c_2 = 1$, as it was observed in [25], but in this case also the last term in β_1, β_2 vanishes, and we have that $\beta_1, \beta_2 = \mathcal{O}(\tau_\theta t^6)$ while $\beta_3 = \mathcal{O}(\tau_\theta t^4)$ (for h fixed). Therefore, the stability estimate (6.8) does not provide a balanced control of the shear term and the derivatives of the rotation when $t \rightarrow 0$. We will see in the next subsection that the OSGS formulation does not suffer from this misbehavior.

6.2. Stability and convergence analysis of the Orthogonal Sub-Grid Scale formulation

The numerical analysis of the method considering the OSGS approach is presented next. For positive and dimensionally correct α_i ($i = 1, 2, 3$), the norm in which the results are presented is:

$$\|\mathbf{v}_h\|^2 := \alpha_1 \|\nabla \phi_h\|^2 + \alpha_2 \|\nabla \cdot \phi_h\|^2 + \alpha_3 \|\gamma_h\|^2. \quad (6.13)$$

This norm has the same form as for the Galerkin method and the ASGS formulation (see estimate (6.8)), but with the advantage that the constants α_i are such that locking is no longer possible, because they are designed in a manner that none of them can become dominant over the others. In particular, it will be shown that they behave as $\alpha_1, \alpha_2 = \mathcal{O}(\tau_\theta t^4)$ and $\alpha_3 = \mathcal{O}(\tau_\theta t^4 h^{-2})$.

In the following, it will be proved that the formulation is stable under the norm (6.13), in the form of the inf-sup condition.

Theorem 6.1 (Stability). *There is a constant $C > 0$ such that*

$$\inf_{\mathbf{u}_h \in \mathcal{X}_h} \sup_{\mathbf{v}_h \in \mathcal{X}_h} \frac{B_{\text{stab}}(\mathbf{u}_h, \mathbf{v}_h)}{\|\mathbf{u}_h\| \|\mathbf{v}_h\|} \geq C. \quad (6.14)$$

Proof. Let us start noting that for any function $\mathbf{u}_h \in \mathcal{X}_h$ we have

$$\begin{aligned} B_{\text{stab}}(\mathbf{u}_h, \mathbf{u}_h) &= k_1 \|\nabla \boldsymbol{\theta}_h\|^2 + k_2 \|\nabla \cdot \boldsymbol{\theta}_h\|^2 + \varepsilon^{-1} \|\gamma_h\|^2 \\ &\quad - \tau_\theta \sum_K \|P^\perp(k_1 \Delta \boldsymbol{\theta}_h + k_2 \nabla(\nabla \cdot \boldsymbol{\theta}_h) + \varepsilon^{-1} \gamma_h)\|_K^2 - \tau_w \sum_K \|P^\perp(\varepsilon^{-1} \nabla \cdot \gamma_h)\|_K^2. \end{aligned} \quad (6.15)$$

It is important to note that the Galerkin terms of the bilinear form B have already the necessary terms to have control over the $\nabla \boldsymbol{\theta}_h$ and γ_h . However, the problem arises when $t \rightarrow 0$ and becomes shear dominant. Because of this, the main idea is to obtain a stability estimate in which the shear dominance can be prevented. This estimate comes from the terms whose orthogonal projections appear in B_{stab} , which is obtained by bounding the bilinear form term by term as follows.

Let us consider $\mathbf{v}_{h1} := (\tau_\theta \phi_1, 0)$, where $\phi_1 = P_h(k_1 \Delta \boldsymbol{\theta}_h + k_2 \nabla(\nabla \cdot \boldsymbol{\theta}_h) + \varepsilon^{-1} \boldsymbol{\gamma}_h)$ and P_h is the L^2 projection onto the FE space. It is understood that the term inside the projection is evaluated element-wise. Taking \mathbf{v}_{h1} as test function in the bilinear form, and integrating by parts the Galerkin terms, yields

$$\begin{aligned} B_{\text{stab}}(\mathbf{u}_h, \mathbf{v}_{h1}) &= - \sum_K \tau_\theta \langle k_1 \Delta \boldsymbol{\theta}_h + k_2 \nabla(\nabla \cdot \boldsymbol{\theta}_h) + \varepsilon^{-1} \boldsymbol{\gamma}_h, \phi_1 \rangle_K + \tau_\theta \sum_E \langle \llbracket k_1 \mathbf{n} \cdot \nabla \boldsymbol{\theta}_h \rrbracket + \llbracket k_2 \mathbf{n} \nabla \cdot \boldsymbol{\theta}_h \rrbracket, \phi_1 \rangle_E \\ &\quad - \tau_\theta^2 \sum_K \langle P^\perp(k_1 \Delta \boldsymbol{\theta}_h + k_2 \nabla(\nabla \cdot \boldsymbol{\theta}_h) + \varepsilon^{-1} \boldsymbol{\gamma}_h), k_1 \Delta \phi_1 + k_2 \nabla(\nabla \cdot \phi_1) - \varepsilon^{-1} \phi_1 \rangle_K \\ &\quad - \tau_\theta \tau_w \sum_K \langle P^\perp(\varepsilon^{-1} \nabla \cdot \boldsymbol{\gamma}_h), -\varepsilon^{-1} \nabla \cdot \phi_1 \rangle_K. \end{aligned} \quad (6.16)$$

Note that the equation contains terms projected in the space that is orthogonal to the FE space, which disappear when tested with ϕ_1 because it belongs to the FE space itself. Then, by using Schwarz's inequality and the inverse estimate (6.1), we obtain:

$$\begin{aligned} B_{\text{stab}}(\mathbf{u}_h, \mathbf{v}_{h1}) &\geq - \tau_\theta \sum_K \|\phi_1\|_K^2 - \tau_\theta \sum_E \|\llbracket k_1 \mathbf{n} \cdot \nabla \boldsymbol{\theta}_h \rrbracket\|_E \|\phi_1\|_E - \tau_\theta \sum_E \|\llbracket k_2 \mathbf{n} \nabla \cdot \boldsymbol{\theta}_h \rrbracket\|_E \|\phi_1\|_E \\ &\quad - \tau_\theta^2 k \frac{C_{\text{inv}}^2}{h^2} \sum_K \|P^\perp(k_1 \Delta \boldsymbol{\theta}_h + k_2 \nabla(\nabla \cdot \boldsymbol{\theta}_h) + \varepsilon^{-1} \boldsymbol{\gamma}_h)\|_K \|\phi_1\|_K \\ &\quad - \tau_\theta \tau_w \varepsilon^{-1} \frac{C_{\text{inv}}}{h} \sum_K \|P^\perp(\varepsilon^{-1} \nabla \cdot \boldsymbol{\gamma}_h)\|_K \|\phi_1\|_K. \end{aligned} \quad (6.17)$$

