Finite element approximation of the Stokes problem

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1. **Introduction and problem statement**
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Boundary value problem

The Stokes problem in differential form consists in finding a velocity (or displacement) $\boldsymbol{u}$ and a pressure p such that

$$
\begin{aligned}
-\nu\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } \Omega \\
\nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega \\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega
\end{aligned}
$$

where:

$\boldsymbol{f}$: Vector of body forces

ν : Kinematic viscosity (fluids) or shear modulus (solids)

$\Omega \subset \mathbb{R}^d$: Computational domain

Weak form 1

Let

$V = H_0^1(\Omega)^d$	Velocity space
$Q = L^2(\Omega)/\mathbb{R}$	Pressure space
$(\cdot,\cdot)$	Inner product in Q
$\langle\cdot,\cdot\rangle$	Duality pairing in $V \times V'$, $V' = H^{-1}(\Omega)^d$

The weak form of the problem is: Find $\boldsymbol{u} \in V$, $p \in Q$ such that

$$\begin{aligned} \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (p, \nabla\cdot\boldsymbol{v}) &= \langle \boldsymbol{f}, \boldsymbol{v}\rangle \qquad && \forall \boldsymbol{v} \in V \\ (q, \nabla\cdot\boldsymbol{u}) &= 0 && \forall q \in Q \end{aligned}$$

Defining

$$\begin{aligned} a &: V \times V \longrightarrow \mathbb{R}, & a(\boldsymbol{u}, \boldsymbol{v}) &= \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) \\ b &: Q \times V \longrightarrow \mathbb{R}, & b(q, \boldsymbol{v}) &= (q, \nabla \cdot \boldsymbol{v}) \\ l &: V \longrightarrow \mathbb{R}, & l(v) &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle \end{aligned}$$

the problem can be written as: Find $\boldsymbol{u} \in V$, $p \in Q$ such that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) - b(p, \boldsymbol{v}) &= l(\boldsymbol{v}) & \forall \boldsymbol{v} \in V \\ b(q, \boldsymbol{u}) &= 0 & \forall q \in Q \end{aligned}$$

Not that this problem is not symmetric (but can be symmetrized).

Defining

$$B : (V \times Q) \times (V \times Q) \longrightarrow \mathbb{R}$$
$$B([\boldsymbol{u}, p], [\boldsymbol{v}, q]) := a(\boldsymbol{u}, \boldsymbol{v}) - b(p, \boldsymbol{v}) + b(q, \boldsymbol{u})$$
$$L : V \times Q \longrightarrow \mathbb{R}$$
$$L([\boldsymbol{v}, q]) = l(\boldsymbol{v})$$

it can also be written as: Find $[\boldsymbol{u}, p] \in V \times Q$ such that

$$B([\boldsymbol{u}, p], [\boldsymbol{v}, q]) = L([\boldsymbol{v}, q]) \quad \forall [\boldsymbol{v}, q] \in V \times Q$$

Weak form 2

Define the space

$$J := \{\boldsymbol{v} \in V \mid \nabla \cdot \boldsymbol{v} = 0\}$$

The weak form of the problem can also be stated as: Find $\boldsymbol{u} \in J$ such that

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \qquad \forall \boldsymbol{v} \in J$$

or

$$a(\boldsymbol{u}, \boldsymbol{v}) = l(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in J$$

Optimization associated to weak form 2

Let us define the quadratic functional

$$
\begin{aligned}
F :& J \longrightarrow \mathbb{R} \\
F(\boldsymbol{v}) :=& \frac{1}{2}\nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{v}) - \langle \boldsymbol{f}, \boldsymbol{v} \rangle = \frac{1}{2}a(\boldsymbol{u}, \boldsymbol{v}) - l(\boldsymbol{v})
\end{aligned}
$$

If $\boldsymbol{u} \in J$ is the solution of the Stokes problem, then

$$
\boldsymbol{u} = \arg\inf_{\boldsymbol{v} \in J} F(\boldsymbol{v})
$$

since it is immediately checked that

$$
\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} F(\boldsymbol{u} + \epsilon \boldsymbol{v}) &= a(\boldsymbol{u}, \boldsymbol{v}) - l(\boldsymbol{v}) = 0 \\
\left.\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\right|_{\epsilon=0} F(\boldsymbol{u} + \epsilon \boldsymbol{v}) &= a(\boldsymbol{v}, \boldsymbol{v}) \geq 0
\end{aligned}
$$

Optimization associated to weak form 1

The Lagrangian associated to the condition $\nabla \cdot \boldsymbol{u} = 0$ is

$$L : V \times Q \longrightarrow \mathbb{R}$$

$$L(\boldsymbol{v}, q) := \frac{1}{2} a(\boldsymbol{v}, \boldsymbol{v}) - b(q, \boldsymbol{v}) - l(\boldsymbol{v})$$

If $[\boldsymbol{u}, p]$ is the solution of weak form 1, then

$$[\boldsymbol{u}, p] = \arg \inf_{\boldsymbol{v} \in V} \sup_{q \in Q} L(\boldsymbol{v}, q)$$

since it is immediately checked that

$$\left.\frac{\partial}{\partial \epsilon_1}\right|_{\epsilon_1 = 0, \epsilon_2 = 0} L(\boldsymbol{u} + \epsilon_1 \boldsymbol{v}, p + \epsilon_2 q) = a(\boldsymbol{u}, \boldsymbol{v}) - b(p, \boldsymbol{v}) - l(\boldsymbol{v}) = 0$$

$$\left.\frac{\partial}{\partial \epsilon_2}\right|_{\epsilon_1 = 0, \epsilon_2 = 0} L(\boldsymbol{u} + \epsilon_1 \boldsymbol{v}, p + \epsilon_2 q) = b(q, \boldsymbol{u}) = 0$$

