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ANALYSIS OF A FINITE ELEMENT METHOD TO PRESCRIBE ESSENTIAL BOUNDARY CONDITIONS ON NON-MATCHING MESHES FOR ELLIPTIC PROBLEMS

JOAN BAIGES AND RAMON CODINA[1]

Abstract. The purpose of this paper is to present the convergence analysis of an approximate method to prescribe essential boundary conditions for elliptic problems. We assume that these are approximated using the finite element method, and that the finite element mesh does not match the domain boundary. The description of the method can be based on the Lagrange multiplier technique, although the Lagrange multiplier is not taken as independent, but related to the unknown through a least-squares term. This allows one to solve a problem in which only the degrees of freedom of the original unknown need to be computed. The method has been presented in (R. Codina and J. Baiges, *Int. J. Numer. Meth. Engng.,* (2015) vol. 104, 624–654), where its stability properties are analyzed. Here we present the numerical analysis, emphasizing its application to mixed elliptic problems (Darcy, Stokes and Maxwell) approximated using the Galerkin method.

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1. INTRODUCTION

Consider an elliptic problem of the form: find $u : \Omega \longrightarrow \mathbb{R}^n$ such that $\mathcal{L}u = f$ in a bounded domain $\Omega \subset \mathbb{R}^d$ and $\mathcal{D}u = \mathcal{D}\bar{u}$ on $\Gamma = \partial\Omega$, where $\mathcal{L}$ is an elliptic differential operator, $\mathcal{D}u$ is a trace operator and f and $\bar{u}$ are given data. For conciseness, we assume that $\mathcal{L}$ is symmetric, so that the differential equations to be solved are the Euler-Lagrange equations for the optimization of a functional F defined on $X_{\bar{u}} = \{v \in X \mid \mathcal{D}v = \mathcal{D}\bar{u} \text{ on } \Gamma\}$, where X is the space where $\mathcal{L}$ is well defined in a distributional sense.

Suppose now that the problem is approximated using the finite element method. To this end, let us construct a finite element *covering* of Ω, $\mathcal{T}_h = \{K\}$, with $h = \max_K\{h_K = \text{diam}(K), K \in \mathcal{T}_h\}$. Let then $\Omega \subset \Omega_h := \bigcup_{K\in\mathcal{T}_h} K$, and therefore $\Gamma \neq \partial\Omega_h$ in general. It is in this sense that we consider the mesh as non-matching the domain Ω (the particular case $\Omega = \Omega_h$ is also allowed). We will assume that all $K \in \mathcal{T}_h$ have an interior with a non-empty intersection with Ω. The question now is how to impose essential boundary conditions on Γ with the discretization of Ω described.

There may be several reasons for using non-matching meshes when approximating partial differential equations, in essence all related to the convenience of using a *fixed mesh* independently of the shape of Ω (see the reviews in [13, 28, 29], for example). Perhaps the most relevant example of such a situation is the case of moving domains,

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[1] Universitat Politècnica de Catalunya, Jordi Girona 1-3, Edifici C1, 08034, Barcelona, Spain

with time varying boundaries Γ. Several methods exist in which what is discretized is a *landscape* domain where Ω moves, and the same mesh is used for several (if not all) configurations. An example of these techniques is presented in [31] (see also references therein). The imposition of essential boundary conditions on non-matching meshes is one of the key ingredients in this type of methods. It is required not only in the finite element approximation, but also using mesh-less methods, for example (see [17]).

Nitsche's method is a simple and effective way to prescribe essential boundary conditions weakly (see for example [27, 31]). It consists in penalizing the difference between the unknown and its prescribed value on the boundary, with a proper scaling. In contrast to the penalty method, Nitsche's method preserves consistency with the problem in the strong form, and can be used with medium valued penalty parameters. However, perhaps the cleanest way to impose weakly essential boundary conditions is through the use of Lagrange multipliers. In this case, the boundary conditions are treated as any other restriction that could be posed to the solution, and no penalty terms or algorithmic parameters need to be chosen. In the context of non-matching mesh methods, it is the basis of the Fictitious Domain Method [19, 20]. More recently, stabilized Lagrange multiplier methods have been presented in [1, 6, 9], where a ghost penalty approach is used to stabilize the imposition of boundary conditions.

If the original problem consists of optimizing F over $X_{\bar{u}}$, the Lagrange multiplier approach consists in optimizing the functional

$$G(v, \mu) = F(v) - \langle \mu, \mathcal{D}v - \mathcal{D}\bar{u} \rangle_\Gamma \tag{1}$$

over X, with μ in the appropriate space. Here and in what follows we will use the following notation. If X is the space where the unknown is sought, its dual will be denoted by X', and the duality by $\langle \cdot, \cdot \rangle_\Omega$. If Λ is the space of traces on Γ of functions in X and Λ' is its dual, the duality in this case will be written as $\langle \cdot, \cdot \rangle_\Gamma$. The L^2-inner product in a domain ω will be denoted by $(\cdot, \cdot)_\omega$. The $L^2(\Omega)$-projection onto a space X will be written as P_X. With this notation in mind, the problem to solve is the optimization of (1) over $X \times \Lambda'$ instead of the optimization of F over $X_{\bar{u}}$. If for example F is positive definite, the solution $[u, \lambda]$ will be a saddle point of G, i.e., $[u, \lambda] = \arg \inf_{v \in X} \sup_{\mu \in \Lambda'} G(v, \mu)$. $\| \cdot \|_Y$ will stand for the norm in a functional space Y. Moreover, inequalities up to *dimensionless* constants, independent also of the discretization, will be written as $\lesssim$ and $\gtrsim$ for $\leq$ and $\geq$, respectively.

To optimize F, we may take its first variation (or Gâteaux derivative) in the direction of $\delta u \in X_0$. This leads to the problem of finding $u \in X_{\bar{u}}$ such that $B(u, \delta u) = L(\delta u)$ for all possible δu, where $B(u, \delta u) = \langle \mathcal{L}u, \delta u \rangle_\Omega$, $L(\delta u) = \langle f, \delta u \rangle_\Omega$. On the other hand, the optimization of (1) consists in seeking the stationary point of its first variation in the direction of $[\delta u, \delta\lambda] \in X \times \Lambda'$, which leads to the problem of finding $[u, \lambda] \in X \times \Lambda'$ such that

$$B(u, \delta u) - \langle \lambda, \mathcal{D}\delta u \rangle_\Gamma = L(\delta u) \qquad \forall\, \delta u \in X, \tag{2}$$

$$\langle \delta\lambda, \mathcal{D}u \rangle_\Gamma = \langle \delta\lambda, \mathcal{D}\bar{u} \rangle_\Gamma \qquad \forall\, \delta\lambda \in \Lambda'. \tag{3}$$

The finite element approximation of this problem is standard, although the choice of the interpolating spaces for u and λ is restricted by the satisfaction of an adequate inf-sup condition.

From the practical point of view, the main inconvenience of the Lagrange multiplier technique over Nitsche's method is that it introduces new degrees of freedom. For some interpolations it can be eliminated locally, but in general this is not possible because of the inf-sup condition just mentioned. This condition can be circumvented using for example the technique presented in [5]; in this case, if the Lagrange multiplier can be condensed locally, the resulting method is closely related to Nische's method [31].

It would be desirable to prescribe boundary conditions using only the degrees of freedom to approximate u, but trying to avoid penalty terms, as in Nitsche's method. This is for example the goal of the method proposed in [12], which shows optimal convergence rates without the need of a penalty parameter, even though it is not variationally based and yields always a non-symmetric problem. This is also the shortcoming of the method proposed in [18], which is based on the introduction of an additional element-wise discontinuous flux whose trace of Γ is used to test the essential boundary condition. The main advantage is that this flux can be condensed at the element level, yielding a method that involves only the degrees of freedom of u. A modification was proposed in [4] that yields a variationally based method while maintaining the possibility of condensing the additional flux variable. The stability analysis of the method for the convection-diffusion equation and the incompressible Navier-Stokes equations was already presented in [4], and was further extended in [14], where the general methodology was described and applied to several versions of Darcy's and Stokes' problems. The general method was termed *Linked Lagrange Multiplier* (LLM) method, as the essential idea is to impose that the classical Lagrange multiplier be linked (in a certain sense) to the main unknown through a least-squares term.

The aim of this paper is to present a complete a priori *convergence analysis* of the method proposed in [14], with some modifications for the Darcy and the Stokes problems, and also to extend it to the Maxwell problem. The LLM method will thus be applied and analyzed for three classical *mixed* elliptic problems. The application to other elliptic problems should be straightforward from the analysis presented in this work.

The LLM method proposed in [14] is recalled in Section 2, where the general convergence analysis strategy is also presented. This strategy is then applied to the Darcy problem (Section 3), to the Stokes problem (Section 4) and to the Maxwell problem (Section 5). Numerical examples are not included, since they can be found elsewhere [4, 14]. The paper concludes with some final observations in Section 6.

2. The linked Lagrange multiplier method for an abstract problem

2.1. Description of the method

Let us recall the LLM presented in [14]. Let X_h and Λ_h be *conforming* finite element spaces to approximate X and Λ, respectively, constructed from the finite element partition $\mathcal{T}_h$ of Ω_h. In what follows, all finite element spaces are assumed to be defined on Ω_h, and likewise for the continuous spaces, even if the functions belonging to them are only required in $\Omega \subset \Omega_h$. The finite element integrals of the variational forms of the problem will be done only over the Ω domain, without including the $\Omega_h \setminus \Omega$ region. Therefore the method falls into the category of what are known as *cut-FEM methods*.

Assuming F positive definite, the approximate optimization of functional G in (1) yields the problem

$$[u_h, \lambda_h] = \arg \inf_{v_h \in X_h} \sup_{\mu_h \in \Lambda'_h} G(v_h, \mu_h), \tag{4}$$

or, equivalently, the discrete variational problem: find $[u_h, \lambda_h] \in X_h \times \Lambda'_h$ such that

$$B(u_h, \delta u_h) - \langle \lambda_h, \mathcal{D}\delta u_h \rangle_\Gamma = L(\delta u_h) \qquad \forall\, \delta u_h \in X_h, \tag{5}$$

$$-\langle \delta\lambda_h, \mathcal{D} u_h \rangle_\Gamma = -\langle \delta\lambda_h, \mathcal{D}\bar{u} \rangle_\Gamma \qquad \forall\, \delta\lambda_h \in \Lambda'_h. \tag{6}$$

Here and in what follows, δf will stand for the test function associated to a function f when a variational problem is considered. It needs to be remarked that integrals involved in both (4) and (5) need to be performed over Ω, not over Ω_h, and that Γ needs to be represented by finite element functions. However, we shall not consider the approximation error introduced by this representation, and in fact we will assume Γ to be piecewise affine (see below).

Problem (5)-(6) is well posed if the following inf-sup condition holds: there exists $\beta > 0$ such that for all $\lambda_h \in \Lambda_h$ there is $v_h \in X_h$ such that

$$\beta \|\lambda_h\|_{\Lambda'} \|v_h\|_X \leq \langle \lambda_h, \mathcal{D} v_h \rangle_\Gamma. \tag{7}$$

(7) can be written as $\|\lambda_h\|_{\Lambda'} \|v_h\|_X \lesssim \langle \lambda_h, \mathcal{D} v_h \rangle_\Gamma$. In order to avoid the need for satisfying (7), some additional mesh-dependent terms may be added to (5)-(6), as in [5, 10].

The LLM method can be explained as a modification of (4) in which the Lagrange multiplier is not an independent variable, but it is linked to u_h through a least-squares term. If integrating by parts we obtain the formal identity

$$\langle \mathcal{L} u, \delta u \rangle_\Omega = B(u, \delta u) - \langle \mathcal{F}_n u, \mathcal{D} \delta u \rangle_\Gamma, \tag{8}$$

where $\mathcal{F}_n u$ is a certain trace on Γ of a field $\mathcal{F} u$ defined on Ω, *we impose that λ_h be equal to the trace on Γ of a field σ_h which is equal to $\mathcal{F} u_h$ in a least-squares sense*. If $\sigma_{n,h}$ is this trace and Σ_h the space where σ_h is sought, we replace (4) by

$$[u_h, \sigma_h] = \arg \inf_{v_h \in X_h} \sup_{\tau_h \in \Sigma_h} \hat{G}([v_h, \tau_h]), \tag{9}$$

$$\hat{G}([v_h, \tau_h]) := F(v_h) - \langle \tau_{n,h}, \mathcal{D} v_h - \mathcal{D} \bar{u} \rangle_\Gamma - \frac{1}{2N} \|\tau_h - \mathcal{F} v_h\|^2_{L^2(\Omega)}, \tag{10}$$

with N a sufficiently large numerical parameter. In the case of several boundary conditions to be enforced, each will be associated with a new variable which will be determined by a term as the last one in (10), and with a sign determined by stability reasons (see Section 5). This term resembles the one introduced in [5] to circumvent the inf-sup condition (7), the main difference being that our least-squares term is defined in the computational domain, not on the boundary; the benefit of our approach is twofold, namely, we will be able to condense the Lagrange multiplier and there will be very mild stability conditions to be satisfied.

Equivalently, problem (9) can be written as: find $[u_h, \sigma_h] \in X_h \times \Sigma_h$ such that

$$B(u_h, \delta u_h) - \langle \sigma_{n,h}, \mathcal{D} \delta u_h \rangle_\Gamma - \frac{1}{N} (-\mathcal{F} \delta u_h, \sigma_h - \mathcal{F} u_h)_\Omega = L(\delta u_h), \tag{11}$$

$$-\langle \delta \sigma_{n,h}, \mathcal{D} u_h \rangle_\Gamma - \frac{1}{N} (\delta \sigma_h, \sigma_h - \mathcal{F} u_h)_\Omega = -\langle \delta \sigma_{n,h}, \mathcal{D} \bar{u} \rangle_\Gamma, \tag{12}$$

for all $[\delta u_h, \delta \sigma_h] \in X_h \times \Sigma_h$. *We will assume throughout that Σ_h is made of piecewise discontinuous functions*. In this case, it is seen from (12) that *only the value of functions in Σ_h on elements cut by Γ is needed*. Thus, functions in Σ_h can be considered to be zero outside these elements (except when they are interpolants of the unknowns).

2.2. Analysis strategy

The numerical analysis of the methods to be proposed is standard, and follows the same pattern in all cases. In order to fix notation and for completeness, let us explain it here.