Then, using Young's inequality and the trace inequalities (6.3) it follows that:

$$\begin{aligned} B_{\text{stab}}(\mathbf{u}_h, \mathbf{v}_{h1}) &\geq - \tau_\theta \sum_K \|\phi_1\|_K^2 - \tau_\theta \sum_K \|\phi_1\|_K^2 - \tau_\theta \frac{C_{\text{trace}}^2}{2h^2} \sum_K \|k_1 \nabla \boldsymbol{\theta}_h\|_K^2 - \tau_\theta \frac{C_{\text{trace}}^2}{2h^2} \sum_K \|k_2 \nabla \cdot \boldsymbol{\theta}_h\|_K^2 \\ &\quad - \frac{\tau_\theta^2 k}{2} \frac{C_{\text{inv}}^2}{h^2} \sum_K \left(\|\phi_1\|_K^2 + \|P^\perp(k_1 \Delta \boldsymbol{\theta}_h + k_2 \nabla(\nabla \cdot \boldsymbol{\theta}_h) + \varepsilon^{-1} \boldsymbol{\gamma}_h)\|_K^2 \right) \\ &\quad - \tau_\theta \tau_w \frac{\varepsilon^{-1} C_{\text{inv}}^2}{2} \sum_K \|\phi_1\|_K^2 - \tau_\theta \tau_w \frac{\varepsilon^{-1}}{2} \sum_K \|P^\perp(\varepsilon^{-1} \nabla \cdot \boldsymbol{\gamma}_h)\|_K^2. \end{aligned} \quad (6.18)$$

Similarly, consider $\mathbf{v}_{h2} = (\mathbf{0}, \tau_w v_2)$, where $v_2 = P_h(\varepsilon^{-1} \nabla \cdot \boldsymbol{\gamma}_h)$, as test function in the bilinear form. We have that

$$\begin{aligned} B_{\text{stab}}(\mathbf{u}_h, \mathbf{v}_{h2}) &= - \tau_w (\varepsilon^{-1} \nabla \cdot \boldsymbol{\gamma}_h, v_2) + \tau_w \sum_E \langle \varepsilon^{-1} \llbracket \mathbf{n} \cdot \boldsymbol{\gamma}_h \rrbracket, v_2 \rangle_E \\ &\quad - \tau_w^2 \varepsilon^{-1} \sum_K \langle P^\perp(\varepsilon^{-1} \nabla \cdot \boldsymbol{\gamma}_h), \nabla \cdot \nabla v_2 \rangle_K \\ &\quad - \tau_\theta \tau_w \varepsilon^{-1} \sum_K \langle P^\perp(k_1 \Delta \boldsymbol{\theta}_h + k_2 \nabla(\nabla \cdot \boldsymbol{\theta}_h) + \varepsilon^{-1} \boldsymbol{\gamma}_h), \nabla v_2 \rangle_K. \end{aligned} \quad (6.19)$$

Then following the same procedure as before, it follows that

$$\begin{aligned} B_{\text{stab}}(\mathbf{u}_h, \mathbf{v}_{h2}) &\geq - \tau_w \sum_K \|v_2\|_K^2 - \tau_w \sum_K \|v_2\|_K^2 - \tau_w \frac{C_{\text{trace}}^2}{4h^2} \sum_K \|\varepsilon^{-1} \boldsymbol{\gamma}_h\|_K^2 \\ &\quad - \tau_w^2 \frac{\varepsilon^{-1} C_{\text{inv}}^2}{2} \sum_K \left(\|v_2\|_K^2 + \|P^\perp(\varepsilon^{-1} \nabla \cdot \boldsymbol{\gamma}_h)\|_K^2 \right) \\ &\quad - \tau_\theta \tau_w \frac{\varepsilon^{-1}}{2} \sum_K \|v_2\|_K^2 - \tau_\theta \tau_w \frac{\varepsilon^{-1} C_{\text{inv}}^2}{2} \sum_K \|P^\perp(k_1 \Delta \boldsymbol{\theta}_h + k_2 \nabla(\nabla \cdot \boldsymbol{\theta}_h) + \varepsilon^{-1} \boldsymbol{\gamma}_h)\|_K^2. \end{aligned} \quad (6.20)$$

Lastly, consider $\mathbf{v}_h = \mathbf{u}_h + \frac{1}{2}\mathbf{v}_{h1} + \frac{1}{2}\mathbf{v}_{h2}$, which is equivalent to adding up (6.15), $\frac{1}{2}$ (6.18), and $\frac{1}{2}$ (6.20). This yields

$$\begin{aligned} B_{\text{stab}}(\mathbf{u}_h, \mathbf{v}_h) &\geq \left(k_1 - \tau_\theta k_1^2 \frac{C_{\text{trace}}^2}{4h^2}\right) \|\nabla \boldsymbol{\theta}_h\|^2 + \left(k_2 - \tau_\theta k_2^2 \frac{C_{\text{trace}}^2}{4h^2}\right) \|\nabla \cdot \boldsymbol{\theta}_h\|^2 + \left(\varepsilon^{-1} - \tau_w \varepsilon^{-2} \frac{C_{\text{trace}}^2}{8h^2}\right) \|\boldsymbol{\gamma}_h\|^2 \\ &\quad - \left(\tau_\theta + \frac{\tau_\theta^2 k}{2} \frac{C_{\text{inv}}^2}{h^2} + \tau_\theta \tau_w \frac{\varepsilon^{-1}}{2} \frac{C_{\text{inv}}^2}{h^2}\right) \sum_K \|k_1 \Delta \boldsymbol{\theta}_h + k_2 \nabla(\nabla \cdot \boldsymbol{\theta}_h) + \varepsilon^{-1} \boldsymbol{\gamma}_h\|_K^2 \\ &\quad - \left(\tau_w + \tau_\theta \tau_w \frac{\varepsilon^{-1}}{2} + \tau_w^2 \frac{\varepsilon^{-1}}{2} \frac{C_{\text{inv}}^2}{h^2}\right) \sum_K \|\varepsilon^{-1} \nabla \cdot \boldsymbol{\gamma}_h\|_K^2. \end{aligned} \quad (6.21)$$