Penalized "Lagrangian"

Instead of introducing a Lagrange multiplier, one can penalize the constraint $\nabla \cdot \boldsymbol{u} = 0$. Define

$$\begin{aligned} L_1^\epsilon &: V \longrightarrow \mathbb{R} \\ L_1^\epsilon(\boldsymbol{v}) &:= \frac{1}{2} a(\boldsymbol{v}, \boldsymbol{v}) - l(v) + \frac{1}{2\epsilon} \|\nabla \cdot \boldsymbol{v}\|^2 \end{aligned}$$

with $\epsilon > 0$. The associated variational problem is: Find $\boldsymbol{u}^\epsilon \in V$ such that

$$a(\boldsymbol{u}^\epsilon, \boldsymbol{v}) + \frac{1}{\epsilon}(\nabla \cdot \boldsymbol{u}^\epsilon, \nabla \cdot \boldsymbol{v}) = l(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in V$$

and the associated boundary value problem:

$$\begin{aligned} -\nu \Delta \boldsymbol{u}^\epsilon - \frac{1}{\epsilon} \nabla(\nabla \cdot \boldsymbol{u}^\epsilon) &= \boldsymbol{f} && \text{in } \Omega \\ \boldsymbol{u}^\epsilon &= \boldsymbol{0} && \text{on } \partial\Omega \end{aligned}$$

Perturbed Lagrangian

In fact, the optimization of the Lagrangian with respect to the multiplier is not defined (neither positive nor negative). One can define the perturbed Lagrangian:

$$L_2^\epsilon : V \times Q \longrightarrow \mathbb{R}$$

$$L_2^\epsilon(\boldsymbol{v}, q) := \frac{1}{2} a(\boldsymbol{v}, \boldsymbol{v}) - b(q, \boldsymbol{v}) - l(\boldsymbol{v}) - \epsilon \|q\|^2$$

with $\epsilon > 0$. Now it holds:

$$\left.\frac{\partial^2}{\partial \epsilon_1^2}\right|_{\epsilon_1=0,\epsilon_2=0} L_2^\epsilon(\boldsymbol{u} + \epsilon_1 \boldsymbol{v}, p + \epsilon_2 q) = a(\boldsymbol{v}, \boldsymbol{v}) \geq 0$$

$$\left.\frac{\partial^2}{\partial \epsilon_2^2}\right|_{\epsilon_1=0,\epsilon_2=0} L_2^\epsilon(\boldsymbol{u} + \epsilon_1 \boldsymbol{v}, p + \epsilon_2 q) = -\epsilon \|q\|^2 \leq 0$$

so the problem is really a saddle point problem.

The associated variational problem is: Find $[\boldsymbol{u}^\epsilon, p^\epsilon] \in V \times Q$ such that

$$\begin{aligned} a(\boldsymbol{u}^\epsilon, \boldsymbol{v}) - b(p^\epsilon, \boldsymbol{v}) &= l(\boldsymbol{v}) && \forall \boldsymbol{v} \in V \\ b(q, \boldsymbol{u}^\epsilon) + \epsilon(p^\epsilon, q) &= 0 && \forall q \in Q \end{aligned}$$

and the associated boundary value problem:

$$\begin{aligned} -\nu \Delta \boldsymbol{u}^\epsilon + \nabla p^\epsilon &= \boldsymbol{f} && \text{in } \Omega \\ \epsilon p^\epsilon + \nabla \cdot \boldsymbol{u}^\epsilon &= 0 && \text{in } \Omega \\ \boldsymbol{u}^\epsilon &= \boldsymbol{0} && \text{on } \partial\Omega \end{aligned}$$

Remarks

- Note that if q is constant:

$$\epsilon \int_\Omega p^\epsilon + \int_\Omega \nabla \cdot \boldsymbol{u}^\epsilon = \epsilon \int_\Omega p^\epsilon = 0$$

  and thus the pressure space is

$$Q = \left\{ q \in L^2(\Omega) \mid \int_\Omega q = 0 \right\} \cong L^2(\Omega)/\mathbb{R}$$

- At the differential level, the penalized and the perturbed Lagrangian lead to the same BVP:

$$\left. \begin{aligned} -\nu \Delta \boldsymbol{u}^\epsilon + \nabla p^\epsilon &= \boldsymbol{f} \\ \epsilon p^\epsilon + \nabla \cdot \boldsymbol{u}^\epsilon &= 0 \end{aligned} \right\} \Leftrightarrow -\nu \Delta \boldsymbol{u}^\epsilon - \frac{1}{\epsilon} \nabla (\nabla \cdot \boldsymbol{u}^\epsilon) = \boldsymbol{f}$$

- In elasticity

$$\frac{1}{\epsilon} = \frac{E}{1 - 2\nu} = K \quad \text{Bulk modulus}$$

Objectives

- Theoretical: existence and uniqueness of $\boldsymbol{u}$ and p.
- Theoretical: convergence of $\boldsymbol{u}^\epsilon$ and p^ϵ to $\boldsymbol{u}$ and p as $\epsilon \to 0$.
- Approximation of V by $V_h \subset V$ and Q by $Q_h \subset Q$. Conditions on Q_h and V_h.
- Is it possible to avoid the conditions on Q_h and V_h?