Let $\boldsymbol{U}_h = [u_h, \sigma_h]$, $\boldsymbol{U} = [u, \mathcal{F}u]$, $\delta\boldsymbol{U}_h = [\delta u_h, \delta\sigma_h]$. Problem (11)-(12) can be written as: find $\boldsymbol{U}_h \in X_h \times \Sigma_h$ such that

$$B_{\rm G}(\boldsymbol{U}_h, \delta\boldsymbol{U}_h) = L_{\rm G}(\delta\boldsymbol{U}_h) \quad \forall \delta\boldsymbol{U}_h \in \boldsymbol{U}_h,$$

with the obvious identification of the bilinear form $B_{\rm G}$ and the linear form $L_{\rm G}$ with the terms in (11)-(12). Let $|\!|\!|\cdot|\!|\!|_{\rm G}$ be a norm defined on $X_h \times \Sigma_h$. We then have:

Theorem 1. *Suppose that the following conditions hold:*

(1) *For all $\boldsymbol{U}_h \in X_h \times \Sigma_h$ there exists $\delta\boldsymbol{U}_h \in X_h \times \Sigma_h$ such that*

$$B_{\rm G}(\boldsymbol{U}_h, \delta\boldsymbol{U}_h) \gtrsim |\!|\!|\boldsymbol{U}_h|\!|\!|_{\rm G} |\!|\!|\delta\boldsymbol{U}_h|\!|\!|_{\rm G} \quad \textit{(Stability)}. \tag{13}$$

(2) *There exists a scalar function $\mathcal{E}_{\rm G}(\boldsymbol{U}, h) \geq 0$, tending to zero as $h \to 0$, such that*

$$B_{\rm G}(\boldsymbol{U} - \tilde{\boldsymbol{U}}_h, \delta\boldsymbol{U}_h) \lesssim \mathcal{E}_{\rm G}(\boldsymbol{U}, h) |\!|\!|\delta\boldsymbol{U}_h|\!|\!|_{\rm G}, \tag{14}$$

$$|\!|\!|\boldsymbol{U} - \tilde{\boldsymbol{U}}_h|\!|\!|_{\rm G} \lesssim \mathcal{E}_{\rm G}(\boldsymbol{U}, h), \tag{15}$$

where $\tilde{\boldsymbol{U}}_h \in X_h \times \Sigma_h$ is an interpolant of the solution of the continuous problem $\boldsymbol{U}$.

Then, the following a priori estimate holds:

$$|\!|\!|\boldsymbol{U} - \boldsymbol{U}_h|\!|\!|_{\rm G} \lesssim \mathcal{E}_{\rm G}(\boldsymbol{U}, h). \tag{16}$$

Proof. The proof is standard. The key point is that the approximation given by (11)-(12) is consistent, in the sense that $B_{\rm G}(\boldsymbol{U}, \delta\boldsymbol{U}_h) = L_{\rm G}(\delta\boldsymbol{U}_h)$ for all $\delta\boldsymbol{U}_h \in X_h \times \Sigma_h$, and therefore $B_{\rm G}(\boldsymbol{U} - \boldsymbol{U}_h, \delta\boldsymbol{U}_h) = 0$ for all $\delta\boldsymbol{U}_h \in X_h \times \Sigma_h$. Using this and (13) we have that there exists $\delta\boldsymbol{U}_h \in X_h \times \Sigma_h$ such that

$$\begin{aligned} |\!|\!|\boldsymbol{U}_h - \tilde{\boldsymbol{U}}_h|\!|\!|_{\rm G} |\!|\!|\delta\boldsymbol{U}_h|\!|\!|_{\rm G} &\lesssim B_{\rm G}(\boldsymbol{U}_h - \tilde{\boldsymbol{U}}_h, \delta\boldsymbol{U}_h) + B_{\rm G}(\boldsymbol{U} - \boldsymbol{U}_h, \delta\boldsymbol{U}_h) \\ &\lesssim \mathcal{E}_{\rm G}(\boldsymbol{U}, h) |\!|\!|\delta\boldsymbol{U}_h|\!|\!|_{\rm G} \qquad \text{(because of (14))} \end{aligned}$$

The result follows from this, the triangle inequality and (15). Note that (14) is not exactly a continuity statement for $B_{\rm G}$, but it is the condition that is really needed. □

In essence, for the different problems to be presented below we will give the norm $|\!|\!|\cdot|\!|\!|_{\rm G}$, check the stability condition (13) and determine the *error function* $\mathcal{E}_{\rm G}$ that satisfies conditions (14) and (15). In all cases we will see that the error function obtained is optimal, in the sense that tends to zero at the best rate allowed by the finite element interpolation. For all the problems analyzed we will present a single convergence theorem, which will include stability (in fact, the step that requires more work).

For simplicity, we shall assume that the family of finite element partitions $\{\mathcal{T}_h\}$ of Ω_h is quasi-uniform, and from it we construct a conforming finite element space $X_h \subset X$ and Σ_h, the space for the LLM. For any finite element function f_h in any of these spaces, the inverse estimates

$$\|f_h\|_{L^2(\Gamma_K)} \leq C_{h,K,\Gamma} h^{-1/2} \|f_h\|_{L^2(K\cap\Omega)}, \qquad \|f_h\|_{L^2(\Gamma)} \lesssim h^{-1/2} \|f_h\|_{L^2(\Omega)}, \tag{17}$$

will hold, $\Gamma_K = K \cap \Gamma$, $K \in \mathcal{T}_h$. Note that the constant $C_{h,K,\Gamma}$ may grow as the measure of $K \cap \Omega$ tends to zero. This yields a deterioration in the stability and convergence estimates to be obtained that is shared by most methods to prescribe essential boundary conditions on non-matching methods. A method to alleviate it in a particular setting and for Nitsche's method is presented in [11], consisting in the introduction of a term that penalizes the jumps of the gradients of the unknown along the edges of the badly cut elements. As another example, in [23] a modification of the formulation introduced in [5] applied to embedded boundaries is proposed, in this case consisting in extrapolating the normal derivative of the unknown on the boundaries of badly cut elements from neighboring elements. The extension of any of these techniques to the Linked Lagrange Multiplier framework seems to be possible, although we have not considered this in this work. Thus, we explicitly assume that:

Assumption 1. *There exists a constant $C_\Gamma < \infty$ such that the constants in (17) satisfy $C_{h,K,\Gamma} \leq C_\Gamma$ for all $K \in \mathcal{T}_h$ and for all $h > 0$. Therefore, we may simply write the first inequality in (17) as $\|f_h\|_{L^2(\Gamma_K)} \lesssim h^{-1/2} \|f_h\|_{L^2(K\cap\Omega)}$.*

Let us stress that this assumption is not a restriction of our analysis, but it is common to all methods that do not employ any of the stabilization techniques mentioned above. In particular, it is the classical assumption required in Nitsche's method or the standard Lagrange multiplier technique.

In order to present our results in scale independent expressions, we shall need a characteristic length of the domain Ω (for example its diameter), that we will denote by L_0.

Finally, we will consider that each Γ_K, $K \in \mathcal{T}_h$, has a constant normal $\boldsymbol{n}$, i.e., it is affine. The proofs below could generalized to smooth Γ_K by controlling the difference between $\boldsymbol{n}$ and its projection onto an appropriate finite element space.

3. Darcy's problem

Let us apply now the method described before to Darcy's problem. It consists of finding $u : \Omega \longrightarrow \mathbb{R}$ and $\boldsymbol{q} : \Omega \longrightarrow \mathbb{R}^d$ such that

$$-\frac{1}{k}\boldsymbol{q} - \nabla u = \boldsymbol{f} \quad \text{in } \Omega, \tag{18}$$

$$\nabla \cdot \boldsymbol{q} = g \quad \text{in } \Omega, \tag{19}$$

where $\boldsymbol{f}$ and g are given functions and $k > 0$ is a physical parameter, here assumed constant. The essential boundary conditions depend on the functional setting of the problem described next. From now on, vector or tensor fields will be represented by boldface characters.

3.1. Primal and dual forms of Darcy's problem

Let us identify Darcy's problem within the general frame presented before. The number of components of the unknown is now $n = d + 1$, which we arrange as $[\boldsymbol{q}, u]$. The differential operator is $\mathcal{L}([\boldsymbol{q}, u]) = [-\frac{1}{k}\boldsymbol{q} - \nabla u, \nabla \cdot \boldsymbol{q}]$. If

	Primal form	Dual form
V	$H^1(\Omega)$	$L^2(\Omega)$
R	$L^2(\Omega)^d$	$H(\mathrm{div};\Omega)$
$B([\boldsymbol{q},u],[\delta\boldsymbol{q},\delta u])$	$-\frac{1}{k}(\boldsymbol{q},\delta\boldsymbol{q})_\Omega$ $-(\nabla u,\delta\boldsymbol{q})_\Omega-(\nabla\delta u,\boldsymbol{q})_\Omega$	$-\frac{1}{k}(\boldsymbol{q},\delta\boldsymbol{q})_\Omega$ $+(u,\nabla\cdot\delta\boldsymbol{q})_\Omega+(\delta u,\nabla\cdot\boldsymbol{q})_\Omega$
$\mathcal{F}[\boldsymbol{q},u]$	$-\boldsymbol{q}$	u
$\mathcal{D}[\boldsymbol{q},u]$	u	$\boldsymbol{n}\cdot\boldsymbol{q}$
Λ	$H^{1/2}(\Gamma)$	$H^{-1/2}(\Gamma)$
$F[\boldsymbol{q},u]$	$-\frac{1}{2k}\Vert\boldsymbol{q}\Vert^2_{L^2(\Omega)}-(\nabla u,\boldsymbol{q})_\Omega$ $-\langle\boldsymbol{f},\boldsymbol{q}\rangle_\Omega-\langle g,u\rangle_\Omega$	$-\frac{1}{2k}\Vert\boldsymbol{q}\Vert^2_{L^2(\Omega)}+(u,\nabla\cdot\boldsymbol{q})_\Omega$ $-\langle\boldsymbol{f},\boldsymbol{q}\rangle_\Omega-\langle g,u\rangle_\Omega$

TABLE 1. Primal and dual forms of Darcy's problem

we test it against $[\delta\boldsymbol{q},\delta u]$, we formally obtain an expression of the general form (8), which may change depending on the term we integrate by parts. Considering smooth enough functions, we have

$$\langle\mathcal{L}([\boldsymbol{q},u],[\delta\boldsymbol{q},\delta u]\rangle_\Omega=\begin{cases}-\frac{1}{k}(\boldsymbol{q},\delta\boldsymbol{q})_\Omega-(\nabla u,\delta\boldsymbol{q})_\Omega-(\nabla\delta u,\boldsymbol{q})_\Omega+\langle\boldsymbol{n}\cdot\boldsymbol{q},\delta u\rangle_\Gamma, & \text{(P)}\\ -\frac{1}{k}(\boldsymbol{q},\delta\boldsymbol{q})_\Omega+(u,\nabla\cdot\delta\boldsymbol{q})_\Omega+(\delta u,\nabla\cdot\boldsymbol{q})_\Omega-\langle\boldsymbol{n}\cdot\delta\boldsymbol{q},u\rangle_\Gamma. & \text{(D)}\end{cases}$$

Let V and R the spaces where u and $\boldsymbol{q}$ are defined, respectively. These depend on whether expression (P) or expression (D) is used. The first leads to the primal form of Darcy's problem and the second to the dual form. The spaces in play, the bilinear form B, operators $\mathcal{F}$ and $\mathcal{D}$ satisfying (8), the trace space Λ and the functional F whose optimization leads to the Euler-Lagrange equations (18)-(19) are all given in Table 3.1 (see [2, 7] for background). The notation used in this table is standard; in particular, $H(\mathrm{div};\Omega)$ is the spaces of vector fields in $L^2(\Omega)^d$ with divergence in $L^2(\Omega)$. The right-hand-side of the variational form of the problem is $L([\delta\boldsymbol{q},\delta u])=\langle\boldsymbol{f},\delta\boldsymbol{q}\rangle_\Omega+\langle g,\delta u\rangle_\Omega$, which is assume to be well-defined, that is to say, $\boldsymbol{f}$ belongs to the dual of R and g to the dual of V. Both the primal and the dual forms of Darcy's problem are well posed; the former can be reduced to the classical Poisson's problem, and therefore it is not worth to study it in mixed form, and the latter is the prototype of mixed problem, for which an inf-sup condition has to be checked (see [7]).

In view of the expression of the boundary operator $\mathcal{D}$, the essential boundary conditions that we shall consider are

$$u=\bar{u}\quad\text{on }\Gamma\qquad\text{for the primal form,}$$
$$\boldsymbol{n}\cdot\boldsymbol{q}=\bar{q}_n\quad\text{on }\Gamma\qquad\text{for the dual form.}$$

In what follows we present the results of the convergence analysis of the finite element approximation of both the primal and dual forms of the problem. We will analyze the Galerkin finite element approximation of the problem, but a similar analysis applies to a stabilized finite element method, which allows one to circumvent the compatibility conditions between the finite element space $V_h\subset V$ and $R_h\subset R$ (see [2, 14]).

3.2. Galerkin finite element approximation

According to the general methodology described in Section 2 and considering the description of the primal and dual problems presented above, the problem we propose to solve is the optimization of the functional

$$\hat{G}([\boldsymbol{r}_h, v_h, \boldsymbol{\tau}_h]) = \begin{cases} F([\boldsymbol{r}_h, v_h]) - \langle \tau_{n,h}, v_h - \bar{u} \rangle_\Gamma - \frac{1}{2N_0 k} \|\boldsymbol{\tau}_h + \boldsymbol{r}_h\|^2_{L^2(\Omega)}, & \text{(P)} \\ F([\boldsymbol{r}_h, v_h]) - \langle \tau_h, \boldsymbol{n} \cdot \boldsymbol{r}_h - \bar{q}_n \rangle_\Gamma - \frac{k}{2N_0 L_0^2} \|\tau_h - v_h\|^2_{L^2(\Omega)}, & \text{(D)} \end{cases}$$

over $R_h \times V_h \times \Sigma_h$, corresponding to the primal (P) and the dual (D) forms of the problem. Note that in the first case $\boldsymbol{\tau}_h$ is a vector field, with the physical meaning of a flux, and $\tau_{n,h}$ is its normal trace on Γ, whereas in the second case τ_h is a scalar, with the same physical meaning as the primal variable. The factors k and L_0^2 in the least-squares terms have been introduced to leave only one dimensionless parameter N_0 in the formulation.