The last two terms can be separated using the triangular inequality, enabling us to write the expression in terms of the original variables only:

$$B_{\text{stab}}(\mathbf{u}_h, \mathbf{v}_h) \geq \alpha_1 \|\nabla \boldsymbol{\theta}_h\|^2 + \alpha_2 \|\nabla \cdot \boldsymbol{\theta}_h\|^2 + \alpha_3 \|\boldsymbol{\gamma}_h\|^2 \equiv \|\mathbf{u}_h\|^2, \quad (6.22)$$

where

$$\alpha_1 = \tau_\theta \left[c_1 \frac{kk_1}{h^2} + c_2 \frac{k_1}{\varepsilon} - k_1^2 \frac{C_{\text{inv}}^2}{4h^2} \left(\frac{C_{\text{inv}} k \varepsilon}{c_1 k \varepsilon + c_2 h^2} \right) - k_1^2 \frac{C_{\text{inv}}^2}{4h^2} \left(\frac{C_{\text{inv}}^2 k \varepsilon}{c_3 k \varepsilon + c_4 h^2} \right) - k_1^2 \frac{C_{\text{trace}}^2}{4h^2} \right], \quad (6.23)$$

$$\alpha_2 = \tau_\theta \left[c_1 \frac{kk_2}{h^2} + c_2 \frac{k_2}{\varepsilon} - k_2^2 \frac{C_{\text{inv}}^2}{4h^2} \left(\frac{C_{\text{inv}} k \varepsilon}{c_1 k \varepsilon + c_2 h^2} \right) - k_2^2 \frac{C_{\text{inv}}^2}{4h^2} \left(\frac{C_{\text{inv}}^2 k \varepsilon}{c_3 k \varepsilon + c_4 h^2} \right) - k_2^2 \frac{C_{\text{trace}}^2}{4h^2} \right], \quad (6.24)$$

$$\begin{aligned} \alpha_3 = \tau_\theta &\left[c_1 \frac{k \varepsilon^{-1}}{h^2} + \frac{c_2}{\varepsilon^2} - \frac{1}{\varepsilon^2} - \frac{k}{\varepsilon} \frac{C_{\text{inv}}^2}{4h^2} \left(\frac{h^2}{c_1 k \varepsilon + c_2 h^2} \right) - \frac{k}{\varepsilon} \frac{C_{\text{inv}}^2}{4h^2} \left(\frac{h^2}{c_3 k \varepsilon + c_4 h^2} \right) - \frac{k}{\varepsilon} \frac{C_{\text{trace}}^2}{8h^2} \left(\frac{c_1 k \varepsilon + c_2 h^2}{c_3 k \varepsilon + c_4 h^2} \right) \right. \\ &\quad \left. - \frac{k}{\varepsilon} \frac{C_{\text{inv}}^2}{h^2} \left(\frac{c_1 k \varepsilon + c_2 h^2}{c_3 k \varepsilon + c_4 h^2} \right) - \frac{k}{\varepsilon} \frac{C_{\text{inv}}^2}{2h^2} \left(\frac{c_1 k \varepsilon + c_2 h^2}{c_3 k \varepsilon + c_4 h^2} \right) \left(\frac{C_{\text{inv}}^2 k \varepsilon^{-1}}{c_3 k \varepsilon^{-1} + c_4 \varepsilon^{-2} h^2} \right) \right]. \end{aligned} \quad (6.25)$$

From the expression in (6.25) it can be verified again that the value $c_2 = 1$ is necessary to eliminate shear dominance and that the constants must satisfy $c_3 \geq c_1$ and $c_4 \geq c_2$. It is also needed that $c_1 > 2C_{\text{inv}}^2 + \frac{1}{4}C_{\text{trace}}^2$. In fact, for linear elements it suffices to take $c_1 > 0$, as the integration by parts in (6.16) is not needed (the terms multiplied by C_{trace} do not show up) and one may take the constant associated to the inverse estimate for second derivatives as $C_{\text{inv}} = 0$. Under all these conditions, it is readily checked that

$$\alpha_1, \alpha_2 \sim \tau_\theta t^4 \left[A'_1 \frac{t^2}{h^2} + A'_2 c_2 \right], \quad \alpha_3 \sim \tau_\theta t^4 \left[B'_1 \frac{1}{h^2} + \frac{B'_2}{t^2} (c_2 - 1) \right], \quad (6.26)$$

where A'_1, A'_2, B'_1, B'_2 are independent of t and h . Comparing (6.26) with (6.12) it is observed that the last term in α_1, α_2 does not vanish when $c_2 = 1$, contrary to what happens for β_1, β_2 . Thus, when $t \rightarrow 0$ we have that $\alpha_1, \alpha_2 = \mathcal{O}(\tau_\theta t^4)$ and $\alpha_3 = \mathcal{O}(\tau_\theta t^4 h^{-2})$, as claimed earlier.

Therefore, we have proved that there is a positive constant C for which the following inequality holds:

$$B_{\text{stab}}(\mathbf{u}_h, \mathbf{v}_h) \geq C \|\mathbf{u}_h\|^2. \quad (6.27)$$

It is also easy to check that

$$\|\mathbf{v}_{h1}\|^2 \leq \tau_\theta^2 \frac{C_{\text{inv}}^4}{h^4} (\alpha_1 + \alpha_2) (k_1^2 \|\nabla \boldsymbol{\theta}_h\|_K^2 + k_2^2 \|\nabla \cdot \boldsymbol{\theta}_h\|_K^2) + \tau_w^2 \varepsilon^{-2} \frac{C_{\text{inv}}^2}{h^2} (\alpha_1 + \alpha_2) \|\boldsymbol{\gamma}_h\|_K^2 \leq C \|\mathbf{u}_h\|^2, \quad (6.28)$$

$$\|\mathbf{v}_{h2}\|^2 \leq \tau_w^2 \varepsilon^{-2} \alpha_3 \frac{C_{\text{inv}}^4}{h^4} \|\boldsymbol{\gamma}_h\|_K^2 \leq C \|\mathbf{u}_h\|^2, \quad (6.29)$$

and therefore $\|\mathbf{v}_h\|^2 \leq C \|\mathbf{u}_h\|^2$. From this result and (6.27) it follows that for each $\mathbf{u}_h \in \mathcal{X}_h$ there exists $\mathbf{v}_h \in \mathcal{X}_h$ such that $B_{\text{stab}}(\mathbf{u}_h, \mathbf{v}_h) \geq C \|\mathbf{u}_h\| \|\mathbf{v}_h\|$, from where the theorem follows. \square

Once the stability is established, a standard procedure follows to prove convergence. There are two preliminary lemmas that are needed to achieve it, concerning the consistency and the interpolation error.