Other non-touched (and important!) issues:

- Is it possible to approximate J by $J_h \subset J$? Solenoidal velocity interpolations.
- What about approximations where $V_h \not\subset V$ and/or $Q_h \not\subset Q$? Non-conforming approximations and, in particular, discontinuous Galerkin methods.

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Existence and uniqueness in J

Theorem. There exists a unique $\boldsymbol{u} \in J$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) = l(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in J$$

PROOF: Use Lax-Milgram's Lemma (a is coercive). □

REMARK. Sometimes it is convenient to define $a(\boldsymbol{u}, \boldsymbol{v})$ as

$$a(\boldsymbol{u}, \boldsymbol{v}) = \nu \int_\Omega \nabla^S \boldsymbol{u} : \nabla^S \boldsymbol{v}, \quad \nabla^S \boldsymbol{v} := \frac{1}{2}\left[\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T\right]$$

In this case, coercivity of a follows from Korn's inequality instead of Poincaré-Friedrics inequality.

Helmholtz decomposition

Theorem. Given a vector field $\boldsymbol{w} \in L^2(\Omega)^d$, there exists a unique decomposition

$$
\begin{aligned}
&\boldsymbol{w} = \boldsymbol{v} + \nabla q \\
&\boldsymbol{v} \in H(\mathrm{div}, \Omega),\ \nabla \cdot \boldsymbol{v} = 0 \text{ in } \Omega,\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega \\
&q \in H^1(\Omega)/\mathbb{R},\ \int_\Omega \boldsymbol{v} \cdot \nabla q = 0
\end{aligned}
$$

PROOF: If the decomposition exists, it is easily checked that it is orthogonal:

$$\int_\Omega \boldsymbol{v} \cdot \nabla q = \int_\Omega \nabla \cdot (\boldsymbol{v} q) = \int_{\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{n} q = 0$$

and from this it follows that it is unique:

$$\boldsymbol{w} = \boldsymbol{v}_1 + \nabla q_1 = \boldsymbol{v}_2 + \nabla q_2 \Rightarrow \boldsymbol{v}_1 - \boldsymbol{v}_2 = \nabla q_1 - \nabla q_2$$
$$\Rightarrow \boldsymbol{v}_1 = \boldsymbol{v}_2, \; q_1 = q_2 + \text{constant}$$

Next, it can be seen that q and $\boldsymbol{v}$ can be found as the solution of

$$-\Delta q = \nabla \cdot \boldsymbol{w} \quad \text{in } \Omega, \qquad \frac{\partial q}{\partial n} = 0 \quad \text{on } \partial\Omega$$
$$\boldsymbol{v} = \boldsymbol{w} - \nabla q$$

Existence and uniqueness in $V \times Q$

Theorem. If $N : V \longrightarrow \mathbb{R}$ is linear and continuous and $N(\boldsymbol{v}) = 0$ for all $\boldsymbol{v} \in J$ then there exists $p \in L^2(\Omega)/\mathbb{R}$ such that $N(\boldsymbol{v}) = (p, \nabla \cdot \boldsymbol{v})$.

PROOF: Use a representation theorem to show that $N(\boldsymbol{v}) = \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ and the Helmholtz decomposition of $\boldsymbol{w}$. □

Theorem. There exists a unique $[\boldsymbol{u}, p] \in V \times Q$ solution of the Stokes problem in velocity-pressure form.

PROOF: Define $N : V \longrightarrow \mathbb{R}$ by $N(\boldsymbol{v}) = a(\boldsymbol{u}, \boldsymbol{v}) - l(\boldsymbol{v})$. Then $N(\boldsymbol{v}) = 0$ for $\boldsymbol{v} \in J$ and thus there exists $p \in L^2(\Omega)/\mathbb{R}$ such that $a(\boldsymbol{u}, \boldsymbol{v}) - l(\boldsymbol{v}) = (p, \nabla \cdot \boldsymbol{v})$. □

Abstract problem and proof by regularization

Theorem. Let V, Q be Hilbert spaces, $a : V \times V \longrightarrow \mathbb{R}$ continuous and V-elliptic:

$$a(\boldsymbol{u}, \boldsymbol{v}) \leq N_a \|\boldsymbol{u}\|_V \|\boldsymbol{v}\|_V, \quad a(\boldsymbol{v}, \boldsymbol{v}) \geq K_a \|\boldsymbol{v}\|_V^2$$

$b : Q \times V \longrightarrow \mathbb{R}$ continuous and satisfying the inf-sup condition

$$\inf_{q \in Q} \sup_{\boldsymbol{v} \in V} \frac{b(q, \boldsymbol{v})}{\|q\|_Q \|\boldsymbol{v}\|_V} \geq K_b$$

and $l : V \longrightarrow \mathbb{R}$ continuous: $l(\boldsymbol{v}) \leq N_l \|\boldsymbol{v}\|_V$. Then, there exists a unique solution to the problem of finding $[\boldsymbol{u}, p] \in V \times Q$ such that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) - b(p, \boldsymbol{v}) &= l(\boldsymbol{v}) & \forall \boldsymbol{v} \in V \\ b(q, \boldsymbol{u}) &= 0 & \forall q \in Q \end{aligned}$$