3.2.1. *Discrete primal form*

Problem (11)-(12) now reads: find $[\boldsymbol{q}_h, u_h, \boldsymbol{\sigma}_h] \in R_h \times V_h \times \Sigma_h$ such that

$$-\frac{1}{k}(\boldsymbol{q}_h, \delta\boldsymbol{q}_h)_\Omega - (\nabla u_h, \delta\boldsymbol{q}_h)_\Omega - \frac{1}{N_0 k}(\delta\boldsymbol{q}_h, \boldsymbol{\sigma}_h + \boldsymbol{q}_h)_\Omega = \langle \boldsymbol{f}, \delta\boldsymbol{q}_h \rangle_\Omega, \tag{20}$$

$$-(\nabla \delta u_h, \boldsymbol{q}_h)_\Omega - \langle \sigma_{n,h}, \delta u_h \rangle_\Gamma = \langle g, \delta u_h \rangle_\Omega, \tag{21}$$

$$-\langle \delta\sigma_{n,h}, u_h \rangle_\Gamma - \frac{1}{N_0 k}(\delta\boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h + \boldsymbol{q}_h)_\Omega = -\langle \delta\sigma_{n,h}, \bar{u} \rangle_\Gamma, \tag{22}$$

for all $\delta\boldsymbol{q}_h \in R_h$, $\delta u_h \in V_h$ and $\delta\boldsymbol{\sigma}_h \in \Sigma_h$. This is the LLM method in the case we are considering. From (22) it follows that the degrees of freedom of $\boldsymbol{\sigma}_h$ can be condensed at the element level *if its interpolation is discontinuous*. If fact, this is precisely one of the ingredients that we need to prove stability of the method, which can be expressed as a compatibility condition between the spaces V_h and Σ_h:

Lemma 1. *If Σ_h is made of piecewise discontinuous (vector) functions, for all $v_h \in V_h$ there exists $\boldsymbol{\tau}_h \in \Sigma_h$ and $\delta_0 > 0$ such that*

$$\|v_h\|^2_{L^2(\Gamma)} \lesssim \langle \boldsymbol{\tau}_h \cdot \boldsymbol{n}, v_h \rangle_\Gamma + \delta_0 h \|\nabla v_h\|^2_{L^2(\Omega)}, \tag{23}$$

$$\|\boldsymbol{\tau}_h\|_{L^2(\Gamma)} = \|v_h\|_{L^2(\Gamma)}, \quad \|\boldsymbol{\tau}_h\|^2_{L^2(\Omega)} \lesssim h \|v_h\|^2_{L^2(\Gamma)}. \tag{24}$$

Proof. Let $K \in \mathcal{T}_h$ be such that $\Gamma_K := \Gamma \cap K \neq \emptyset$. Given $v_h \in V_h$, let us define its linear and quadratic means over Γ_K as

$$v_{\mathrm{lm},K} = \frac{1}{|\Gamma_K|} \int_{\Gamma_K} v_h, \quad v_{\mathrm{qm},K} = \frac{1}{|\Gamma_K|^{1/2}} \Big(\int_{\Gamma_K} v_h^2 \Big)^{1/2},$$

where $|\Gamma_K|$ is the measure of Γ_K. Using Schwarz's inequality it follows that $v_{\mathrm{qm},K} \geq |v_{\mathrm{lm},K}|$. Furthermore, Poincaré's inequality and a scaling argument allow one to prove that

$$\int_{\Gamma_K} v_h^2 \lesssim \int_{\Gamma_K} v_{\mathrm{lm},K}^2 + \delta_0 h \int_K |\nabla v_h|^2, \tag{25}$$

for a certain $\delta_0 > 0$. Let us define now $\boldsymbol{\tau}^*_{h,K} = \boldsymbol{n}\, v_{\mathrm{lm},K}$, constant over K, $\boldsymbol{n}$ being the normal to Γ_K. We have that

$$\int_{\Gamma_K} \boldsymbol{\tau}^*_{h,K} \cdot \boldsymbol{n}\, v_h = \int_{\Gamma_K} v_{\mathrm{lm},K} v_h = v^2_{\mathrm{lm},K} |\Gamma_K| = \int_{\Gamma_K} v^2_{\mathrm{lm},K},$$

and therefore from (25) we get

$$\int_{\Gamma_K} v_h^2 \lesssim \int_{\Gamma_K} \boldsymbol{\tau}^*_{h,K} \cdot \boldsymbol{n}\, v_h + \delta_0 h \int_K |\nabla v_h|^2. \tag{26}$$

Let us define now $\boldsymbol{\tau}_{h,K} = \boldsymbol{n}\, v_{\mathrm{qm},K} \operatorname{sgn}(v_{\mathrm{lm},K})$ for K such that $K \cap \Gamma \neq \emptyset$, and $\boldsymbol{\tau}_{h,K} = \boldsymbol{0}$ otherwise. Using the fact that $v_{\mathrm{qm},K} \geq |v_{\mathrm{lm},K}|$ we can see that

$$\int_{\Gamma_K} \boldsymbol{\tau}_{h,K} \cdot \boldsymbol{n}\, v_h = v_{\mathrm{qm},K} \operatorname{sgn}(v_{\mathrm{lm},K}) \int_K v_h \geq |v_{\mathrm{lm},K}|^2 |\Gamma_K| = \int_{\Gamma_K} \boldsymbol{\tau}^*_{h,K} \cdot \boldsymbol{n}\, v_h.$$

Condition (23) follows using this in (26) and summing for all K. From the construction of $\boldsymbol{\tau}_h$, it is easily checked that it satisfies also (24). □

The second condition we need to prove stability is an inf-sup condition between R_h and V_h, the same as in the classical Galerkin approximation of Darcy's problem:

Assumption 2. *For all $v_h \in V_h$ there exists $\boldsymbol{r}_h \in R_h \setminus \{\boldsymbol{0}\}$ such that*

$$\|\nabla v_h\|_{L^2(\Omega)} \|\boldsymbol{r}_h\|_{L^2(\Omega)} \lesssim (\nabla v_h, \boldsymbol{r}_h)_\Omega. \tag{27}$$

Remark 3.1. *Since the finite element partition covers Ω_h, the classical stability condition for Darcy's problem in primal form would be $(\nabla v_h, \boldsymbol{r}_h)_{\Omega_h} \gtrsim \|\nabla v_h\|_{L^2(\Omega_h)} \|\boldsymbol{r}_h\|_{L^2(\Omega_h)} \gtrsim \|\nabla v_h\|_{L^2(\Omega)} \|\boldsymbol{r}_h\|_{L^2(\Omega)}$, instead of (27). However, for any discrete function f_h with positive integral over Ω and over Ω_h that converges (at least in $L^1(\Omega_h)$) to a non-zero function f, since $\mathrm{meas}(\Omega_h \setminus \Omega) \to 0$ as $h \to 0$ we have that*

$$\int_\Omega f_h = \int_{\Omega_h} f_h - \int_{\Omega_h \setminus \Omega} (f_h - f) - \int_{\Omega_h \setminus \Omega} f \gtrsim \int_{\Omega_h} f_h - \varphi(h),$$

where $\varphi(h)$ depends on f, not on f_h, and $\varphi(h) \to 0$ as $h \to 0$. Therefore, for h small enough we have that $\int_\Omega f_h \gtrsim \int_{\Omega_h} f_h$. We have used this in (27) for $f_h = \nabla v_h \cdot \boldsymbol{r}_h$.

For the primal problem we are considering, the continuous version of (27) is trivial to check, since we may simply take $\boldsymbol{r} = \nabla v$. However, at the discrete level we may wish to use different interpolations for u_h and $\boldsymbol{q}_h$, which have to satisfy (27).

The bilinear form associated to problem (20)-(22) is given by

$$\begin{aligned} B_{\mathrm{DP}}([\boldsymbol{q}_h, u_h, \boldsymbol{\sigma}_h], [\delta\boldsymbol{q}_h, \delta u_h, \delta\boldsymbol{\sigma}_h]) &= -\frac{1}{k}(\boldsymbol{q}_h, \delta\boldsymbol{q}_h)_\Omega - (\nabla u_h, \delta\boldsymbol{q}_h)_\Omega - (\nabla \delta u_h, \boldsymbol{q}_h)_\Omega \\ &- \langle \sigma_{n,h}, \delta u_h \rangle_\Gamma - \langle \delta\sigma_{n,h}, u_h \rangle_\Gamma - \frac{1}{N_0 k}(\delta\boldsymbol{\sigma}_h + \delta\boldsymbol{q}_h, \boldsymbol{\sigma}_h + \boldsymbol{q}_h)_\Omega. \end{aligned} \tag{28}$$

Theorem 2. *Suppose that Σ_h is made of discontinuous functions, that $N_0 > 1$ and that Assumption 2 holds. Then, the bilinear form in (28) is stable in the norm*

$$|||[\boldsymbol{r}, v, \boldsymbol{\tau}]|||_{\rm DP}^2 := \frac{1}{k}\|\boldsymbol{r}\|_{L^2(\Omega)}^2 + k\|\nabla v\|_{L^2(\Omega)}^2 + \frac{k}{h}\|v\|_{L^2(\Gamma)}^2 + \frac{1}{k}\|\boldsymbol{\tau}\|_{L^2(\Omega)}^2. \tag{29}$$

Moreover, if the solution of the continuous problem $\boldsymbol{U} = [\boldsymbol{q}, u, -\boldsymbol{q}]$ is such that $\boldsymbol{n}\cdot\boldsymbol{q}$ is bounded in $L^2(\Gamma)$, the solution $\boldsymbol{U}_h = [\boldsymbol{q}_h, u_h, \boldsymbol{\sigma}_h]$ of problem (20)-(22) satisfies the error estimate (16) in the norm (29) with the error function

$$\begin{aligned}\mathcal{E}_{\rm DP}(\boldsymbol{U}, h) &= k^{-1/2}\|\boldsymbol{q} - \tilde{\boldsymbol{q}}_h\|_{L^2(\Omega)} + k^{1/2}\|\nabla u - \nabla \tilde{u}_h\|_{L^2(\Omega)} \\ &\quad + k^{-1/2}\|\boldsymbol{q} + \tilde{\boldsymbol{\sigma}}_h\|_{L^2(\Omega)} + k^{-1/2}h^{1/2}\|\boldsymbol{n}\cdot\boldsymbol{q} + \boldsymbol{n}\cdot\tilde{\boldsymbol{\sigma}}_h\|_{L^2(\Gamma)} \\ &\quad + k^{1/2}h^{-1/2}\|u - \tilde{u}_h\|_{L^2(\Gamma)}\end{aligned} \tag{30}$$

for any $[\tilde{\boldsymbol{q}}_h, \tilde{u}_h, \tilde{\boldsymbol{\sigma}}_h] \in V_h \times R_h \times \Sigma_h$.

Proof. The stability condition (13) applied to the bilinear form (28) in the norm (29) is proved in [14]. Since the method is obviously consistent, we only need to check (14) and (15). The latter is trivial, and thus only the former needs to be verified. It is easily seen that all terms in $B_{\rm DP}([\boldsymbol{q} - \tilde{\boldsymbol{q}}_h, u - \tilde{u}_h, -\boldsymbol{q} - \tilde{\boldsymbol{\sigma}}_h], [\delta\boldsymbol{q}_h, \delta u_h, \delta\boldsymbol{\sigma}_h])$ can be bounded by $\mathcal{E}_{\rm DP}(\boldsymbol{U}, h)|||[\delta\boldsymbol{q}_h, \delta u_h, \delta\boldsymbol{\sigma}_h]|||_{\rm DP}$. Let us check this for the last two terms in the definition (28). First, we have that

$$\begin{aligned}\langle \delta\sigma_{n,h}, u - \tilde{u}_h\rangle_\Gamma &\lesssim k^{-1/2}h^{1/2}\|\boldsymbol{n}\cdot\delta\boldsymbol{\sigma}_h\|_{L^2(\Gamma)}k^{1/2}h^{-1/2}\|u - \tilde{u}_h\|_{L^2(\Gamma)} \\ &\lesssim k^{-1/2}\|\delta\boldsymbol{\sigma}_h\|_{L^2(\Omega)}k^{1/2}h^{-1/2}\|u - \tilde{u}_h\|_{L^2(\Gamma)} \\ &\lesssim \mathcal{E}_{\rm DP}(\boldsymbol{U}, h)|||[\delta\boldsymbol{q}_h, \delta u_h, \delta\boldsymbol{\sigma}_h]|||_{\rm DP},\end{aligned}$$

where the inverse estimate (17) has been used in the second step. For the last term in (28) we have

$$\begin{aligned}&-\frac{1}{N_0 k}(\delta\boldsymbol{\sigma}_h + \delta\boldsymbol{q}_h, (-\boldsymbol{q} - \tilde{\boldsymbol{\sigma}}_h) + (\boldsymbol{q} - \tilde{\boldsymbol{q}}_h))_\Omega \\ &\lesssim k^{-1/2}\left(\|\delta\boldsymbol{\sigma}_h\|_{L^2(\Omega)} + \|\delta\boldsymbol{q}_h\|_{L^2(\Omega)}\right)k^{-1/2}\left(\|\boldsymbol{q} + \tilde{\boldsymbol{\sigma}}_h\|_{L^2(\Omega)} + \|\boldsymbol{q} - \tilde{\boldsymbol{q}}_h\|_{L^2(\Omega)}\right) \\ &\lesssim \mathcal{E}_{\rm DP}(\boldsymbol{U}, h)|||[\delta\boldsymbol{q}_h, \delta u_h, \delta\boldsymbol{\sigma}_h]|||_{\rm DP}.\end{aligned}$$

The rest of terms can be bounded similarly. □

Remark 3.2. *It is seen from (30) that the error estimate obtained is optimal. In particular, if V_h is constructed with elements of order p_V, R_h with elements of order p_R and Σ_h with (discontinuous) elements of order p_Σ, the error in the norm (29) is of order $\min\{p_V, p_R + 1, p_\Sigma + 1\}$. Note that for stability we only require $p_\Sigma \geq 0$, but for optimal convergence it is needed that $p_\Sigma \geq p_V - 1$ (and obviously that $p_R \geq p_V - 1$).*

Let us note that the proof of Theorem 2 is particularly simple because of the consistency property. In [10], for example, a method to stabilize the Lagrange multiplier is proposed which has a consistency error that needs to be controlled. Moreover, the stabilization in [10] prevents from condensing the Lagrange multiplier at the element level, as it involves degrees of freedom on neighboring elements.

Finally, let us observe that $\boldsymbol{n}\cdot\boldsymbol{q}|_\Gamma$ needs to be in $L^2(\Gamma)$. This can only be guaranteed if $\boldsymbol{q} \in H^1(\Omega)^d$, whereas the primal problem is defined for $\boldsymbol{q} \in L^2(\Omega)^d$ only. This increment of regularity required does not affect convergence rate estimates provided by the error function (30), since those rates can only be obtained for $\boldsymbol{q}$ at least in $H^1(\Omega)^d$. However, some care is required if only convergence with minimal regularity (without rates of convergence) needs to be proved. We have not attempted this analysis in this work.