Lemma 6.2 (Consistency). *Let $\mathbf{u} \in \mathcal{X}$ be the solution of the continuous problem and $\mathbf{u}_h \in \mathcal{X}_h$ the FE solution of (6.5). If \mathbf{u} is regular enough so that $B_{\text{stab}}(\mathbf{u}, \mathbf{v}_h)$ is well defined, then*

$$B_{\text{stab}}(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathcal{X}_h. \quad (6.30)$$

Proof. Since the stabilization terms are residual based and the Galerkin method does not contribute to the consistency error, the lemma is satisfied by construction. \square

The following lemma concerns an interpolation error in terms of the working norm $\|\cdot\|$ and the bilinear form B_{stab} . Let \mathcal{W}_h be a generic FE space of degree k_v . The interpolation error $\epsilon_i(v)$ for any function $v \in H^{k_v+1}(\Omega)$ for $i = 0, 1$ is defined as follows

$$\inf_{v_h \in \mathcal{W}_h} \sum_K \|v - v_h\|_{H^i(K)} \leq Ch^{k_v''+1-i} \sum_K \|v\|_{H^{k_v''+1}(K)} =: \epsilon_i(v), \quad (6.31)$$

where $k_v'' = \min(k_v, k_v')$. Also consider \tilde{v}_h to be the best approximation of v in \mathcal{W}_h . Note that $\epsilon_0(v) = h\epsilon_1(v)$. In particular, the notation will be $v = \boldsymbol{\theta}$ for the rotations and $v = w$ for the deflection, with orders of interpolation k_θ and k_w , respectively.

The error function of the method will be proven to be:

$$E(h) := \left(\sqrt{k_1} + \sqrt{k_2} \right) \epsilon_1(\boldsymbol{\theta}) + \frac{1}{\sqrt{\varepsilon}} \epsilon_0(\boldsymbol{\theta}) + \frac{1}{\sqrt{\varepsilon}} \epsilon_1(w). \quad (6.32)$$

Lemma 6.3 (Interpolation error). *Let $\mathbf{u} \in \mathcal{X}$ be the continuous solution, assumed to be regular enough, and $\tilde{\mathbf{u}}_h \in \mathcal{X}_h$ its best FE approximation. Then the following inequalities hold:*

$$B_{\text{stab}}(\mathbf{u} - \tilde{\mathbf{u}}_h, \mathbf{v}_h) \leq CE(h) \|\mathbf{v}_h\|, \quad (6.33)$$

$$\|\mathbf{u} - \tilde{\mathbf{u}}_h\| \leq CE(h). \quad (6.34)$$

Proof. Let us prove (6.34). Consider the definition of the working norm (6.13); it can be easily checked that

$$\|\mathbf{u} - \tilde{\mathbf{u}}_h\|^2 \leq C \left[\alpha_1 \epsilon_1^2(\boldsymbol{\theta}) + \alpha_2 \epsilon_1^2(\boldsymbol{\theta}) + \alpha_3 \epsilon_1^2(w) + \alpha_3 \epsilon_0^2(\boldsymbol{\theta}) \right], \quad (6.35)$$

from where (6.34) follows using (5.17) for the expression of the stabilization parameters and (6.23)-(6.25) for the expression of α_i , $i = 1, 2, 3$.

Then consider $\mathbf{e}_u = \mathbf{u} - \tilde{\mathbf{u}}_h = [\mathbf{e}_\theta, e_w]$, where $\mathbf{e}_\theta = \boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}_h$ and $e_w = w - \tilde{w}_h$; the proof of (6.33) is as follows:

$$\begin{aligned} B_{\text{stab}}(\mathbf{e}_u, \mathbf{v}_h) &= k_1 (\nabla \mathbf{e}_\theta, \nabla \boldsymbol{\phi}_h) + k_2 (\nabla \cdot \mathbf{e}_\theta, \nabla \cdot \boldsymbol{\phi}_h) + \frac{1}{\varepsilon} (\nabla e_w - \mathbf{e}_\theta, \nabla v_h - \boldsymbol{\phi}_h) \\ &\quad + \tau_\theta \sum_K \left\langle P_\theta^\perp \left[k_1 \Delta \mathbf{e}_\theta + k_2 \nabla (\nabla \cdot \mathbf{e}_\theta) + \frac{1}{\varepsilon} (\nabla e_w - \mathbf{e}_\theta) \right], P_\theta^\perp \left[-k_1 \Delta \boldsymbol{\phi}_h - k_2 \nabla (\nabla \cdot \boldsymbol{\phi}_h) - \frac{1}{\varepsilon} \boldsymbol{\gamma}_h \right] \right\rangle_K \\ &\quad + \tau_w \sum_K \left\langle P_w^\perp \left[\frac{1}{\varepsilon} \nabla \cdot (\nabla e_w - \mathbf{e}_\theta) \right], P_w^\perp \left[-\frac{1}{\varepsilon} \nabla \cdot \boldsymbol{\gamma}_h \right] \right\rangle_K \\ &\leq \sqrt{k_1} \|\nabla \mathbf{e}_\theta\| \sqrt{k_1} \|\nabla \boldsymbol{\phi}_h\| + \sqrt{k_2} \|\nabla \cdot \mathbf{e}_\theta\| \sqrt{k_2} \|\nabla \cdot \boldsymbol{\phi}_h\| + \frac{1}{\sqrt{\varepsilon}} \|\nabla e_w - \mathbf{e}_\theta\| \frac{1}{\sqrt{\varepsilon}} \|\boldsymbol{\gamma}_h\| \\ &\quad + \left(\tau_\theta \frac{k_1}{h} \|\nabla \mathbf{e}_\theta\| + \tau_\theta \frac{k_2}{h} \|\nabla \cdot \mathbf{e}_\theta\| + \frac{\tau_\theta}{\varepsilon} (\|\nabla e_w\| + \|\mathbf{e}_\theta\|) \right) \left(k_1 \frac{C_{\text{inv}}}{h} \|\nabla \boldsymbol{\phi}_h\| \right. \\ &\quad \left. + k_2 \frac{C_{\text{inv}}}{h} \|\nabla \cdot \boldsymbol{\phi}_h\| + \frac{1}{\varepsilon} \|\boldsymbol{\gamma}_h\| \right) + \left(\tau_w \frac{1}{\varepsilon} \frac{1}{h} \|\nabla e_w\| + \tau_w \frac{1}{\varepsilon} \|\nabla \cdot \mathbf{e}_\theta\| \right) \frac{C_{\text{inv}}}{\varepsilon h} \|\boldsymbol{\gamma}_h\| \\ &\leq \left[\left(k_1 + \tau_\theta k_1^2 \frac{C_{\text{inv}}}{h^2} \right) \|\nabla \mathbf{e}_\theta\| + \tau_\theta k_1 k_2 \frac{C_{\text{inv}}}{h^2} \|\nabla \cdot \mathbf{e}_\theta\| + \tau_\theta \frac{k_1}{\varepsilon} \frac{C_{\text{inv}}}{h} \|\nabla e_w\| \right. \\ &\quad \left. + \tau_\theta \frac{k_1}{\varepsilon} \frac{C_{\text{inv}}}{h^2} \|\mathbf{e}_\theta\| \right] \|\nabla \boldsymbol{\phi}_h\| + \left[\tau_\theta k_1 k_2 \frac{C_{\text{inv}}}{h^2} \|\nabla \mathbf{e}_\theta\| + \left(k_2 + \tau_\theta k_2^2 \frac{C_{\text{inv}}}{h^2} \right) \|\nabla \cdot \mathbf{e}_\theta\| \right. \end{aligned}$$

$$\begin{aligned}
& + \tau_\theta \frac{k_2}{\varepsilon} \frac{C_{\text{inv}}}{h} \|\nabla e_w\| + \tau_\theta \frac{k_2}{\varepsilon} \frac{C_{\text{inv}}}{h^2} \|\mathbf{e}_\theta\| \Big] \|\nabla \cdot \boldsymbol{\phi}_h\| \\
& + \left[\tau_\theta \frac{k_1}{\varepsilon h} \|\nabla \mathbf{e}_\theta\| + \tau_\theta \frac{k_2}{\varepsilon h} \|\nabla \cdot \mathbf{e}_\theta\| + \left(\frac{1}{\varepsilon} + \frac{\tau_\theta}{\varepsilon^2} + \tau_w \frac{C_{\text{inv}}}{\varepsilon^2 h^2} \right) (\|\nabla e_w\| + \|\mathbf{e}_\theta\|) \right] \|\boldsymbol{\gamma}_h\| \\
\leq & C \left[k_1 \|\nabla \mathbf{e}_\theta\| + k_2 \|\nabla \cdot \mathbf{e}_\theta\| + \frac{h}{\varepsilon} \|\nabla e_w\| + \frac{h}{\varepsilon} \|\mathbf{e}_\theta\| \right] \left(\|\nabla \boldsymbol{\phi}_h\|_K + \|\nabla \cdot \boldsymbol{\phi}_h\| + \frac{1}{h} \|\boldsymbol{\gamma}_h\| \right) \\
\leq & C \left[\left(\sqrt{k_1} + \sqrt{k_2} \right) \epsilon_1(\boldsymbol{\theta}) + \frac{1}{\sqrt{\varepsilon}} \epsilon_0(\boldsymbol{\theta}) + \frac{1}{\sqrt{\varepsilon}} \epsilon_1(w) \right] (\sqrt{\alpha_1} \|\nabla \boldsymbol{\phi}_h\| + \sqrt{\alpha_2} \|\nabla \cdot \boldsymbol{\phi}_h\| + \sqrt{\alpha_3} \|\boldsymbol{\gamma}_h\|). \quad (6.36)
\end{aligned}$$

All the terms have been organized to see that it is clear that they are all bounded by $CE(h)\|\mathbf{v}_h\|$, from where (6.33) follows. \square

With this, it only remains to prove convergence, which proceeds in a standard manner.

Theorem 6.4 (Convergence). *Let $\mathbf{u} \in \mathcal{X}$ be the solution of the continuous problem, assumed to be regular enough. There is a positive constant C such that*

$$\|\mathbf{u} - \mathbf{u}_h\| \leq CE(h). \quad (6.37)$$

Proof. Consider $\tilde{\mathbf{u}}_h - \mathbf{u}_h \in \mathcal{X}_h$, where $\tilde{\mathbf{u}}_h$ is the best FE approximation to \mathbf{u} . From the inf-sup condition (6.14) it follows that there exists $\mathbf{v}_h \in \mathcal{X}_h$ such that

$$\begin{aligned}
C \|\tilde{\mathbf{u}}_h - \mathbf{u}_h\| \|\mathbf{v}_h\| & \leq B_{\text{stab}}(\tilde{\mathbf{u}}_h - \mathbf{u}_h, \mathbf{v}_h) \\
& = B_{\text{stab}}(\tilde{\mathbf{u}}_h - \mathbf{u}, \mathbf{v}_h) \quad (\text{from the consistency (6.30)}) \\
& \leq CE(h) \|\mathbf{v}_h\| \quad (\text{from (6.33)}),
\end{aligned}$$

from where $\|\tilde{\mathbf{u}}_h - \mathbf{u}_h\| \leq CE(h)$. Subsequently, the theorem follows from the triangle inequality $\|\mathbf{u} - \mathbf{u}_h\| \leq \|\mathbf{u} - \tilde{\mathbf{u}}_h\| + \|\tilde{\mathbf{u}}_h - \mathbf{u}_h\|$ and the interpolation error estimate (6.34). \square

From this result and the expression of the error function in (6.32) it follows that when t is ‘large’ the optimal combination of interpolation orders for rotations and displacements is $k_\theta = k_w$. However, when t is small, say $t < h$, the best is to take $k_\theta = k_w - 1$, since in this case rotations and deflection contribute with the same order of h to the error of the formulation.