PROOF: Consider the regularized problem

$$\begin{aligned} a(\boldsymbol{u}^\epsilon, \boldsymbol{v}) - b(p^\epsilon, \boldsymbol{v}) &= l(\boldsymbol{v}) && \forall \boldsymbol{v} \in V \\ \epsilon(p^\epsilon, q) + b(q, \boldsymbol{u}^\epsilon) &= 0 && \forall q \in Q \end{aligned}$$

Step 1. Lax Milgram's Theorem guarantees that there exists a unique $[\boldsymbol{u}^\epsilon, p^\epsilon]$ for all $\epsilon > 0$.
Step 2. It is easily shown that $\|\boldsymbol{u}^\epsilon\|_V$ and $\|p^\epsilon\|_Q$ are bounded independently of ϵ (use is made of the inf-sup condition).
Step 3. As a consequence, $\boldsymbol{u}^\epsilon \rightharpoonup \boldsymbol{u}^*$, $p^\epsilon \rightharpoonup p^*$, as $\epsilon \to 0$, and $[\boldsymbol{u}^*, p^*]$ turns out to be solution of the problem with $\epsilon = 0$.
Step 4. Uniqueness is easy (again, the inf-sup condition is needed). □

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Babuška's Theorem (Babuška-Banach-Nečas)

Theorem. Let $B : V \times W \longrightarrow \mathbb{R}$, $L : W \longrightarrow \mathbb{R}$ and consider the abstract variational problem: Find $u \in V$ such that

$$B(u, v) = L(v) \quad \forall v \in W$$

Consider also the following conditions:

$$B(u, v) \leq N_B \|u\|_V \|v\|_W \qquad \text{Continuity of } B$$

$$L(v) \leq N_L \|v\|_W \qquad \text{Continuity of } L$$

$$\inf_{u \in V} \sup_{v \in W} \frac{B(u, v)}{\|u\|_V \|v\|_W} \geq K_B \qquad V\text{-Stability of } B$$

Then, these are necessary and sufficient conditions for the variational problem to have a unique solution that depends continuously on the data as $\|u\|_V \leq \frac{N_L}{K_B}$

Coercivity vs V-stability

If $B : V \times V \longrightarrow \mathbb{R}$ and $B(v, v) \geq K_B \|v\|_V^2$, the previous theorem reduces to Lax-Milgram's Lemma. Coercivity is sufficient, but not necessary for the problem to be well posed. The following analogy holds:

Finite dimensional problem:		Variational problem:
$v^T B u = v^T f$		$B(u, v) = L(v)$
Matrix B positive definite	$\leftrightarrow$	Bilinear form B coercive
Matrix B non-singular	$\leftrightarrow$	Bilinear form B V-stable

Why Ladyzhenskaya-Babuška-Brezzi ?

Consider the abstract form of the Stokes problem

$$B([\boldsymbol{u},p],[\boldsymbol{v},q]) = L([\boldsymbol{v},q]) \quad \forall [\boldsymbol{v},q] \in V \times Q$$
$$B([\boldsymbol{u},p],[\boldsymbol{v},q]) := a(\boldsymbol{u},\boldsymbol{v}) - b(p,\boldsymbol{v}) + b(q,\boldsymbol{u}), \quad L([\boldsymbol{v},q]) = l(\boldsymbol{v})$$

- If a is coercive in V, B is $V \times Q$-stable if, and only if (Brezzi):

$$\inf_{q \in Q} \sup_{\boldsymbol{v} \in V} \frac{b(q,\boldsymbol{v})}{\|q\|_Q \|\boldsymbol{v}\|_V} \geq K_b > 0$$

- If $b(q,\boldsymbol{v}) = (p, \nabla \cdot \boldsymbol{v})$, $Q = L^2(\Omega)/\mathbb{R}$, $V = H^1_0(\Omega)^d$, the previous condition holds true (Ladyzhenskaya).

Discrete variational form

Let $\mathcal{T} = \{K\}$ be a finite element partition of Ω of size h and consider the finite element spaces:

$$V_h := \left\{ \boldsymbol{v}_h \in C^0(\Omega)^d \mid \boldsymbol{v}_h|_K \in R_k(K)^d \right\}$$
$$Q_h := \left\{ q_h \in L^2(\Omega)/\mathbb{R} \mid q_h|_K \in R_l(K) \right\}$$

where $R_p(K)$ is the space of complete polynomials of degree p on K. By construction

$$V_h \subset V, \quad Q_h \subset Q \quad \text{Conforming approximation}$$

The discrete Stokes problem is: Find $\boldsymbol{u}_h \in V_h$, $p_h \in Q_h$ such that

$$\begin{aligned} a(\boldsymbol{u}_h, \boldsymbol{v}_h) - b(p_h, \boldsymbol{v}_h) &= l(\boldsymbol{v}_h) & \forall \boldsymbol{v}_h \in V_h \\ b(q_h, \boldsymbol{u}_h) &= 0 & \forall q_h \in Q_h \end{aligned}$$

Matrix version

The matrix form of this problem is:

$$\begin{aligned} \mathsf{K}\mathsf{U} - \mathsf{D}^T\mathsf{P} &= \mathsf{F} \\ \mathsf{D}\mathsf{U} &= 0 \end{aligned}$$

where U and P are the arrays of nodal velocities and pressures, respectively and

$$\begin{aligned} \mathsf{K}_{ij}^{ab} &= \nu(\nabla N_u^a, \nabla N_u^b)\delta_{ij} \\ \mathsf{D}_j^{cb} &= (N_p^c, \partial_j N_u^b) \end{aligned}$$

N_u and N_p denote the velocity and pressure shape functions, respectively, and the superscript the node to which they are associated.