3.2.2. *Discrete dual form*

Problem (11)-(12) reads: find $[\boldsymbol{q}_h, u_h, \sigma_h] \in R_h \times V_h \times \Sigma_h$ such that

$$-\frac{1}{k}(\boldsymbol{q}_h, \delta\boldsymbol{q}_h)_\Omega + (u_h, \nabla\cdot\delta\boldsymbol{q}_h)_\Omega - \langle \sigma_h, \boldsymbol{n}\cdot\delta\boldsymbol{q}_h\rangle_\Gamma = \langle \boldsymbol{f}, \delta\boldsymbol{q}_h\rangle_\Omega, \tag{31}$$

$$(\delta u_h, \nabla\cdot\boldsymbol{q}_h)_\Omega - \frac{k}{N_0 L_0^2}(\delta u_h, \sigma_h - u_h)_\Omega = \langle g, \delta u_h\rangle_\Omega, \tag{32}$$

$$-\langle \delta\sigma_h, \boldsymbol{n}\cdot\boldsymbol{q}_h\rangle_\Gamma + \frac{k}{N_0 L_0^2}(\delta\sigma_h, \sigma_h - u_h)_\Omega = -\langle \delta\sigma_h, \bar{q}_n\rangle_\Gamma, \tag{33}$$

for all $\delta\boldsymbol{q}_h \in R_h$, $\delta u_h \in V_h$ and $\delta\sigma_h \in \Sigma_h$. This is the LLM for the dual form of Darcy's problem. As for the primal version, the linked Lagrange multiplier σ_h can be condensed at the element level from (33) if its interpolation is discontinuous. However, contrary to the primal form, this is not enough to prove stability (and hence convergence). More precisely, we will show below that one of the ingredients to prove stability that we would need is to assume that for all $\boldsymbol{r}_h \in R_h$ there exists $\tau_h \in \Sigma_h$ and $\delta_0 > 0$ such that

$$\|\boldsymbol{n}\cdot\boldsymbol{r}_h\|_{L^2(\Gamma)}^2 \lesssim \langle \tau_h, \boldsymbol{n}\cdot\boldsymbol{r}_h\rangle_\Gamma + \delta_0 h\|\nabla\cdot\boldsymbol{r}_h\|_{L^2(\Omega)}^2, \tag{34}$$

$$\|\tau_h\|_{L^2(\Gamma)} = \|\boldsymbol{n}\cdot\boldsymbol{r}_h\|_{L^2(\Gamma)}, \quad \|\tau_h\|_{L^2(\Omega)}^2 \lesssim h\|\boldsymbol{n}\cdot\boldsymbol{r}_h\|_{L^2(\Gamma)}^2. \tag{35}$$

However, this does not hold with the only condition that Σ_h is made of piecewise discontinuous functions, i.e., we do not have the counterpart of Lemma 1. For example, take $\boldsymbol{r}_h$ as piecewise linear, divergence-free and with zero mean over each $\Gamma_K = \Gamma\cap K$, $K \in \{\mathcal{T}_h\}$; there is no constant $\tau_h \in \Sigma_h$ satisfying (34), as the right-hand-side would be zero. A sufficient condition for (34)-(35) to hold is that the order of interpolation of Σ_h, p_Σ, is *at least* that of R_h, p_R (and functions in Σ_h are discontinuous). To check it, it suffices to take $\tau_h|_{\Gamma_K} = \boldsymbol{n}\cdot\boldsymbol{r}_h|_{\Gamma_K}$ for each $K \in \{\mathcal{T}_h\}$ and extend it to K by taking it constant in the direction of $\boldsymbol{n}$. In this case, $\delta_0 = 0$ in (34).

As for the primal problem, using the Galerkin method we also need the well posedness condition of Darcy's problem when boundary conditions are imposed in a classical way. We state it as follows:

Assumption 3. *The following two conditions hold:*

(1) *For all $v_h \in V_h$ there exists $\boldsymbol{r}_h \in R_h \setminus \{\boldsymbol{0}\}$ such that $\boldsymbol{n}\cdot\boldsymbol{r}_h = 0$ on $\partial\Omega_h$ and*

$$\|\boldsymbol{n}\cdot\boldsymbol{r}_h\|_{L^2(\Gamma)} = \varphi(h)h^{1/2}L_0^{-1}\|\boldsymbol{r}_h\|_{L^2(\Omega)},\ \textit{with}\ \varphi(h)\to 0\ \textit{as}\ h\to 0, \tag{36}$$

$$\|v_h\|_{L^2(\Omega)}\|\boldsymbol{r}_h\|_{H(\mathrm{div},\Omega)} \lesssim (v_h, \nabla\cdot\boldsymbol{r}_h)_\Omega. \tag{37}$$

(2) *For all $\boldsymbol{r}_h \in R_h$, there exists $v_h \in V_h \setminus \{0\}$ such that*

$$\|\nabla\cdot\boldsymbol{r}_h\|_{L^2(\Omega)}\|v_h\|_{L^2(\Omega)} \lesssim (\nabla\cdot\boldsymbol{r}_h, v_h)_\Omega. \tag{38}$$

Condition (36) implies that $\boldsymbol{n}\cdot\boldsymbol{r}_h$ can be chosen to tend to zero on Γ at a rate higher than $h^{1/2}$ (note that it is zero on $\partial\Omega_h$), which simply avoids pathological families $\{\mathcal{T}_h\}$, $h > 0$, in which there would be a non-uniform mesh refinement with the element sizes close to Γ going to zero at a rate much smaller than the rest of the mesh. Factor $L_0^{-1}\|\boldsymbol{r}_h\|_{L^2(\Omega)}$ has been introduced only for normalization purposes. Note that in inequality (37) integrals extend over Ω; see Remark 3.1. Condition (38) states that the term $(\boldsymbol{q}_h, \delta\boldsymbol{q}_h)_\Omega$ understood as a bilinear form in $R_h \times R_h$ is coercive in $Z_h = \{\boldsymbol{r}_h \in R_h \mid (v_h, \nabla\cdot\boldsymbol{r}_h)_\Omega = 0\ \forall v_h \in V_h\}$ with respect to the norm in $H(\mathrm{div},\Omega)$. Examples of pairs V_h-R_h that satisfy conditions (37) and (38) are the Raviart-Thomas or the Brezzi-Douglas-Marini elements (see [7], for example). In general, if $H_0(\mathrm{div};\Omega_h)$ is the space of vector fields in $H(\mathrm{div};\Omega_h)$ with zero normal trace on $\partial\Omega_h$, we assume that the spaces R_h and V_h are such that the following diagram

$$\begin{array}{ccc} H_0(\mathrm{div};\Omega_h) & \xrightarrow{\nabla\cdot} & L^2(\Omega_h) \\ \big\downarrow P_{R_h} & & \big\downarrow P_{V_h} \\ R_h & \xrightarrow{\nabla\cdot} & V_h \end{array}$$

commutes. This is part of the simplest de Rham's diagram. In fact, also the diagram with $H(\mathrm{div};\Omega_h)$ instead of $H_0(\mathrm{div};\Omega_h)$ commutes; we could have replaced (38) by the simpler condition $P_{V_h}(\nabla\cdot\boldsymbol{r}_h) = \nabla\cdot\boldsymbol{r}_h$ for all $\boldsymbol{r}_h \in R_h$, but the use of (38) highlights the parallelism in the proof of our results. When dealing with Maxwell's problem we will assume that the corresponding part of de Rham's diagram also commutes (see e.g. [7]).

The bilinear form associated to problem (31)-(33) is given by

$$\begin{aligned} B_{\mathrm{DD}}([\boldsymbol{q}_h, u_h, \sigma_h], [\delta\boldsymbol{q}_h, \delta u_h, \delta\sigma_h]) &= -\frac{1}{k}(\boldsymbol{q}_h, \delta\boldsymbol{q}_h)_\Omega + (u_h, \nabla\cdot\delta\boldsymbol{q}_h)_\Omega + (\delta u_h, \nabla\cdot\boldsymbol{q}_h)_\Omega \\ &- \langle\delta\sigma_h, \boldsymbol{n}\cdot\boldsymbol{q}_h\rangle_\Gamma - \langle\sigma_h, \boldsymbol{n}\cdot\delta\boldsymbol{q}_h\rangle_\Gamma + \frac{k}{N_0 L_0^2}(\delta\sigma_h - \delta u_h, \sigma_h - u_h)_\Omega. \end{aligned} \tag{39}$$

The stability and convergence result for the problem we are considering is the following:

Theorem 3. *Suppose that Σ_h is made of discontinuous functions of order $p_\Sigma \geq p_R$, that $N_0 > 1$ is large enough and that Assumption 3 holds. Then, the bilinear form in (39) is stable in the norm*

$$\begin{aligned} |\!|\!|[\boldsymbol{r}, v, \tau]|\!|\!|_{\mathrm{DD}}^2 := \frac{1}{k}\|\boldsymbol{r}\|_{L^2(\Omega)}^2 + \frac{L_0^2}{k}\|\nabla\cdot\boldsymbol{r}\|_{L^2(\Omega)}^2 + \frac{L_0^2}{kh}\|\boldsymbol{n}\cdot\boldsymbol{r}\|_{L^2(\Gamma)}^2 \\ + \frac{k}{L_0^2}\|u\|_{L^2(\Omega)}^2 + \frac{k}{L_0^2}\|\tau\|_{L^2(\Omega)}^2. \end{aligned} \tag{40}$$

Moreover, if the solution of the continuous problem is $\boldsymbol{U} = [\boldsymbol{q}, u, u]$, the solution $\boldsymbol{U}_h = [\boldsymbol{q}_h, u_h, \sigma_h]$ of problem (31)-(33) satisfies the error estimate (16) in the norm (40) with the error function

$$\begin{aligned} \mathcal{E}_{\mathrm{DD}}(\boldsymbol{U}, h) = k^{-1/2}\|\boldsymbol{q} - \tilde{\boldsymbol{q}}_h\|_{L^2(\Omega)} + k^{-1/2}L_0\|\nabla\cdot\boldsymbol{q} - \nabla\cdot\tilde{\boldsymbol{q}}_h\|_{L^2(\Omega)} \\ + k^{-1/2}h^{-1/2}L_0\|\boldsymbol{n}\cdot\boldsymbol{q} - \boldsymbol{n}\cdot\tilde{\boldsymbol{q}}_h\|_{L^2(\Gamma)} + k^{1/2}L_0^{-1}\|u - \tilde{u}_h\|_{L^2(\Omega)} \\ + k^{1/2}L_0^{-1}\|u - \tilde{\sigma}_h\|_{L^2(\Omega)} + k^{1/2}h^{1/2}L_0^{-1}\|u - \tilde{u}_h\|_{L^2(\Gamma)} \\ + k^{1/2}h^{1/2}L_0^{-1}\|u - \tilde{\sigma}_h\|_{L^2(\Gamma)} \end{aligned} \tag{41}$$

for any $[\tilde{\boldsymbol{q}}_h, \tilde{u}_h, \tilde{\boldsymbol{\sigma}}_h] \in V_h \times R_h \times \Sigma_h$.

Proof. The proof of the stability condition (13) follows the same steps as in [14]. The difference is that in the above reference we did not assume (36) and this led us to require some additional terms in (39) to control the gradients of σ_h.

We thus give the full proof for completeness. Let $\boldsymbol{U}_h = [\boldsymbol{q}_h, u_h, \sigma_h]$. For $N_0 > 1$ we have that

$$B_{\mathrm{DD}}(\boldsymbol{U}_h, [-\boldsymbol{q}_h, u_h, \sigma_h]) \gtrsim \frac{1}{k}\|\boldsymbol{q}_h\|^2_{L^2(\Omega)} + \frac{k}{N_0 L_0^2}\|\sigma_h - u_h\|^2_{L^2(\Omega)}. \tag{42}$$

Let v_h^q the element in V_h for which (38) holds for $\boldsymbol{r}_h = \boldsymbol{q}_h$, normalized so that $\|v_h^q\|_{L^2(\Omega)} = \|\nabla\cdot\boldsymbol{q}_h\|_{L^2(\Omega)}$. We have that

$$B_{\mathrm{DD}}(\boldsymbol{U}_h, [\boldsymbol{0}, k^{-1}L_0^2 v_h^q, 0]) \gtrsim \frac{L_0^2}{k}\|\nabla\cdot\boldsymbol{q}_h\|^2_{L^2(\Omega)} - \frac{k}{N_0 L_0^2}\|\sigma_h - u_h\|^2_{L^2(\Omega)}. \tag{43}$$

Let $\boldsymbol{r}_h^u$ be the function that allows one to guarantee (37) for $v_h = u_h$, normalized so that

$$\|\boldsymbol{r}_h^u\|_{H(\mathrm{div},\Omega)} := \|\nabla\cdot\boldsymbol{r}_h^u\|_{L^2(\Omega)} + \frac{1}{L_0}\|\boldsymbol{r}_h^u\|_{L^2(\Omega)} = \|u_h\|_{L^2(\Omega)}.$$

It is easily checked that

$$B_{\mathrm{DD}}(\boldsymbol{U}_h, [kL_0^{-2}\boldsymbol{r}_h^u, 0, \boldsymbol{0}]) \gtrsim \frac{k}{L_0^2}\|u_h\|^2_{L^2(\Omega)} - \frac{1}{k}\|\boldsymbol{q}_h\|^2_{L^2(\Omega)} - \frac{k}{L_0^2}\langle\sigma_h, \boldsymbol{n}\cdot\boldsymbol{r}_h^u\rangle_\Gamma.$$

The last term can be bounded as follows:

$$\begin{aligned}\langle\sigma_h, \boldsymbol{n}\cdot\boldsymbol{r}_h^u\rangle_\Gamma &\le \|\boldsymbol{n}\cdot\boldsymbol{r}_h^u\|_{L^2(\Gamma)}\|\sigma_h\|_{L^2(\Gamma)} \\ &\lesssim \varphi(h) h^{1/2} L_0^{-1}\|\boldsymbol{r}_h^u\|_{L^2(\Omega)} h^{-1/2}\|\sigma_h\|_{L^2(\Omega)} \qquad \text{(From (36) and (17))} \\ &\lesssim \varphi(h)\|u_h\|_{L^2(\Omega)}\|\sigma_h\|_{L^2(\Omega)},\end{aligned} \tag{44}$$

and therefore, for h small enough

$$B_{\mathrm{DD}}(\boldsymbol{U}_h, [kL_0^{-2}\boldsymbol{r}_h^u, 0, \boldsymbol{0}]) \gtrsim \frac{k}{L_0^2}\|u_h\|^2_{L^2(\Omega)} - \frac{1}{k}\|\boldsymbol{q}_h\|^2_{L^2(\Omega)} - \frac{k}{L_0^2}\varphi(h)\|\sigma_h\|^2_{L^2(\Omega)}. \tag{45}$$