7. Numerical results

All the examples of this section have been run considering linear continuous interpolation for both rotations and displacements, i.e., $k_\theta = k_w = 1$.

7.1. Shear-locking

In this section, the behavior of the ASGS and OSGS formulations is evaluated for the plate and beam problems. From the physical point of view, the formulations must be able to represent thin behavior, or in other words, the effects of shear deformations must become negligible for decreasing thickness. From the numerical point of view, this can be verified if the numerical results are free from shear-locking. This implies that the solution obtained using the Reissner-Mindlin and Timoshenko theories should coincide with the solution of the Kirchhoff and the Euler-Bernoulli beam theory, respectively, for small thicknesses.

To assess this, let us consider a square plate of domain $\Omega = (0, L)^2$ clamped on all its sides and subject to a uniform load $q = 1$ and a cantilever beam of domain $\Omega = (0, L)$ with a point load $P = -1$ at the right end and clamped at the left one. For the beam geometry we define a rectangular cross section of side $b = 1$ and thickness t , thus the inertia is $I = bt^3/12$. For the material properties we consider $E = 10^6$ and $\nu = 0.2$ in all cases (SI units can

be assumed to fix ideas). The analytical deflection at the center of the plate and at the end of the beam according to the Kirchhoff and Euler-Bernoulli theories are, respectively:

$$w_K = 0.01524 \frac{qL^4(1-\nu^2)}{Et^3}, \quad w_E = \frac{PL^3}{3EI}. \quad (7.1)$$

For the numerical computations, we consider a mesh of 20×20 square elements for the plate and 50 linear elements for the beam. The ratio between the deflection obtained numerically from the stabilized formulations and the analytical solutions respect to the thickness is plotted in Fig. 2. Results show that both formulations are able to represent then thin limit behavior.

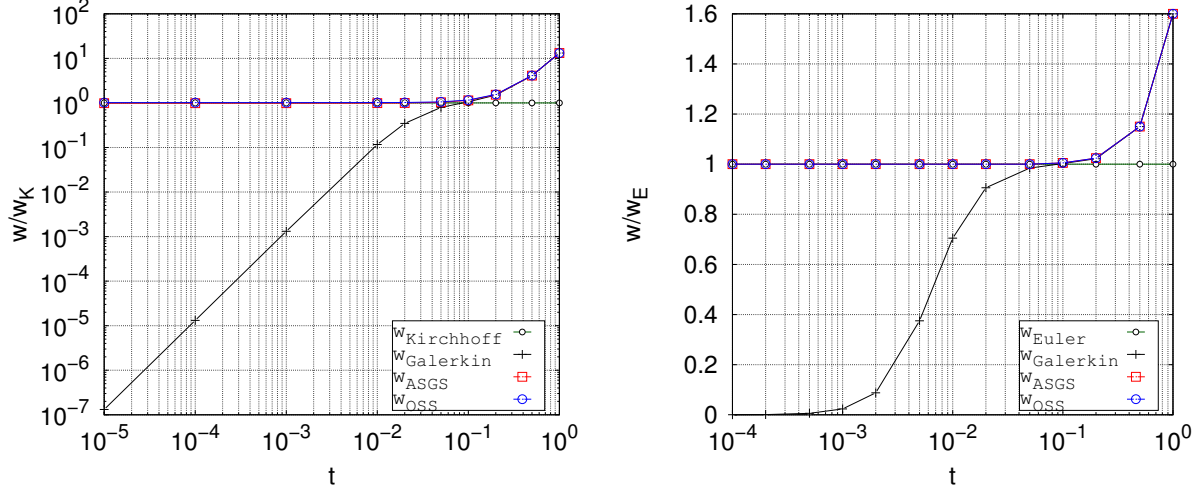


Figure 2: Comparison of deflection w from numerical results vs analytical solutions of plate (left) and beam (right).

7.2. Convergence tests

7.2.1. Convergence for beams

To assess the convergence of the stabilized Timoshenko beam formulation, consider a beam oriented in the x direction clamped at both sides, a homogeneous load $q = 1$ on its entire length $L = 1$, and the same material and geometrical properties as in the previous beam example. Since the formulation has to be able to represent the thin behavior, the solutions of the numerical method is compared to the analytical solution of the Euler-Bernoulli beam theory:

$$\theta_z(x) = \frac{qx}{12EI}(L-x)(1-2x), \quad w(x) = \frac{qx^2}{24EI}(L-x)^2.$$

The problem then is numerically solved for decreasing element sizes h , thickness $t = 10^{-4}$ and the error is evaluated in the L^2 norm. For comparison purposes, results are shown for the standard Galerkin, ASGS and OSS formulations, as presented in Fig. 3. Results show that deflections and rotations are optimally convergent respect to the element size using any of the two stabilized formulations.

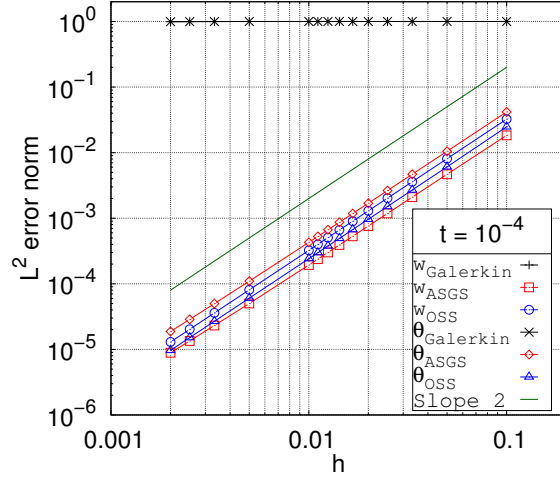


Figure 3: L^2 -error norm for Galerkin, ASGS and OSGS formulations.

For the numerical tests, the stabilization constants chosen for the ASGS formulation are $c_1 = c_3 = 12$, as found in section 5. In the case of the OSGS formulation, the values of the constants have no justification from the exact solution, and were chosen by testing different values. Nevertheless, this selection is not arbitrary; it is known that the zero shear constraint is difficult to handle, and selecting small values of c_1 can alleviate it. Under this assumption, it was found that values of $c_1 \leq 10^{-3}$ have to be used to obtain optimally convergent results. Recall that for linear elements the only condition needed for stability is that $c_1 > 0$, as discussed in section 6.