Stability and discrete inf-sup condition

Taking $\boldsymbol{v}_h = \boldsymbol{u}_h$, $q_h = p_h$ it follows that:

$$K_a \|\boldsymbol{u}_h\|_V^2 \leq a(\boldsymbol{u}_h, \boldsymbol{u}_h) = l(\boldsymbol{u}_h) \leq N_l \|\boldsymbol{u}_h\|_V \Rightarrow \|\boldsymbol{u}_h\|_V \leq \frac{N_l}{K_a}$$

Pressure stability can only be obtained from the only place where pressure occurs:

$$b(p_h, \boldsymbol{v}_h) = l(\boldsymbol{v}_h) - a(\boldsymbol{u}_h, \boldsymbol{v}_h) \leq \left(N_l + N_a \frac{N_l}{K_a} \right) \|\boldsymbol{v}_h\|_V$$

The only way to control the pressure is to guarantee that

$$\forall p_h \in Q_h \; \exists \; \boldsymbol{v}_h \in V_h \; \text{ such that } \; K_b \|p_h\| \|\boldsymbol{v}_h\|_V \leq b(p_h, \boldsymbol{v}_h)$$

$$\iff \inf_{p_h \in Q_h} \sup_{\boldsymbol{v}_h \in V_h} \frac{b(p_h, \boldsymbol{v}_h)}{\|p_h\|_Q \|\boldsymbol{v}_h\|_V} \geq K_b$$

Compatible velocity-pressure pairs

Main problem: if $V_h \subset V$

$$\inf_{p\in Q}\sup_{\mathbf{v}\in V}\frac{b(p,\mathbf{v})}{\|p\|_Q\|\mathbf{v}\|_V}\geq K_b \not\Rightarrow \inf_{p_h\in Q_h}\sup_{\mathbf{v}_h\in V_h}\frac{b(p_h,\mathbf{v}_h)}{\|p_h\|_Q\|\mathbf{v}_h\|_V}\geq K_b'$$

Pairs V_h-Q_h satisfying the inf-sup condition will be called compatible, admissible or, in the present context, div-stable.

Methods to check compatibility:

- Verfürth (1984)
- Boland-Nicolaides (1983)
- Stenberg (1984). Macroelement technique.

Example to check the inf-sup condition: Fortin's trick

Theorem. If there exists an operator $\Pi_h : V \longrightarrow V_h$ such that

- $\forall \boldsymbol{v} \in V, \forall q_h \in Q_h, b(q_h, \boldsymbol{v} - \Pi_h \boldsymbol{v}) = 0$
- $\|\Pi_h \boldsymbol{v}\|_V \leq C \|\boldsymbol{v}\|_V$

then the pair V_h-Q_h satisfies the inf-sup condition.

PROOF:

$$\sup_{\boldsymbol{v}_h \in V_h} \frac{b(q_h, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_V} \geq \sup_{\boldsymbol{v} \in V} \frac{b(q_h, \Pi_h \boldsymbol{v})}{\|\Pi_h \boldsymbol{v}\|_V} = \sup_{\boldsymbol{v} \in V} \frac{b(q_h, \boldsymbol{v})}{\|\Pi_h \boldsymbol{v}\|_V}$$

$$\geq \frac{1}{C} \sup_{\boldsymbol{v} \in V} \frac{b(q_h, \boldsymbol{v})}{\|\boldsymbol{v}\|_V} \geq \frac{1}{C} K_b \|q_h\|_Q$$

Counting variables

If the discrete problem has a unique solution:

$$\mathsf{U} = \mathsf{K}^{-1}(\mathsf{F} + \mathsf{D}^T\mathsf{P})$$
$$\mathsf{D}\mathsf{K}^{-1}\mathsf{F} + \mathsf{D}\mathsf{K}^{-1}\mathsf{D}^T\mathsf{P} = 0$$

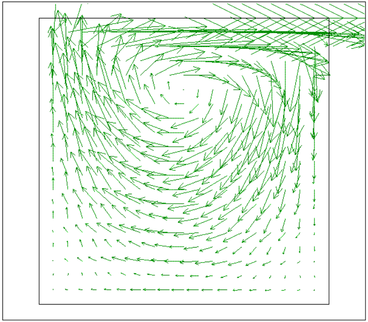
from where

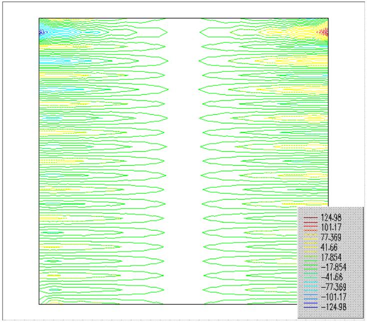
$$\mathrm{rank}(\mathsf{D}\mathsf{K}^{-1}\mathsf{D}^T) = n_p \leq \mathrm{rank}(\mathsf{K}^{-1}) = n_u$$

If there is a partition for which $n_p > n_u$, it is impossible to fulfill the inf-sup condition with K_b independent of h.