Let us control the boundary term in (40), which can be done using (34)-(35) or using the fact that $p_\Sigma \ge p_R$, as explained above. If $\tau_h^q \in \Sigma_h$ is the function that allows one to guarantee (34) for $\boldsymbol{r}_h = \boldsymbol{q}_h$, we have that

$$\begin{aligned}&B_{\mathrm{DD}}(\boldsymbol{U}_h, [\boldsymbol{0}, 0, -L_0^2 k^{-1} h^{-1}\tau_h^q]) \\ &\quad\gtrsim \frac{L_0^2}{kh}(\|\boldsymbol{n}\cdot\boldsymbol{q}_h\|^2_{L^2(\Gamma)} - \delta_0 h\|\nabla\cdot\boldsymbol{q}\|^2_{L^2(\Omega)}) - \frac{1}{N_0 h^{1/2}}\|\boldsymbol{n}\cdot\boldsymbol{q}_h\|_{L^2(\Gamma)}\|\sigma_h - u_h\|_{L^2(\Omega)} \\ &\quad\gtrsim \frac{L_0^2}{kh}\|\boldsymbol{n}\cdot\boldsymbol{q}_h\|^2_{L^2(\Gamma)} - \frac{L_0^2}{k}\|\nabla\cdot\boldsymbol{q}\|^2_{L^2(\Omega)} - \frac{k}{N_0 L_0^2}\|\sigma_h - u_h\|^2_{L^2(\Omega)}.\end{aligned} \tag{46}$$

From (42), (43), (45) and (46) it follows that if we take as test function $\delta\boldsymbol{U}_h^0 = [-\boldsymbol{q}_h + \beta_1 k L_0^{-2}\boldsymbol{r}_h^u, u_h + \beta_2 k^{-1} L_0^2\, v_h^q, \sigma_h - \beta_3 L_0^2 k^{-1} h^{-1}\tau_h^q]$, for appropriate β_1, β_2 and β_3,

$$B_{\mathrm{DD}}(\boldsymbol{U}_h, \delta\boldsymbol{U}_h^0) \gtrsim |||\boldsymbol{U}_h|||^2_{\mathrm{DD}},$$

for h sufficiently small and N_0 sufficiently large. The proof of (13) concludes checking that $\|\!|\boldsymbol{U}_h|\!\|_{\rm DD} \gtrsim \|\!|\delta\boldsymbol{U}_h^0|\!\|_{\rm DD}$.

To prove the error estimate (16), we need to check (14) and (15). The latter follows directly form the definition of the norm (40) and the error function (41). For the former, it is easy to see that

$$\begin{aligned}
&B_{\rm DD}([\boldsymbol{q}-\tilde{\boldsymbol{q}}_h, u-\tilde{u}_h, u-\tilde{\sigma}_h],[\delta\boldsymbol{q}_h,\delta u_h,\delta\sigma_h]) \\
&\quad\lesssim \mathcal{E}_{\rm DD}(\boldsymbol{U},h)\|\!|[\delta\boldsymbol{q}_h,\delta u_h,\delta\sigma_h]|\!\|_{\rm DD} - \langle\delta\sigma_h, \boldsymbol{n}\cdot\boldsymbol{q}-\boldsymbol{n}\cdot\tilde{\boldsymbol{q}}_h\rangle_\Gamma \\
&\quad\lesssim \mathcal{E}_{\rm DD}(\boldsymbol{U},h)\|\!|[\delta\boldsymbol{q}_h,\delta u_h,\delta\sigma_h]|\!\|_{\rm DD} + h^{-1/2}\|\delta\sigma_h\|_{L^2(\Omega)}\|\boldsymbol{n}\cdot\boldsymbol{q}-\boldsymbol{n}\cdot\tilde{\boldsymbol{q}}_h\|_{L^2(\Gamma)}.
\end{aligned}$$

where the inverse estimate (17) has been used in the last step. □

Remark 3.3. *As for the primal problem, it is observed from (41) that the error estimate obtained is optimal. In fact, for accuracy reasons we would need $p_R = p_V + 1 = p_\Sigma + 1$ to obtain an estimate of order h^{p_R}, but for stability we need $p_\Sigma = p_R$. From the computational point of view, this has very little impact, as the degrees of freedom of σ_h can be condensed element-wise.*

4. Stokes' problem

Let us apply now the methodology described in Section 2 to the Stokes problem, which can model either stationary viscous fluids (at negligible Reynolds number) or linear incompressible elasticity. Using the terminology of fluid mechanics, let $\boldsymbol{u} : \Omega \longrightarrow \mathbb{R}^d$ be the velocity and $p : \Omega \longrightarrow \mathbb{R}$ the pressure of a fluid moving in Ω of (kinematic) viscosity $\nu > 0$ and subject to a body force $\boldsymbol{f}$. The problem to be solved consists of finding $\boldsymbol{u}$ and p as the solution to the boundary value problem

$$-2\nu\nabla\cdot\nabla^S\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{47}$$

$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{48}$$

$$\boldsymbol{u} = \bar{\boldsymbol{u}} \quad \text{on } \Gamma, \tag{49}$$

where $\bar{\boldsymbol{u}}$ is a given velocity prescribed on Γ and $\nabla^S\boldsymbol{u}$ stands for the symmetrical part of $\nabla\boldsymbol{u}$. Calling $\mathcal{L}([\boldsymbol{u},p]) = [-2\nu\nabla\cdot\nabla^S\boldsymbol{u}+\nabla p, \nabla\cdot\boldsymbol{u}]$, we have that

$$\langle\mathcal{L}([\boldsymbol{u},p],[\delta\boldsymbol{u},\delta p]\rangle_\Omega = B([\boldsymbol{u},p],[\delta\boldsymbol{u},\delta p]) - \langle\mathcal{F}_n[\boldsymbol{u},p],\mathcal{D}[\delta\boldsymbol{u},\delta p]\rangle_\Gamma,$$

where

$$\begin{aligned}
&B([\boldsymbol{u},p],[\delta\boldsymbol{u},\delta p]) = 2\nu(\nabla^S\boldsymbol{u},\nabla^S\delta\boldsymbol{u})_\Omega - (p,\nabla\cdot\delta\boldsymbol{u})_\Omega - (\delta p,\nabla\cdot\boldsymbol{u})_\Omega \\
&\mathcal{F}_n[\boldsymbol{u},p] = \boldsymbol{n}\cdot\mathcal{F}([\boldsymbol{u},p]), \quad \mathcal{F}([\boldsymbol{u},p]) = -p\boldsymbol{I} + 2\nu\nabla^S\boldsymbol{u}, \\
&\mathcal{D}[\delta\boldsymbol{u},\delta p] = \delta\boldsymbol{u},
\end{aligned}$$

where $\boldsymbol{I}$ is the identity in $\mathbb{R}^d$. The problem is well posed in the space $X = V\times Q = H^1(\Omega)^d \times L^2(\Omega)/\mathbb{R}$, and the space of traces is $\Lambda = H^{1/2}(\Gamma)^d$, the trace operator being $[\boldsymbol{u},p]\mapsto\boldsymbol{u}|_\Gamma$.

Let $V_h \subset V$ and $Q_h \subset Q$ be the finite element spaces to approximate V and Q, respectively, and let Σ_h be the space of the new variable to be introduced for enforcing (49) weakly. Using the ideas presented in Section 2 and choosing

$N = 2N_0\nu$, with N_0 dimensionless, the method we propose consists of optimizing the functional

$$\hat{G}([\boldsymbol{v}_h, q_h, \boldsymbol{\xi}_h]) = F([\boldsymbol{v}_h, q_h]) - \langle \boldsymbol{n}\cdot\boldsymbol{\xi}_h, \boldsymbol{v}_h - \bar{\boldsymbol{u}}\rangle_\Gamma - \frac{1}{4N_0\nu}\|\boldsymbol{\xi}_h + q_h\boldsymbol{I} - 2\nu\nabla^S\boldsymbol{v}_h\|^2_{L^2(\Omega)},$$

over $V_h \times Q_h \times \Sigma_h$, where

$$F([\boldsymbol{v}, q]) = \nu\|\nabla^S\boldsymbol{v}\|^2_{L^2(\Omega)} - (q, \nabla\cdot\boldsymbol{v})_\Omega - \langle \boldsymbol{f}, \boldsymbol{v}\rangle_\Omega,$$

which is well defined if $\boldsymbol{f} \in V'$, the dual of V. Problem (11)-(12) now reads: find $[\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h] \in V_h \times Q_h \times \Sigma_h$ such that

$$\begin{aligned} 2\nu(\nabla^S\boldsymbol{u}_h, \nabla^S\delta\boldsymbol{u}_h)_\Omega - (p_h, \nabla\cdot\delta\boldsymbol{u}_h)_\Omega - \langle \boldsymbol{n}\cdot\boldsymbol{\sigma}_h, \delta\boldsymbol{u}_h\rangle_\Gamma \\ -\frac{1}{2N_0\nu}(-2\nu\nabla^S\delta\boldsymbol{u}_h, \boldsymbol{\sigma}_h + p_h\boldsymbol{I} - 2\nu\nabla^S\boldsymbol{u}_h)_\Omega = \langle \boldsymbol{f}, \delta\boldsymbol{u}_h\rangle_\Omega, \end{aligned} \tag{50}$$

$$-(\delta p_h, \nabla\cdot\boldsymbol{u}_h)_\Omega - \frac{1}{2N_0\nu}(\delta p_h\boldsymbol{I}, \boldsymbol{\sigma}_h + p_h\boldsymbol{I} - 2\nu\nabla^S\boldsymbol{u}_h)_\Omega = 0, \tag{51}$$

$$-\langle \boldsymbol{n}\cdot\delta\boldsymbol{\sigma}_h, \boldsymbol{u}_h\rangle_\Gamma - \frac{1}{2N_0\nu}(\delta\boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h + p_h\boldsymbol{I} - 2\nu\nabla^S\boldsymbol{u}_h)_\Omega = -\langle \boldsymbol{n}\cdot\delta\boldsymbol{\sigma}_h, \bar{\boldsymbol{u}}\rangle_\Gamma, \tag{52}$$

for all $[\delta\boldsymbol{u}_h, \delta p_h, \delta\boldsymbol{\sigma}_h] \in V_h \times Q_h \times \Sigma_h$.

In [14] we considered a slightly different version of the problem, with very similar stability and convergence properties, which was motivated by the method originally proposed in [4]. In [14], the linked Lagrange multiplier $\boldsymbol{\sigma}_h$ is an approximation of only the viscous (deviatoric) part of the stress and some additional terms involving the pressure are added to have consistency, whereas in the method we consider now it also includes the pressure (spheric) component. Problem (50)-(52) is in fact what results from the straightforward application of the concepts in Section 2. Because of the differences explained, we shall provide below a full stability and convergence analysis.

Similarly to Darcy's problem, to prove stability we shall need two conditions, one between Σ_h and V_h and the other the classical inf-sup condition between V_h and Q_h. As for Darcy's problem in primal form, the former holds as soon as Σ_h is made of discontinuous functions, with no further requirements. Indeed, we have the following counterpart of Lemma 1:

Lemma 2. *If Σ_h is made of piecewise discontinuous (tensor) functions, for all $\boldsymbol{v}_h \in V_h$ there exists $\boldsymbol{\tau}_h \in \Sigma_h$ and $\delta_0 > 0$ such that*

$$\|\boldsymbol{v}_h\|^2_{L^2(\Gamma)} \lesssim \langle \boldsymbol{\tau}_h\cdot\boldsymbol{n}, \boldsymbol{v}_h\rangle_\Gamma + \delta_0 h\|\nabla^S\boldsymbol{v}_h\|^2_{L^2(\Omega)}, \tag{53}$$

$$\|\boldsymbol{\tau}_h\|_{L^2(\Gamma)} = \|\boldsymbol{v}_h\|_{L^2(\Gamma)}, \quad \|\boldsymbol{\tau}_h\|^2_{L^2(\Omega)} \lesssim h\|\boldsymbol{v}_h\|^2_{L^2(\Gamma)}. \tag{54}$$

Proof. Let $\mathrm{Ker}_h\nabla^S := \{\boldsymbol{v}_h \in V_h \mid \boldsymbol{v}_h|_K = \boldsymbol{a}_K + \boldsymbol{\omega}_K \times \boldsymbol{x},\ K \in \mathcal{T}_h\}$, with $\boldsymbol{a}_K$ and $\boldsymbol{\omega}_K$ constant vectors in each K and $\boldsymbol{x}$ being the position vector. For $\boldsymbol{v}_h \in \mathrm{Ker}_h^\perp\nabla^S$, the result holds because of Lemma 1 and Korn's inequality, i.e., $\|\nabla\boldsymbol{v}_h\|^2_{L^2(\Omega)} \lesssim \|\nabla^S\boldsymbol{v}_h\|^2_{L^2(\Omega)}$. Since the space of normal components of elements in Σ_h is dense in the space of piecewise linear polynomials $\mathrm{Ker}_h\nabla^S \subset \mathcal{C}^0(\Omega)$, for all $\boldsymbol{v}_h \in \mathrm{Ker}_h\nabla^S$ there exists $\boldsymbol{\tau}_h \in \Sigma_h$ such that

$\|\boldsymbol{v}_h - \boldsymbol{\tau}_h \cdot \boldsymbol{n}\|_{L^2(\Gamma)} \lesssim \varphi(h)\|\boldsymbol{v}_h\|_{L^2(\Gamma)}$, with $\varphi(h) \to 0$ as $h \to 0$. Thus

$$\begin{aligned}\|\boldsymbol{v}_h\|^2_{L^2(\Gamma)} &= \langle \boldsymbol{\tau}_h \cdot \boldsymbol{n}, \boldsymbol{v}_h\rangle_\Gamma + \langle \boldsymbol{v}_h - \boldsymbol{\tau}_h \cdot \boldsymbol{n}, \boldsymbol{v}_h\rangle_\Gamma \\ &\lesssim \langle \boldsymbol{\tau}_h \cdot \boldsymbol{n}, \boldsymbol{v}_h\rangle_\Gamma + \varphi(h)\|\boldsymbol{v}_h\|^2_{L^2(\Gamma)},\end{aligned}$$

from where for h small enough it follows that (53) holds also for elements in $\mathrm{Ker}_h \nabla^S$ with $\delta_0 = 0$. Normalizing $\boldsymbol{\tau}_h \cdot \boldsymbol{n}$ we also have (54). □

The second condition is that the velocity-pressure pair V_h-Q_h is inf-sup stable for the problem imposing the essential boundary conditions in a classical way on Ω_h:

Assumption 4. *For all $q_h \in Q_h$ there exists $\boldsymbol{v}_h \in V_h \setminus \{\boldsymbol{0}\}$ such that $\boldsymbol{v}_h = \boldsymbol{0}$ on $\partial\Omega_h$ and*

$$\|\boldsymbol{v}_h\|_{L^2(\Gamma)} = \varphi(h)h^{1/2}L_0^{-1}\|\boldsymbol{v}_h\|_{L^2(\Omega)}, \text{ with } \varphi(h) \to 0 \text{ as } h \to 0, \tag{55}$$

$$\|q_h\|_{L^2(\Omega)}\|\boldsymbol{v}_h\|_{H^1(\Omega)} \lesssim -(q_h, \nabla \cdot \boldsymbol{v}_h)_\Omega. \tag{56}$$

As for Assumptions 2 and 3, in inequality (56) integrals extend over Ω; see Remark 3.1. Instead this assumption between V_h and Q_h, we could have used stabilized finite element methods (see [2, 14]).