One important aspect of a locking-free formulation is that its convergence should be independent of the thickness, at least to some degree. To assess this dependence, the convergence curves of the displacements are plotted for different thicknesses. As presented in Fig. 4, the ASGS formulation becomes noticeable dependent on the thickness for low enough values, while the OSGS formulation shows a robust behavior since it is almost independent of it.

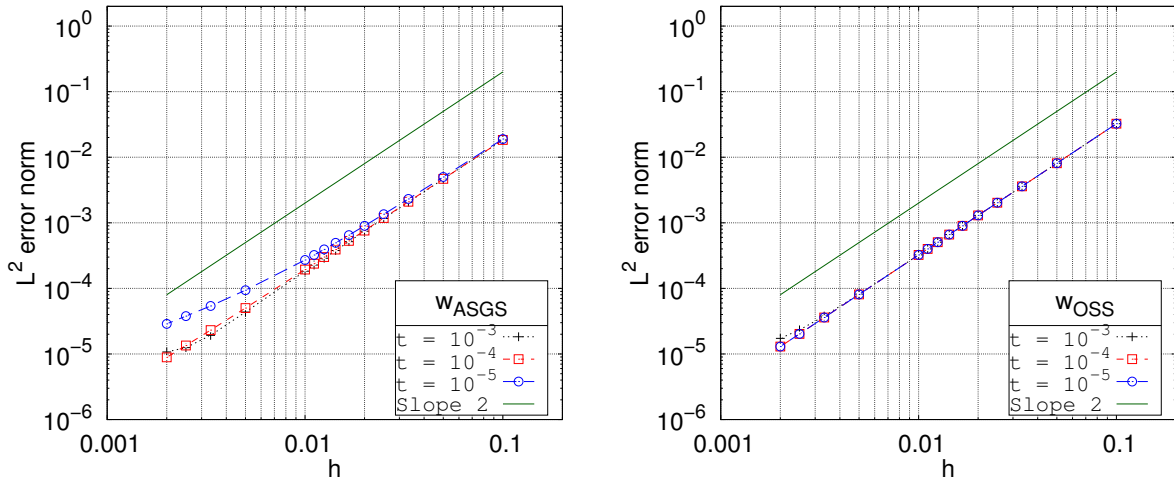


Figure 4: L^2 -error norm for different thickness, ASGS (left) and OSGS (right).

7.2.2. Convergence for plates

To assess the convergence of the stabilized Reissner-Mindlin plate formulation, consider a square plate of domain $\Omega = (0, L)^2$ clamped on all sides. The test is computed applying a load that follows the function proposed in [13]:

$$q(x, y) = \frac{E}{12(1 - \nu^2)} [12y(y - 1)(5x^2 - 5x + 1)(2y^2(y - 1)^2 + x(x - 1)(5y^2 - 5y + 1))]$$

$$+ 12x(x-1)(5y^2 - 5y + 1)(2x^2(x-1)^2 + y(y-1)(5x^2 - 5x + 1)],$$

for which the exact solutions of displacements and rotations are given by

$$\begin{aligned} w(x, y) &= \frac{x^3 y^3}{3t^3} (x-1)^3 (y-1)^3 - \frac{2}{5t(1-\nu)} [y^3 (y-1)^3 x (x-1) (5x^2 - 5x + 1) \\ &\quad + x^3 (x-1)^3 y (y-1) (5y^2 - 5y + 1)], \\ \theta_x(x, y) &= \frac{y^3 x^2}{t^3} (y-1)^3 (x-1)^2 (2x-1), \\ \theta_y(x, y) &= \frac{x^3 y^2}{t^3} (x-1)^3 (y-1)^2 (2y-1). \end{aligned}$$

The load, displacement and rotation fields are showed graphically in Fig. 5.

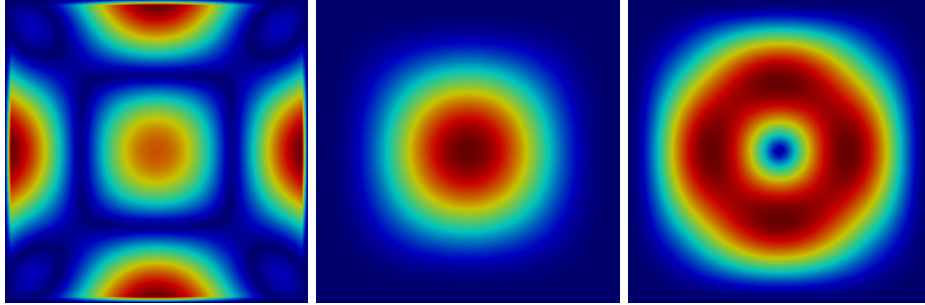


Figure 5: Applied Load (left) and numerical solutions of displacement (center) and norm of the rotation vector (right).

The problem then is solved for decreasing element sizes h , considering a constant thickness of $t = 10^{-4}$, and the error is evaluated in the L^2 norm. Numerical tests have been computed to evaluate which stabilization parameters are best suited to obtain optimal convergence. In regard of this, the only constant that must remain fixed is $c_2 = 1$, as proven in the convergence analysis. On this regard, optimal constants have not been found for the ASGS formulation, while in the OSGS formulation $c_1 \leq 10^{-3}$ has proven to give good results. As presented in Fig. 6 for the OSGS formulation, results prove to converge optimally respect to the element size.

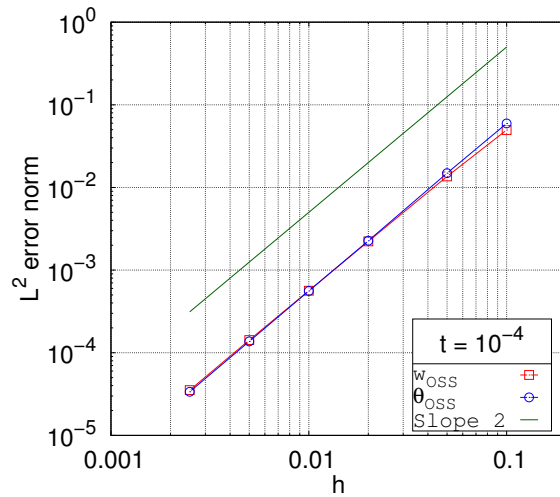


Figure 6: L^2 -error norm for OSGS formulation.