$n_p \leq n_u$ for all partitions is a necessary, but not sufficient condition for the inf-sup condition to hold.

Example: Galerkin method for cavity flow problem (Q_1/Q_1 interpolation)





Abstract convergence theorem

Consider the abstract variational problem: Find $u \in V$ such that

$$B(u, v) = L(v) \quad \forall v \in W,$$

Theorem. Suppose that the conditions of Babuška's theorem hold and, moreover,

- $\inf_{u_h \in V_h} \sup_{v_h \in W_h} \frac{B(u_h, v_h)}{\|u_h\|_V \|v_h\|_W} \geq K_B$: V_h-Stability of B
- $B(u - u_h, v_h) \leq C\varepsilon(h)\|v_h\|_W$: Consistency error
- $\inf_{\hat{u}_h \in V_h} \|u - \hat{u}_h\|_V \leq C\varepsilon(h)$: Interpolation error

for a certain error function $\varepsilon(h)$ (with $\varepsilon(h) \to 0$ as $h \to 0$). Then

$$\|u - u_h\|_V \leq C\varepsilon(h)$$

PROOF: For any $\hat{u}_h \in V_h$ there holds:

$$\begin{aligned} K_B \|\hat{u}_h - u_h\|_V \|v_h\|_W &\leq B(\hat{u}_h - u_h, v_h) \\ &= B(\hat{u}_h - u, v_h) + B(u - u_h, v_h) \\ &\leq N_B \|\hat{u}_h - u\|_V \|v_h\|_W + C\varepsilon(h)\|v_h\|_W \end{aligned}$$

On the other hand:

$$\begin{aligned} \|u - u_h\|_V &\leq \|u - \hat{u}_h\|_V + \|\hat{u}_h - u_h\|_V \\ &\leq \|u - \hat{u}_h\|_V + \frac{N_B}{K_B}\|\hat{u}_h - u\|_V + \frac{C}{K_B}\varepsilon(h) \end{aligned}$$

The result follows taking the inf over $\hat{u}_h \in V_h$. □

Application to the Stokes problem

In this case:

- $u_h = [\boldsymbol{u}_h, p_h]$
- $V_h = W_h \equiv V_h \times Q_h$ (Recall: $V \times Q = H_0^1(\Omega)^d \times L^2(\Omega)/\mathbb{R}$)
- $B(u - u_h, v_h) =$
  $a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) - b(p - p_h, \boldsymbol{v}_h) + b(q_h, \boldsymbol{u} - \boldsymbol{u}_h) = 0$
- $\inf_{\hat{u}_h \in V_h} \|u - \hat{u}_h\|_V = \inf_{\hat{\boldsymbol{u}}_h \in V_h, \hat{q}_h \in Q_h} (\|\boldsymbol{u} - \hat{\boldsymbol{u}}_h\|_V + \|p - \hat{p}_h\|_Q)$.
  For $\boldsymbol{u}$ and p regular enough:
  $\inf_{\hat{u}_h \in V_h} \|u - \hat{u}_h\|_V \leq C \left(h^k \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + h^{l+1} \|p\|_{H^l(\Omega)} \right)$

The previous theorem reads:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_V + \|p - p_h\|_Q \leq C \left(h^k \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + h^{l+1} \|p\|_{H^l(\Omega)} \right) \equiv \varepsilon(h)$$

L^2 estimate for the velocity via a duality argument

Theorem. Suppose that the domain Ω is regular enough, so that the Stokes problem

$$\begin{aligned} -\nu\Delta \boldsymbol{z} + \nabla\xi &= \boldsymbol{f} && \text{in } \Omega \\ \nabla\cdot\boldsymbol{z} &= 0 && \text{in } \Omega \\ \boldsymbol{z} &= \boldsymbol{0} && \text{on } \partial\Omega \end{aligned}$$

is such that $\|\boldsymbol{z}\|_{H^2} \leq C\|\boldsymbol{f}\|_{L^2}$, $\|\xi\|_{H^1} \leq C\|\boldsymbol{f}\|_{L^2}$. Then

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2} \leq Ch\,\varepsilon(h)$$