Let $\boldsymbol{U}_h = [\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h]$, $\delta\boldsymbol{U}_h = [\delta\boldsymbol{u}_h, \delta p_h, \delta\boldsymbol{\sigma}_h]$. The bilinear form of this problem is:

$$\begin{aligned} B_{\mathrm{S}}(\boldsymbol{U}_h, \delta\boldsymbol{U}_h) &= 2\nu(\nabla^S \boldsymbol{u}_h, \nabla^S \delta\boldsymbol{u}_h)_\Omega - (p_h, \nabla \cdot \delta\boldsymbol{u}_h)_\Omega - (\delta p_h, \nabla \cdot \boldsymbol{u}_h)_\Omega \\ &- \langle \boldsymbol{n} \cdot \boldsymbol{\sigma}_h, \delta\boldsymbol{u}_h\rangle_\Gamma - \langle \boldsymbol{n} \cdot \delta\boldsymbol{\sigma}_h, \boldsymbol{u}_h\rangle_\Gamma \\ &- \frac{1}{2N_0\nu}(\delta\boldsymbol{\sigma}_h + \delta p_h\boldsymbol{I} - 2\nu\nabla^S\delta\boldsymbol{u}_h, \boldsymbol{\sigma}_h + p_h\boldsymbol{I} - 2\nu\nabla^S\boldsymbol{u}_h)_\Omega. \end{aligned} \tag{57}$$

Our stability and convergence result is the following:

Theorem 4. *Suppose that Σ_h is made of discontinuous functions, that $N_0 > 1$ is sufficiently large and that Assumption 4 holds. Then, for h sufficiently small the bilinear form in (57) is stable in the norm*

$$|||[\boldsymbol{v}, q, \boldsymbol{\tau}]|||^2_{\mathrm{S}} := \nu\|\nabla^S\boldsymbol{v}\|^2_{L^2(\Omega)} + \frac{1}{\nu}\|q\|^2_{L^2(\Omega)} + \frac{\nu}{h}\|\boldsymbol{v}\|^2_{L^2(\Gamma)} + \frac{1}{\nu}\|\boldsymbol{\tau}\|^2_{L^2(\Omega)}. \tag{58}$$

Moreover, if $\boldsymbol{U} = [\boldsymbol{u}, p, \boldsymbol{\sigma}]$ is the solution of the continuous problem, with $\boldsymbol{\sigma} := -p\boldsymbol{I} + 2\nu\nabla^S\boldsymbol{u}$, the solution $\boldsymbol{U}_h = [\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h]$ of problem (50)-(52) satisfies the error estimate (16) in the norm (58) with the error function

$$\begin{aligned}\mathcal{E}_{\mathrm{S}}(\boldsymbol{U}, h) &= \nu^{1/2}\|\nabla^S\boldsymbol{u} - \nabla^S\tilde{\boldsymbol{u}}_h\|_{L^2(\Omega)} + \nu^{-1/2}\|p - \tilde{p}_h\|_{L^2(\Omega)} + \nu^{-1/2}\|\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h\|_{L^2(\Omega)} \\ &+ \nu^{1/2}h^{-1/2}\|\boldsymbol{u} - \tilde{\boldsymbol{u}}_h\|_{L^2(\Gamma)} + \nu^{-1/2}h^{1/2}\|\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h\|_{L^2(\Gamma)}\end{aligned} \tag{59}$$

for any $[\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{\boldsymbol{\sigma}}_h] \in V_h \times Q_h \times \Sigma_h$.

Proof. Let us start noting that, for $N_0 > 1$,

$$B_{\mathrm{S}}(\boldsymbol{U}_h, [\boldsymbol{u}_h, -p_h, -\boldsymbol{\sigma}_h])$$

$$= 2\nu\|\nabla^S \boldsymbol{u}_h\|^2_{L^2(\Omega)} + \frac{1}{2N_0\nu}\Big(\|\boldsymbol{\sigma}_h + p_h\boldsymbol{I}\|^2_{L^2(\Omega)} - 4\nu^2\|\nabla^S \boldsymbol{u}_h\|^2_{L^2(\Omega)}\Big)$$
$$\gtrsim \nu\|\nabla^S \boldsymbol{u}_h\|^2_{L^2(\Omega)} + \frac{1}{N_0\nu}\|\boldsymbol{\sigma}_h + p_h\boldsymbol{I}\|^2_{L^2(\Omega)}. \tag{60}$$

Let now $\boldsymbol{\tau}_h^u$ be the function that verifies (53)-(54) for $\boldsymbol{v}_h = \boldsymbol{u}_h$. We have that

$$B_{\rm S}(\boldsymbol{U}_h, [\boldsymbol{0}, 0, -\nu h^{-1}\boldsymbol{\tau}_h^u])$$
$$\gtrsim \frac{\nu}{h}\|\boldsymbol{u}_h\|^2_{L^2(\Gamma)} - \nu\|\nabla^S \boldsymbol{u}_h\|^2_{L^2(\Omega)} - \frac{1}{2N_0 h^{1/2}}\|\boldsymbol{u}_h\|_{L^2(\Gamma)}\|\boldsymbol{\sigma}_h + p_h\boldsymbol{I} - 2\nu\nabla^S\boldsymbol{u}_h\|_{L^2(\Omega)}$$
$$\gtrsim \frac{\nu}{h}\|\boldsymbol{u}_h\|^2_{L^2(\Gamma)} - \nu\|\nabla^S \boldsymbol{u}_h\|^2_{L^2(\Omega)} - \frac{1}{N_0\nu}\|\boldsymbol{\sigma}_h + p_h\boldsymbol{I}\|^2_{L^2(\Omega)}. \tag{61}$$

In order to get control on the pressure, we make use of Assumption 4. Let $\boldsymbol{v}_h^p$ be the element in V_h for which (56) holds for $q_h = p_h$. Normalizing it so that $\|\boldsymbol{v}_h^p\|_{H^1(\Omega)} = \|\nabla \boldsymbol{v}_h^p\|_{L^2(\Omega)} + L_0^{-1}\|\boldsymbol{v}_h^p\|_{L^2(\Omega)} = \nu^{-1}\|p_h\|_{L^2(\Omega)}$, we have

$$B_{\rm S}(\boldsymbol{U}_h, [\boldsymbol{v}_h^p, 0, \boldsymbol{0}]) \gtrsim \frac{1}{\nu}\|p_h\|^2_{L^2(\Omega)} - \nu\|\nabla^S \boldsymbol{u}_h\|^2_{L^2(\Omega)}$$
$$- \frac{1}{N_0\nu}\|\boldsymbol{\sigma}_h + p_h\boldsymbol{I}\|^2_{L^2(\Omega)} - \langle \boldsymbol{n}\cdot\boldsymbol{\sigma}_h, \boldsymbol{v}^p\rangle_\Gamma. \tag{62}$$

The last term can be bounded using (55) as the boundary term in (44) and get

$$\int_\Gamma (\boldsymbol{n}\cdot\boldsymbol{\sigma}_h)\cdot\boldsymbol{v}_h^p \lesssim \varphi(h)\frac{1}{L_0}\|\boldsymbol{v}_h^p\|_{L^2(\Omega)}\|\boldsymbol{\sigma}_h\|_{L^2(\Omega)}$$
$$\lesssim \varphi(h)\frac{1}{\nu}\|p_h\|_{L^2(\Omega)}(\|\boldsymbol{\sigma}_h + p_h\boldsymbol{I}\|_{L^2(\Omega)} + \|p_h\|_{L^2(\Omega)}),$$

which when used in (62) yields

$$B_{\rm S}(\boldsymbol{U}_h, [\boldsymbol{v}_h^p, 0, \boldsymbol{0}]) \gtrsim \frac{1}{\nu}\|p_h\|^2_{L^2(\Omega)} - \nu\|\nabla^S \boldsymbol{u}_h\|^2_{L^2(\Omega)} - \frac{1}{N_0\nu}\|\boldsymbol{\sigma}_h + p_h\boldsymbol{I}\|^2_{L^2(\Omega)}, \tag{63}$$

for h small enough.

Let $\delta\boldsymbol{U}_h = [\boldsymbol{u}_h + \beta_1\boldsymbol{v}_h^p, -p_h, -\boldsymbol{\sigma}_h - \beta_2\nu h^{-1}\boldsymbol{\tau}_h^u]$. From (60), (61) and (63) it follows that there exists β_1 and β_2 for which $B_{\rm S}(\boldsymbol{U}_h, \delta\boldsymbol{U}_h) \gtrsim \|\!|\boldsymbol{U}_h|\!\|^2_{\rm S}$ for N_0 sufficiently large. Checking that $\|\!|\boldsymbol{U}_h|\!\|_{\rm S} \gtrsim \|\!|\delta\boldsymbol{U}_h|\!\|_{\rm S}$ concludes the proof of the stability property (13).

Let us move to the approximation properties (14) and (15) that yield the error estimate (16). The second one is trivially checked for the error function (59), and thus only (14) needs to be verified. As in the proof of Theorems 2 and 3, this is easily done using the inverse estimate (17). Details are omitted. □

Similar observations as in Remarks 3.2 and 3.3 regarding the optimality of the error estimate can be made for the Stokes problem.

5. Maxwell's problem

As a last application of the LLM technique, let us consider Maxwell's problem written in the following form: find a magnetic induction field $\boldsymbol{u} : \Omega \longrightarrow \mathbb{R}^d$ and a scalar field $p : \Omega \longrightarrow \mathbb{R}$ solution of the boundary value problem

$$\nu\nabla \times \nabla \times \boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{64}$$
$$-\nabla \cdot \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{65}$$
$$\boldsymbol{n} \times \boldsymbol{u} = \boldsymbol{n} \times \bar{\boldsymbol{u}} \quad \text{on } \Gamma, \tag{66}$$
$$p = \bar{p} := 0 \quad \text{on } \Gamma, \tag{67}$$

where $\nu > 0$ is a physical parameter (the inverse of the magnetic permeability times the electric conductivity), $\bar{\boldsymbol{u}}$ is given and $\boldsymbol{f}$ is assumed to be solenoidal. As usual, the scalar field p is introduced to impose that the finite element approximation to $\boldsymbol{u}$ be solenoidal, since at the continuous level the solution is $p = 0$. We call this variable pseudo-magnetic pressure.

For generality, we will consider that (67) needs to be prescribed weakly, even though in the finite element approximation it could be easily prescribed in an essential manner by fixing the nodal values of the approximate solution p_h to zero at the nodes closest to Γ; note that we only need to converge to the continuous solution $p = 0$. We could also replace the boundary condition (67) by $\boldsymbol{n} \cdot \nabla p = 0$ on Γ and add the restriction $\int_\Omega p = 0$ to render the solution unique.

The differential operator of the problem is now $\mathcal{L}([\boldsymbol{u}, p]) = [\nu\nabla \times \nabla \times \boldsymbol{u} + \nabla p, -\nabla \cdot \boldsymbol{u}]$. After appropriate integration by parts we get the identity

$$\langle \mathcal{L}([\boldsymbol{u}, p], [\delta\boldsymbol{u}, \delta p]\rangle_\Omega = B([\boldsymbol{u}, p], [\delta\boldsymbol{u}, \delta p]) - \langle \mathcal{F}_n[\boldsymbol{u}, p], \mathcal{D}[\delta\boldsymbol{u}, \delta p]\rangle_\Gamma,$$

where

$$B([\boldsymbol{u}, p], [\delta\boldsymbol{u}, \delta p]) = \nu(\nabla \times \boldsymbol{u}, \nabla \times \delta\boldsymbol{u})_\Omega + (\nabla p, \delta\boldsymbol{u})_\Omega + (\nabla\delta p, \boldsymbol{u})_\Omega$$
$$\mathcal{F}_n[\boldsymbol{u}, p] = [\nu P_t(\nabla \times \boldsymbol{u}), \boldsymbol{n} \cdot \boldsymbol{u}], \quad \mathcal{F}([\boldsymbol{u}, p]) = [\nu\nabla \times \boldsymbol{u}, \boldsymbol{u}],$$
$$\mathcal{D}[\delta\boldsymbol{u}, \delta p] = [\boldsymbol{n} \times \delta\boldsymbol{u}, \delta p],$$

and we have introduced the tangent projection P_t on the boundary Γ, defined for any vector field $\boldsymbol{a}$ as $P_t(\boldsymbol{a}) = \boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{n})\boldsymbol{n}$. Note that we could also have defined $\mathcal{F}_n[\boldsymbol{u}, p] = [\nu\, \boldsymbol{n} \times \nabla \times \boldsymbol{u}, \boldsymbol{n} \cdot \boldsymbol{u}]$, $\mathcal{D}[\delta\boldsymbol{u}, \delta p] = [P_t(\delta\boldsymbol{u}), \delta p]$; the expression chosen is due to the boundary conditions (67) that we wish to impose.