It is important to note that the constants c_3 and c_4 , which are associated to τ_w , have little impact on the solution. This is explained in detail in subsection 7.3. Recall that the theory does not predict locking-free convergence for

the ASGS formulation.

7.3. Sensitivity to stabilization constants

In this subsection, the sensitivity of the numerical solution to the stabilization constants is checked using numerical examples. When evaluating the convergence of the ASGS formulation for beams, the stabilization parameters were set using the constants $c_1 = c_3 = 12$ obtained from the elastic equations, which is not possible in plates. Even though in the beam case the constants are well defined for the ASGS formulation, it is important to check the performance of both ASGS and OSGS formulations for any set of constants. To address this, consider the same beam problem used previously in this section. Then the values of c_1 and c_3 are modified independently.

In the ASGS case, the constant c_1 is tested for slightly perturbed values, namely 12 ± 0.1 and c_3 is tested for a set of values ranged in $[2, 20]$, as shown in Fig. 7. Results show that the best convergence ratio is obtained for $c_1 = c_3 = 12$, however, a slight variation of c_1 is enough to lock the problem in a constant error independently of the element size, while c_3 allows a more flexible range of working values. The numerical tests confirm that the values obtained from the elastic equations provide the best behavior of the solution, which cannot be found in the case of plates. Regarding this, the convergence ratio curves of the ASGS formulation for plates are the same as the first image in Fig. 7, since the constants that provide optimal convergence were not found.

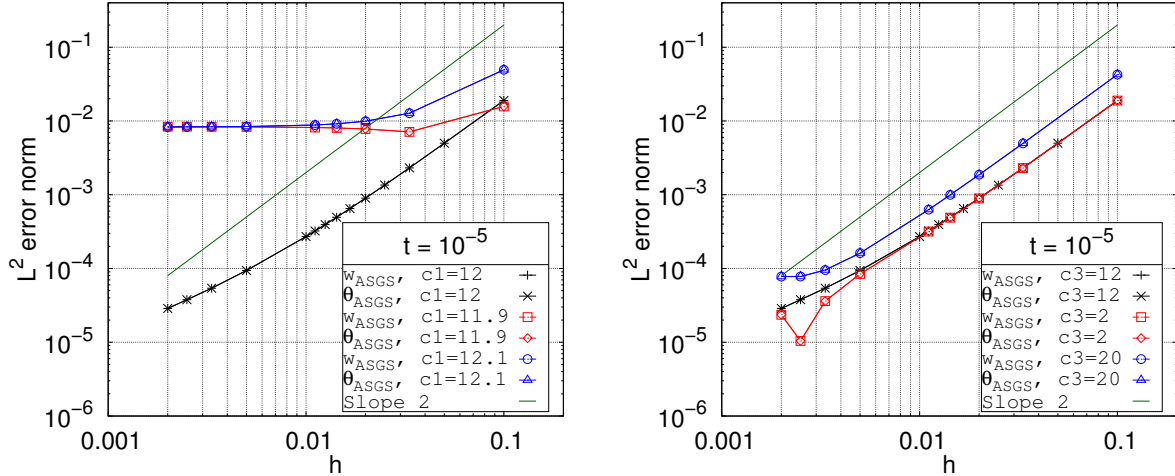


Figure 7: Sensitivity to c_1 (left) and c_3 (right) in L^2 -error norm.

In the case of the OSGS formulation, the stabilization constants do not have a significant impact on the final solution when $c_1 \leq 10^{-3}$. In this regard, the flexibility to choose any value for c_3 has to be assessed properly. Let us compute the convergence curves for the same beam case presented above, but this time to compare the results of extreme values of c_3 , as presented in Fig. 8.

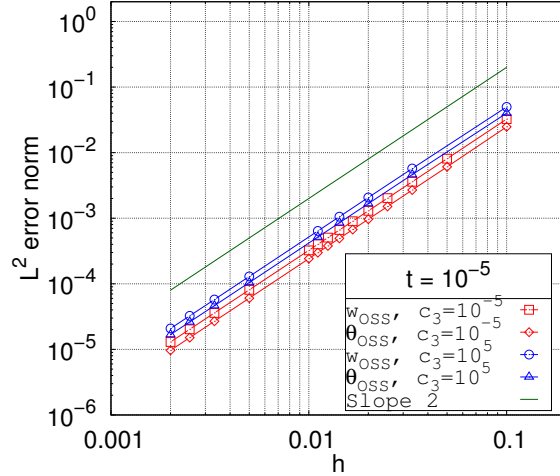


Figure 8: L_2 -error norm for different values of c_3 .

Results show that the curves are almost identical independently of the value of c_3 . This is, however, an important feature of the formulation: for high values of c_3 , the influence of the terms that contain τ_w is reduced, and in fact, it would be more useful not to consider them at all. From the practical point of view, this is equivalent to remove the projection of the force equilibrium equation, or ξ_w , from the formulation, lowering the total number of degrees of freedom. This response to stabilization constants behaves exactly the same for plates as well, and the fact that the number of degrees of freedom of the formulation can be reduced is specially useful for lowering the cost of computations.

8. Conclusions

The numerical locking present in standard Galerkin formulations of Reissner-Mindlin plates and Timoshenko beams has been addressed using the Variational Multiscale Method. The work approaches this issue using the ASGS and the OSGS methods. It has been shown that the norm in which stability can be proved for the ASGS formulation is not free of locking when $t \rightarrow 0$, whereas optimal stability and convergence has been proven for the OSGS approach for arbitrary interpolation of the variables. Nevertheless, for Timoshenko beams there exists a set of algorithmic constants for which the element stiffness matrix of the ASGS formulation coincides with that of the elastic equations, and in this case the method does converge. This, however, has to be considered a singularity rather than a general possibility.

Numerical tests confirm that the theoretical predictions. In particular, the OSGS formulation provides optimally convergent rates, for both displacement and rotations. Moreover, this method is shown to be mildly sensitive to the algorithmic constants. In particular, in the numerical experiments presented it has been unnecessary to activate the stabilization terms corresponding to deflections to obtain optimally convergent results.

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