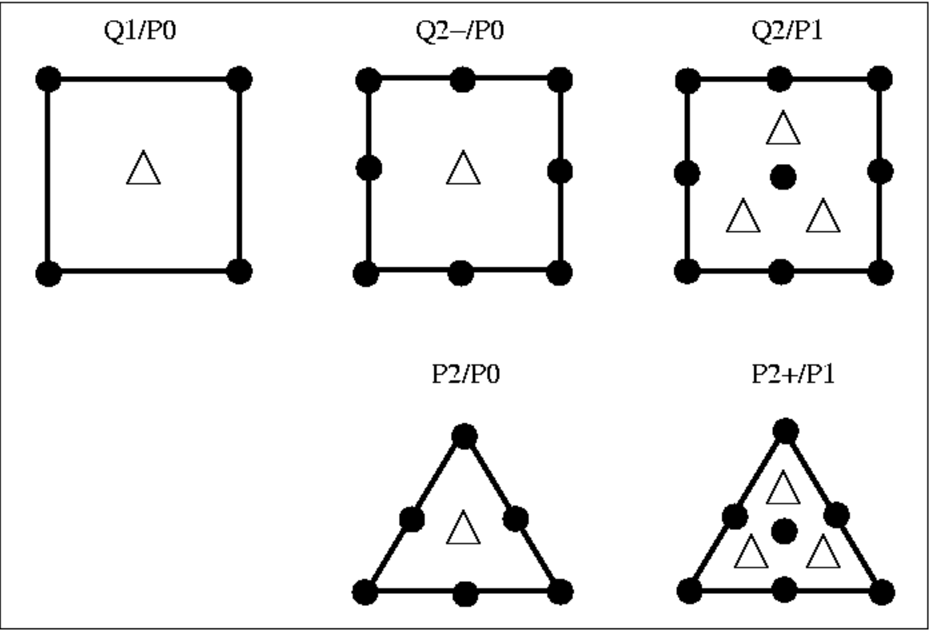
PROOF: Take $\boldsymbol{f} = \boldsymbol{u} - \boldsymbol{u}_h$ in the previous Stokes problem. Then

$$
\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2}^2 &= a(\boldsymbol{z}, \boldsymbol{u} - \boldsymbol{u}_h) - b(\xi, \boldsymbol{u} - \boldsymbol{u}_h) \\
&= a(\boldsymbol{z} - \boldsymbol{z}_h, \boldsymbol{u} - \boldsymbol{u}_h) + a(\boldsymbol{z}_h, \boldsymbol{u} - \boldsymbol{u}_h) - b(\xi - \xi_h, \boldsymbol{u} - \boldsymbol{u}_h) \\
&= a(\boldsymbol{z} - \boldsymbol{z}_h, \boldsymbol{u} - \boldsymbol{u}_h) + b(p - p_h, \boldsymbol{z}_h - \boldsymbol{z}) - b(\xi - \xi_h, \boldsymbol{u} - \boldsymbol{u}_h) \\
&\leq N_a \|\boldsymbol{z} - \boldsymbol{z}_h\|_{H^1} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1} + \|p - p_h\|_{L^2} \|\boldsymbol{z} - \boldsymbol{z}_h\|_{H^1} \\
&\quad + \|\xi - \xi_h\|_{L^2} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1} \\
&\leq C \left(h\|\boldsymbol{z}\|_{H^2} + h\|\xi\|_{H^1} \right) \varepsilon(h) \\
&\leq Ch\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2} \varepsilon(h)
\end{aligned}
$$

□

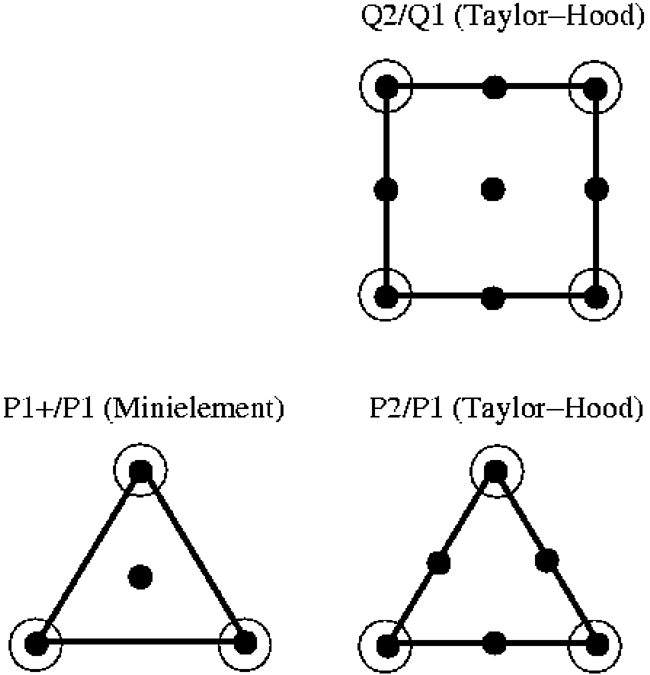
Examples of admissible pairs I: discontinuous pressures

- Q_1/P_0. Multilinear velocity, piecewise constant pressure. It is believed that, except in some particular situations, this element satisfies the inf-sup condition. It gives good results except for the possible appearance of a spurious pressure mode which can be filtered out.
- Q_2^-/P_0. Serendipid second order velocity, piecewise constant pressure.
- Q_2/P_1. Multiquadratic velocity, piecewise linear pressure.
- P_2/P_0. Quadratic velocity, piecewise constant pressure.
- P_2^+/P_0. Quadratic velocity enriched with an internal bubble, piecewise linear pressure.


Q1/P0
Q2–/P0
Q2/P1
P2/P0
P2+/P1

Examples of admissible pairs II: continuous pressures

- P_1^+/P_1. Linear velocity enriched with an internal bubble, linear pressure (minielement).
- P_2/P_1. Quadratic velocity, linear pressure (Taylor-Hood).
- Q_2/Q_1. Multiquadratic velocity, multilinear pressure (Taylor-Hood).