The problem is well posed in the space $X = V \times Q := H(\text{curl}; \Omega) \times H^1(\Omega)$, where $H(\text{curl}; \Omega)$ is the space of vector fields in $L^2(\Omega)^d$ with curl in $L^2(\Omega)^d$. The subspace made of vectors $\boldsymbol{v} \in H(\text{curl}; \Omega)$ such that $\boldsymbol{n} \times \boldsymbol{v} = \boldsymbol{0}$ on $\partial\Omega$ will be denoted by $H_0(\text{curl}; \Omega)$. The space of traces is $\Lambda = H^{-1/2}(\text{div}_\Gamma; \Gamma) \times H^{1/2}(\Gamma)$, the trace operator being $[\boldsymbol{u}, p] \mapsto [\boldsymbol{n} \times \boldsymbol{u}, p]$; for a characterization of $H^{-1/2}(\text{div}_\Gamma; \Gamma)$ for polyhedral domains, see [8] and references therein.

Let $V_h \subset V$ and $Q_h \subset Q$ be finite element spaces to approximate V and Q, respectively. Assumptions on these spaces will be stated later. Now we need to deal with *two* essential boundary conditions, i.e. (66) and (67), and therefore we will need to introduce two linked Lagrange multipliers, $\boldsymbol{\sigma}_{u,h}$ and $\boldsymbol{\sigma}_{p,h}$. Let $\Sigma_{u,h}$ and $\Sigma_{p,h}$ be the finite element spaces where they belong, respectively. The formulation we propose to enforce (66) and (67) consists in optimizing the functional

$$\begin{aligned}\hat{G}([\boldsymbol{v}_h, q_h, \boldsymbol{\xi}_{u,h}, \boldsymbol{\xi}_{p,h}]) &= F([\boldsymbol{v}_h, q_h]) \\ &- \langle P_t(\boldsymbol{\xi}_{u,h}), \boldsymbol{n} \times \boldsymbol{v}_h - \boldsymbol{n} \times \bar{\boldsymbol{u}} \rangle_\Gamma - \langle \boldsymbol{n} \cdot \boldsymbol{\xi}_{p,h}, q_h - \bar{p} \rangle_\Gamma \\ &- \frac{1}{2\nu N_{u,0}} \|\boldsymbol{\xi}_{u,h} - \nu \nabla \times \boldsymbol{v}_h\|^2_{L^2(\Omega)} + \frac{\nu}{2N_{p,0}L_0^2} \|\boldsymbol{\xi}_{p,h} - \boldsymbol{v}_h\|^2_{L^2(\Omega)} \end{aligned} \tag{68}$$

over $V_h \times Q_h \times \Sigma_{u,h} \times \Sigma_{p,h}$, where $N_{u,0} > 0$ and $N_{p,0} > 0$ are dimensionless numerical parameters and

$$F([\boldsymbol{v}, q]) = \frac{\nu}{2} \|\nabla \times \boldsymbol{v}\|^2_{L^2(\Omega)} + (\nabla q, \boldsymbol{v})_\Omega - \langle \boldsymbol{f}, \boldsymbol{v} \rangle_\Omega.$$

As we shall see, the sign of the last term in (68) is determined by stability reasons.

Problem (11)-(12) now reads: find $[\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_{u,h}, \boldsymbol{\sigma}_{p,h}] \in V_h \times Q_h \times \Sigma_{u,h} \times \Sigma_{p,h}$ such that

$$\begin{aligned} &\nu(\nabla \times \boldsymbol{u}_h, \nabla \times \delta \boldsymbol{u}_h)_\Omega + (\nabla p_h, \delta \boldsymbol{u}_h)_\Omega - \langle P_t(\boldsymbol{\sigma}_{u,h}), \boldsymbol{n} \times \delta \boldsymbol{u}_h \rangle_\Gamma \\ &\quad - \frac{1}{\nu N_{u,0}} (\boldsymbol{\sigma}_{u,h} - \nu \nabla \times \boldsymbol{u}_h, -\nu \nabla \times \delta \boldsymbol{u}_h)_\Omega + \frac{\nu}{N_{p,0} L_0^2} (\boldsymbol{\sigma}_{p,h} - \boldsymbol{u}_h, -\delta \boldsymbol{u}_h)_\Omega \\ &\quad = \langle \boldsymbol{f}, \delta \boldsymbol{u}_h \rangle_\Omega, \end{aligned} \tag{69}$$

$$(\nabla \delta p_h, \boldsymbol{u}_h)_\Omega - \langle \boldsymbol{n} \cdot \boldsymbol{\sigma}_{p,h}, \delta p_h \rangle_\Gamma = 0, \tag{70}$$

$$\begin{aligned} &- \langle P_t(\delta \boldsymbol{\sigma}_{u,h}), \boldsymbol{n} \times \boldsymbol{u}_h \rangle_\Gamma - \frac{1}{\nu N_{u,0}} (\boldsymbol{\sigma}_{u,h} - \nu \nabla \times \boldsymbol{u}_h, \delta \boldsymbol{\sigma}_{u,h})_\Omega \\ &\quad = -\langle P_t(\delta \boldsymbol{\sigma}_{u,h}), \boldsymbol{n} \times \bar{\boldsymbol{u}} \rangle_\Gamma, \end{aligned} \tag{71}$$

$$- \langle \boldsymbol{n} \cdot \delta \boldsymbol{\sigma}_{p,h}, p_h \rangle_\Gamma + \frac{\nu}{N_{p,0} L_0^2} (\boldsymbol{\sigma}_{p,h} - \boldsymbol{u}_h, \delta \boldsymbol{\sigma}_{p,h})_\Omega = -\langle \boldsymbol{n} \cdot \delta \boldsymbol{\sigma}_{p,h}, \bar{p} \rangle_\Gamma, \tag{72}$$

for all $[\delta \boldsymbol{u}_h, \delta p_h, \delta \boldsymbol{\sigma}_{u,h}, \delta \boldsymbol{\sigma}_{p,h}] \in V_h \times Q_h \times \Sigma_{u,h} \times \Sigma_{p,h}$.

Let us introduce the conditions we need on the finite element spaces to prove that the solution to this problem is stable and converges to the solution of the continuous problem.

As for the dual form of Darcy's problem, it is not possible to obtain a result as the counterpart of Lemma 1 for V_h. Instead of this, and even if the analogous on Lemma 1 holds for Q_h, we will simply assume that

Assumption 5. *The order of the elements in $\Sigma_{u,h}$ is at least that of the elements in V_h, and the order of the elements in $\Sigma_{p,h}$ is at least that of the elements in Q_h. Both $\Sigma_{u,h}$ and $\Sigma_{p,h}$ are made of discontinuous functions, and thus for each $\boldsymbol{v}_h \in V_h$ and each $q_h \in Q_h$ we may respectively construct $\boldsymbol{\tau}_{u,h} \in \Sigma_{u,h}$ and $\boldsymbol{\tau}_{p,h} \in \Sigma_{p,h}$ such that*

$$\|\boldsymbol{n} \times \boldsymbol{v}_h\|^2_{L^2(\Gamma)} \lesssim \langle P_t(\boldsymbol{\tau}_{u,h}), \boldsymbol{n} \times \boldsymbol{v}_h \rangle_\Gamma, \quad \|\boldsymbol{\tau}_{u,h}\|^2_{L^2(\Omega)} \lesssim h \|\boldsymbol{n} \times \boldsymbol{v}_h\|^2_{L^2(\Gamma)}, \tag{73}$$

$$\|q_h\|^2_{L^2(\Gamma)} \lesssim \langle \boldsymbol{n} \cdot \boldsymbol{\tau}_{p,h}, q_h \rangle_\Gamma, \quad \|\boldsymbol{\tau}_{p,h}\|^2_{L^2(\Omega)} \lesssim h \|q_h\|^2_{L^2(\Gamma)}. \tag{74}$$

The next assumption we need is that spaces V_h and Q_h yield a well posed problem using the Galerkin method when the boundary conditions are prescribed in the standard way on $\partial\Omega_h$. A very important property of the Maxwell problem is that there are situations of interest in which the solution belongs strictly to X, and not to smoother spaces in spite of having smooth data; this happens for example when Ω has re-entrant corners [15], and the solutions obtained are called *singular*. These solutions cannot be approximated if V_h is approximated using continuous functions with uniformly bounded divergence in $L^2(\Omega)$, since this would imply that the solution would belong to $H^1(\Omega)^d$ (see [16, 24], for example). One possibility to overcome this difficulty is to use discontinuous Galerkin methods [21, 22, 26]. Here however we shall restrict ourselves to conforming approximations; these can be achieved for example using Nédelec's elements for V_h and continuous Lagrangian interpolations for Q_h [25, 30]. An alternative would be to use stabilized finite element methods as in [3], although we shall not exploit this possibility in this paper.

Let us state the compatibility requirement between V_h and Q_h that we need. Let $G_h := \{\nabla q_h \mid q_h \in Q_h,\ q_h = 0 \text{ on } \partial\Omega_h\}$, and let $G^\perp_{V_h}$ be the $L^2(\Omega)$-orthogonal complement in V_h, so that $V_h = G_h \oplus G^\perp_{V_h}$. Each $\boldsymbol{v}_h \in V_h$ can be uniquely decomposed as $\boldsymbol{v}_h = \boldsymbol{v}^0_h + \nabla s_h$, with $\boldsymbol{v}^0_h \in G^\perp_{V_h}$ and $s_h \in Q_h$, $s_h = 0$ on $\partial\Omega_h$. We assume that

Assumption 6. *The following two conditions hold:*

(1) *In the decomposition $\boldsymbol{v}_h = \boldsymbol{v}^0_h + \nabla s_h$, there holds:*

$$\|s_h\|_{L^2(\Gamma)} = \varphi(h) h^{1/2} \|\nabla s_h\|_{L^2(\Omega)}, \quad \textit{with } \varphi(h) \to 0 \textit{ as } h \to 0, \tag{75}$$

$$\frac{1}{h}\|\boldsymbol{n} \times \boldsymbol{v}^0_h\|^2_{L^2(\Gamma)} + \|\nabla \times \boldsymbol{v}^0_h\|^2_{L^2(\Omega)} \gtrsim \frac{1}{L_0^2}\|\boldsymbol{v}^0_h\|^2_{L^2(\Omega)}. \tag{76}$$

(2) *For all $q_h \in Q_h$ there exists $\boldsymbol{v}_h \in V_h \setminus \{\boldsymbol{0}\}$ such that $\nabla \times \boldsymbol{v}_h = \boldsymbol{0}$, $\boldsymbol{n} \times \boldsymbol{v}_h = \boldsymbol{0}$ on $\partial\Omega_h$ and*

$$\|\boldsymbol{n} \times \boldsymbol{v}_h\|_{L^2(\Gamma)} = \varphi(h) h^{1/2} L_0^{-1} \|\boldsymbol{v}_h\|_{L^2(\Omega)}, \quad \textit{with } \varphi(h) \to 0 \textit{ as } h \to 0, \tag{77}$$

$$\|\boldsymbol{v}_h\|_{L^2(\Omega)} \|\nabla q_h\|_{L^2(\Omega)} \lesssim (\nabla q_h, \boldsymbol{v}_h)_\Omega. \tag{78}$$

Condition (75) avoids a singular growth of s_h from $\partial\Omega_h$ (where $s_h = 0$) to Γ (see the comments to (36)), whereas (76) is a Poincaré-Friedrics inequality on $G^\perp_{V_h}$ (shifted with the boundary term to account for the fact that $\boldsymbol{n} \times \boldsymbol{v}^0_h$ is not zero on Γ). This, as well as (78), hold for $H(\mathrm{curl};\Omega_h) \times H^1(\Omega_h)$-conforming finite element spaces $V_h \times Q_h$ that make the diagram

$$\begin{array}{ccc} H^1_0(\Omega_h) & \xrightarrow{\nabla} & H_0(\mathrm{curl};\Omega_h) \\ \downarrow P_{Q_h} & & \downarrow P_{V_h} \\ Q_h & \xrightarrow{\nabla} & V_h \end{array}$$

commutative (see [25, Th. 4.7] for the proof of (76) when $\boldsymbol{n} \times \boldsymbol{v}^0_h = \boldsymbol{0}$ on Γ).

Let $\boldsymbol{U}_h = [\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_{u,h}, \boldsymbol{\sigma}_{p,h}]$, $\delta\boldsymbol{U}_h = [\delta\boldsymbol{u}_h, \delta p_h, \delta\boldsymbol{\sigma}_{u,h}, \delta\boldsymbol{\sigma}_{p,h}]$. The bilinear form of problem (69)-(72) is:

$$\begin{aligned} B_{\mathrm{M}}(\boldsymbol{U}_h, \delta\boldsymbol{U}_h) &= \nu(\nabla \times \boldsymbol{u}_h, \nabla \times \delta\boldsymbol{u}_h)_\Omega + (\nabla p_h, \delta\boldsymbol{u}_h)_\Omega + (\nabla \delta p_h, \boldsymbol{u}_h)_\Omega \\ &- \langle P_t(\boldsymbol{\sigma}_{u,h}), \boldsymbol{n} \times \delta\boldsymbol{u}_h\rangle_\Gamma - \langle P_t(\delta\boldsymbol{\sigma}_{u,h}), \boldsymbol{n} \times \boldsymbol{u}_h\rangle_\Gamma \\ &- \langle \boldsymbol{n} \cdot \boldsymbol{\sigma}_{p,h}, \delta p_h\rangle_\Gamma - \langle \boldsymbol{n} \cdot \delta\boldsymbol{\sigma}_{p,h}, p_h\rangle_\Gamma \\ &- \frac{1}{\nu N_{u,0}}(\boldsymbol{\sigma}_{u,h} - \nu\nabla \times \boldsymbol{u}_h, \delta\boldsymbol{\sigma}_{u,h} - \nu\nabla \times \delta\boldsymbol{u}_h)_\Omega \\ &+ \frac{\nu}{N_{p,0}L_0^2}(\boldsymbol{\sigma}_{p,h} - \boldsymbol{u}_h, \delta\boldsymbol{\sigma}_{p,h} - \delta\boldsymbol{u}_h)_\Omega. \end{aligned} \tag{79}$$

Our stability and convergence result is the following:

Theorem 5. *Suppose that Assumptions 5 and 6 hold, that $N_{u,0} > 1$ and $N_{p,0}$ are large enough. Then, for h sufficiently small the bilinear form in (79) is stable in the norm*

$$\begin{aligned} \|[\boldsymbol{v}, q, \boldsymbol{\tau}_u, \boldsymbol{\tau}_p]\|_{\mathrm{M}}^2 &:= \nu\|\nabla\times\boldsymbol{v}\|_{L^2(\Omega)}^2 + \frac{\nu}{L_0^2}\|\boldsymbol{v}\|_{L^2(\Omega)}^2 + \frac{\nu}{h}\|\boldsymbol{n}\times\boldsymbol{v}\|_{L^2(\Gamma)}^2 \\ &+ \frac{L_0^2}{\nu}\|\nabla q\|_{L^2(\Omega)}^2 + \frac{L_0^2}{\nu\, h}\|q\|_{L^2(\Gamma)}^2 + \frac{1}{\nu}\|\boldsymbol{\tau}_u\|_{L^2(\Omega)}^2 + \frac{\nu}{L_0^2}\|\boldsymbol{\tau}_p\|_{L^2(\Omega)}^2. \end{aligned} \tag{80}$$

Moreover, if $\boldsymbol{U} = [\boldsymbol{u}, p, \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_p]$ is the solution of the continuous problem, with $\boldsymbol{\sigma}_u := \nu\nabla\times\boldsymbol{u}$ and $\boldsymbol{\sigma}_p := \boldsymbol{u}$, the solution $\boldsymbol{U}_h = [\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_{u,h}, \boldsymbol{\sigma}_{p,h}]$ of problem (69)-(72) satisfies the error estimate (16) in the norm (80) with the error function

$$\begin{aligned} \mathcal{E}_{\mathrm{M}}(\boldsymbol{U}, h) &= \nu^{1/2}\|\nabla\times\boldsymbol{u} - \nabla\times\tilde{\boldsymbol{u}}_h\|_{L^2(\Omega)} + \nu^{1/2}L_0^{-1}\|\boldsymbol{u} - \tilde{\boldsymbol{u}}_h\|_{L^2(\Omega)} \\ &+ \nu^{1/2}h^{-1/2}\|\boldsymbol{n}\times\boldsymbol{u} - \boldsymbol{n}\times\tilde{\boldsymbol{u}}_h\|_{L^2(\Omega)} \\ &+ L_0\nu^{-1/2}\|\nabla p - \nabla\tilde{p}_h\|_{L^2(\Omega)} + L_0(\nu\, h)^{-1/2}\|p - \tilde{p}_h\|_{L^2(\Gamma)} \\ &+ \nu^{-1/2}\|\boldsymbol{\sigma}_u - \tilde{\boldsymbol{\sigma}}_{u,h}\|_{L^2(\Omega)} + \nu^{1/2}L_0^{-1}\|\boldsymbol{\sigma}_p - \tilde{\boldsymbol{\sigma}}_{p,h}\|_{L^2(\Omega)} \\ &+ \nu^{-1/2}h^{1/2}\|\boldsymbol{\sigma}_u - \tilde{\boldsymbol{\sigma}}_{u,h}\|_{L^2(\Gamma)} + \nu^{1/2}L_0^{-1}h^{1/2}\|\boldsymbol{\sigma}_p - \tilde{\boldsymbol{\sigma}}_{p,h}\|_{L^2(\Gamma)} \end{aligned} \tag{81}$$

for any $[\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{\boldsymbol{\sigma}}_{u,h}, \tilde{\boldsymbol{\sigma}}_{p,h}] \in V_h \times Q_h \times \Sigma_{u,h} \times \Sigma_{p,h}$.

Proof. Using the fact that $N_{u,0} > 1$ we obtain

$$\begin{aligned} B_{\mathrm{M}}(\boldsymbol{U}_h, [\boldsymbol{u}_h, -p_h, -\boldsymbol{\sigma}_{u,h}, \boldsymbol{\sigma}_{p,h}]) &\gtrsim \nu\|\nabla\times\boldsymbol{u}_h\|_{L^2(\Omega)}^2 + \frac{1}{N_{u,0}\nu}\|\boldsymbol{\sigma}_{u,h}\|_{L^2(\Omega)}^2 \\ &+ \frac{\nu}{N_{p,0}L_0^2}\|\boldsymbol{\sigma}_{p,h}\|_{L^2(\Omega)}^2 - \frac{\nu}{N_{p,0}L_0^2}\|\boldsymbol{u}_h\|_{L^2(\Omega)}^2. \end{aligned} \tag{82}$$

Let $\boldsymbol{\tau}_{u,h}^u \in \Sigma_{u,h}$ be the element that allows us to guarantee (73) for $\boldsymbol{v}_h = \boldsymbol{u}_h$. We have that

$$\begin{aligned} &B_{\mathrm{M}}(\boldsymbol{U}_h, [\boldsymbol{0}, 0, -\nu h^{-1}\boldsymbol{\tau}_{u,h}^u, \boldsymbol{0}]) \\ &\quad\gtrsim \frac{\nu}{h}\|\boldsymbol{n}\times\boldsymbol{u}_h\|_{L^2(\Gamma)}^2 - \frac{1}{N_{u,0}\nu^{1/2}}\|\boldsymbol{\sigma}_{u,h} - \nu\nabla\times\boldsymbol{u}_h\|_{L^2(\Omega)}\frac{\nu^{1/2}}{h^{1/2}}\|\boldsymbol{n}\times\boldsymbol{u}_h\|_{L^2(\Gamma)} \\ &\quad\gtrsim \frac{\nu}{h}\|\boldsymbol{n}\times\boldsymbol{u}_h\|_{L^2(\Gamma)}^2 - \frac{1}{N_{u,0}\nu}\|\boldsymbol{\sigma}_{u,h}\|_{L^2(\Omega)}^2 - \frac{\nu}{N_{u,0}}\|\nabla\times\boldsymbol{u}_h\|_{L^2(\Omega)}^2. \end{aligned}$$

Therefore, taking $\delta\boldsymbol{U}_h^0 = [\boldsymbol{u}_h, -p_h, -\boldsymbol{\sigma}_{u,h} - \beta_0\nu h^{-1}\boldsymbol{\tau}_{u,h}^u, \boldsymbol{\sigma}_{p,h}]$ for an appropriate $\beta_0 > 0$ we get

$$\begin{aligned} B_{\mathrm{M}}(\boldsymbol{U}_h, \delta\boldsymbol{U}_h^0) &\gtrsim \nu\|\nabla\times\boldsymbol{u}_h\|_{L^2(\Omega)}^2 + \frac{1}{\nu}\|\boldsymbol{\sigma}_{u,h}\|_{L^2(\Omega)}^2 + \frac{\nu}{L_0^2}\|\boldsymbol{\sigma}_{p,h}\|_{L^2(\Omega)}^2 \\ &+ \frac{\nu}{h}\|\boldsymbol{n}\times\boldsymbol{u}_h\|_{L^2(\Gamma)}^2 - \frac{\nu}{N_{p,0}L_0^2}\|\boldsymbol{u}_h\|_{L^2(\Omega)}^2. \end{aligned} \tag{83}$$

Let us decompose $\boldsymbol{u}_h = \boldsymbol{u}_h^0 + \nabla s_h$, with $\boldsymbol{u}_h^0 \in G_{V_h}^{\perp}$ and $s_h \in Q_h$, $s_h = 0$ on $\partial\Omega_h$. By virtue of (76), from (83) and for $N_{p,0}$ sufficiently large we have that

$$\begin{aligned} B_{\mathrm{M}}(\boldsymbol{U}_h, \delta\boldsymbol{U}_h^0) &\gtrsim \nu\|\nabla\times\boldsymbol{u}_h\|_{L^2(\Omega)}^2 + \frac{1}{\nu}\|\boldsymbol{\sigma}_{u,h}\|_{L^2(\Omega)}^2 + \frac{\nu}{L_0^2}\|\boldsymbol{\sigma}_{p,h}\|_{L^2(\Omega)}^2 \\ &+ \frac{\nu}{h}\|\boldsymbol{n}\times\boldsymbol{u}_h\|_{L^2(\Gamma)}^2 + \frac{\nu}{L_0^2}\|\boldsymbol{u}_h^0\|_{L^2(\Omega)}^2 - \frac{\nu}{N_{p,0}L_0^2}\|\nabla s_h\|_{L^2(\Omega)}^2. \end{aligned} \tag{84}$$

The orthogonality of ∇s_h and $\boldsymbol{u}_h^0$ yields

$$B_{\mathrm{M}}(\boldsymbol{U}_h, [\boldsymbol{0}, \nu L_0^{-2} s_h, \boldsymbol{0}, \boldsymbol{0}]) = \frac{\nu}{L_0^2}\|\nabla s_h\|_{L^2(\Omega)}^2 - \frac{\nu}{L_0^2}\langle \boldsymbol{n}\cdot\boldsymbol{\sigma}_{p,h}, s_h\rangle_\Gamma.$$

Making use of (75) and using the same steps as in (44), the last term can be shown to be arbitrarily small and multiplied by the $L^2(\Omega)$-norm of $\boldsymbol{\sigma}_{p,h}$ and ∇s_h, with adequate scaling coefficients. Thus, from (84) it follows that if $\delta\boldsymbol{U}_h^1 = \delta\boldsymbol{U}_h^0 + \beta_1[\boldsymbol{0}, \nu L_0^{-2} s_h, \boldsymbol{0}, \boldsymbol{0}]$, for an appropriate β_1 we get

$$\begin{aligned} B_{\mathrm{M}}(\boldsymbol{U}_h, \delta\boldsymbol{U}_h^1) &\gtrsim \nu\|\nabla\times\boldsymbol{u}_h\|_{L^2(\Omega)}^2 + \frac{1}{\nu}\|\boldsymbol{\sigma}_{u,h}\|_{L^2(\Omega)}^2 + \frac{\nu}{L_0^2}\|\boldsymbol{\sigma}_{p,h}\|_{L^2(\Omega)}^2 \\ &+ \frac{\nu}{h}\|\boldsymbol{n}\times\boldsymbol{u}_h\|_{L^2(\Gamma)}^2 + \frac{\nu}{L_0^2}\|\boldsymbol{u}_h\|_{L^2(\Omega)}^2, \end{aligned} \tag{85}$$

for h sufficiently small and $N_{p,0}$ large enough. It only remains to get control on p_h. Let us consider first the boundary term, and let $\boldsymbol{\tau}_{p,h}^p$ be the element in $\Sigma_{p,h}$ for which (74) holds. It is easily found that

$$\begin{aligned} &B_{\mathrm{M}}(\boldsymbol{U}_h, [\boldsymbol{0}, 0, \boldsymbol{0}, -L_0^2(\nu h)^{-1}\boldsymbol{\tau}_{p,h}^p]) \\ &\quad\gtrsim \frac{L_0^2}{\nu h}\|p_h\|_{L^2(\Gamma)}^2 - \frac{\nu}{L_0^2}\|\boldsymbol{\sigma}_{p,h}\|_{L^2(\Omega)}^2 - \frac{\nu}{L_0^2}\|\boldsymbol{u}_h\|_{L^2(\Omega)}^2. \end{aligned} \tag{86}$$

Let $\boldsymbol{v}_h^p$ the function that allows us to guarantee (78) for $q_h = p_h$, which we choose such that $\|\boldsymbol{v}_h^p\|_{L^2(\Omega)} = \|\nabla p_h\|_{L^2(\Omega)}$. We have that

$$\begin{aligned} &B_{\mathrm{M}}(\boldsymbol{U}_h, [L_0^2\nu^{-1}\boldsymbol{v}_h^p, 0, \boldsymbol{0}, \boldsymbol{0}]) \gtrsim \frac{L_0^2}{\nu}\|\nabla p_h\|_{L^2(\Omega)}^2 \\ &\quad - \frac{\nu}{L_0^2}\|\boldsymbol{\sigma}_{p,h}\|_{L^2(\Omega)}^2 - \frac{\nu}{L_0^2}\|\boldsymbol{u}_h\|_{L^2(\Omega)}^2 - \frac{L_0^2}{\nu}\langle P_t(\boldsymbol{\sigma}_{u,h}), \boldsymbol{n}\times\boldsymbol{v}_h^p\rangle_\Gamma. \end{aligned} \tag{87}$$

The last term can be made arbitrarily small as $h \to 0$. Thus, from (85), (86) and (87) it follows that if we take $\delta\boldsymbol{U}_h^2 = \delta\boldsymbol{U}_h^1 + [\beta_2 L_0^2\nu^{-1}\boldsymbol{v}_h^p, 0, \boldsymbol{0}, -\beta_3 L_0^2(\nu h)^{-1}\boldsymbol{\tau}_{p,h}^p]$, with appropriate β_2 and β_3, there holds that $B_{\mathrm{M}}(\boldsymbol{U}_h, \delta\boldsymbol{U}_h^2) \gtrsim \|\!|\boldsymbol{U}_h|\!\|_{\mathrm{M}}^2$. It is easily checked that $\|\!|\boldsymbol{U}_h|\!\|_{\mathrm{M}} \gtrsim \|\!|\delta\boldsymbol{U}_h^2|\!\|$, from where we obtain stability of B_{M}.

The approximation properties (14) and (15) that yield the error estimate (16) are once more easily proved using the inverse estimate (17). □

As for the previous problems analyzed, the error estimates obtained are optimal. However, they require smooth enough solutions, which might not exist as explained above. Nevertheless, the stability result proved holds in the case of minimum regularity, and this is enough to prove convergence to singular solutions using the strategy employed for example in [3].

6. Conclusions

In this paper we have presented the analysis of the LLM technique proposed in [14] to impose essential boundary conditions on non-matching meshes. The method can be described as a Lagrange multiplier strategy with the particularity that the multipliers are linked to the unknowns of the problem by imposing that they are equal, in a least-squares sense, to traces of fluxes. These traces make sense for the discrete problem, as the formulation involves fluxes of the unknowns whose traces might not be well defined at the continuous level; for example, in (20)-(22) $\boldsymbol{\sigma}_h$ approximates $-\boldsymbol{q}$, but the normal trace of $\boldsymbol{q}$ is not necessarily well defined on Γ if $\boldsymbol{q}$ belongs only to $L^2(\Omega)^d$.

The general formulation presented has been applied and analyzed for three elliptic problems, namely, Darcy's problem (in primal and dual form), Stokes' problem and Maxwell's problem. In all cases stability and optimal convergence for smooth solutions has been proved. The norm in which the analysis has been made contains all the terms to guarantee stability in the space where the continuous problem is well posed, as well as control on boundary terms required when boundary conditions are not exactly satisfied. Our analysis has been based on the classical stability conditions needed when the Galerkin approximation is used, but could be also extended to cases in which stabilized finite element formulations are chosen using the methodology initiated in [14].

From the computational point of view, the crucial fact is that the new variables introduced by the LLM method are element-wise discontinuous, and this allows one to eliminate their degrees of freedom by static condensation. This process is usually inexpensive. The simplest way to achieve the compatibility requirements we have needed between the spaces of the new variables and the original unknowns of the problem is to take them of the same order, not of the order of the fluxes (one less, in the problems treated), even if in some cases we have been able to prove our results using the second option. A final computational comment is that the formulation depends on algorithmic parameters that need not to be very large, and thus do not spoil the conditioning of the final algebraic system to be solved.

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