Q2/Q1 (Taylor–Hood)
P1+/P1 (Minielement)
P2/P1 (Taylor–Hood)

Boundary value problem

If $\boldsymbol{\sigma}$ is the the deviatoric component of the stress field, the field equations to be solved in the domain Ω can be written as

$$
\begin{aligned}
-\nabla\cdot\boldsymbol{\sigma}+\nabla p &= \boldsymbol{f}\\
\nabla\cdot\boldsymbol{u} &= 0\\
\boldsymbol{\sigma}-2\mu\boldsymbol{\varepsilon} &= 0
\end{aligned}
$$

where $\boldsymbol{\varepsilon}$ the symmetrical part of $\nabla\boldsymbol{u}$.

This is an important model problem in cases where $\boldsymbol{\sigma}$ needs to be interpolated independently.

Variational form

Let $\mathcal{V} = (H^1_0(\Omega))^d$, $\mathcal{Q} = L^2(\Omega)/\mathbb{R}$ and $\mathcal{T} = (L^2(\Omega))^{d\times d}$. If we call $U = (\boldsymbol{u}, p, \boldsymbol{\sigma})$, $\mathcal{X} = \mathcal{V} \times \mathcal{Q} \times \mathcal{T}$, the weak form of the problem consists in finding $U \in \mathcal{X}$ such that

$$B(U, V) = L(V)$$

for all $V = (\boldsymbol{v}, q, \boldsymbol{\tau}) \in \mathcal{X}$, where

$$B(U, V) = (\nabla^S \boldsymbol{v}, \boldsymbol{\sigma}) - (p, \nabla \cdot \boldsymbol{v}) + (q, \nabla \cdot \boldsymbol{u}) + \frac{1}{2\mu}(\boldsymbol{\sigma}, \boldsymbol{\tau}) - (\varepsilon, \boldsymbol{\tau})$$

$$L(V) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle$$

Stability of the Galerkin finite element discretization

Let us consider a finite element spaces $\mathcal{V}_h \subset \mathcal{V}$, $\mathcal{Q}_h \subset \mathcal{Q}$ and $\mathcal{T}_h \subset \mathcal{T}$. If $\mathcal{X}_h = \mathcal{V}_h \times \mathcal{Q}_h \times \mathcal{T}_h$ and $U_h = (\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h)$, the Galerkin finite element approximation consists in finding $U_h \in \mathcal{X}_h$ such that

$$B(U_h, V_h) = L(V_h)$$

for all $V_h = (\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h) \in \mathcal{X}_h$.

If we take $V_h = U_h$, it is found that

$$B(U_h, U_h) = \frac{1}{2\mu} \|\boldsymbol{\sigma}_h\|^2_{L^2}$$

It is seen that B is not coercive in $\mathcal{X}_h$.

Moreover, the inf-sup condition

$$\inf_{U_h \in \mathcal{X}_h} \sup_{V_h \in \mathcal{X}_h} \frac{B(U_h, V_h)}{\|U_h\|_{\mathcal{X}} \|V_h\|_{\mathcal{X}}} \geq K_B$$

which would lead to a well posed problem, is not satisfied for any positive constant K_B unless the two conditions

$$\inf_{q_h \in \mathcal{Q}_h} \sup_{\boldsymbol{v}_h \in \mathcal{V}_h} \frac{(q_h, \nabla \cdot \boldsymbol{v}_h)}{\|q_h\|_{\mathcal{Q}} \|\boldsymbol{v}_h\|_{\mathcal{V}}} \geq C_1 > 0$$

$$\inf_{\boldsymbol{v}_h \in \mathcal{V}_h} \sup_{\boldsymbol{\tau}_h \in \mathcal{T}_h} \frac{(\boldsymbol{\tau}_h, \nabla^S \boldsymbol{v}_h)}{\|\boldsymbol{\tau}_h\|_{\mathcal{T}} \|\boldsymbol{v}_h\|_{\mathcal{V}}} \geq C_2 > 0$$

hold for positive constants C_1 and C_2.

These inf-sup conditions pose stringent requirements on the choice of the finite element spaces.

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Strong and weak penalization

Strong penalization:

$$\text{Optimize } L_1^\epsilon(\boldsymbol{v}) := \frac{1}{2}a(\boldsymbol{v},\boldsymbol{v}) - l(\boldsymbol{v}) + \frac{1}{2\epsilon}\|\nabla\cdot\boldsymbol{v}\|^2$$

$$\text{Variational problem: } a(\boldsymbol{u}^\epsilon,\boldsymbol{v}) + \frac{1}{\epsilon}(\nabla\cdot\boldsymbol{u}^\epsilon,\nabla\cdot\boldsymbol{v}) = l(\boldsymbol{v}) \quad \forall\boldsymbol{v}\in V$$

$$\text{PDE: } -\nu\Delta\boldsymbol{u}^\epsilon - \frac{1}{\epsilon}\nabla(\nabla\cdot\boldsymbol{u}^\epsilon) = \boldsymbol{f}$$

Weak penalization:

$$\text{Optimize } L_2^\epsilon(\boldsymbol{v},q) := \frac{1}{2}a(\boldsymbol{v},\boldsymbol{v}) - b(q,\boldsymbol{v}) - l(\boldsymbol{v}) - \epsilon\|q\|^2$$

$$\text{Variational problem: } a(\boldsymbol{u}^\epsilon,\boldsymbol{v}) - b(p^\epsilon,\boldsymbol{v}) = l(\boldsymbol{v}) \quad \forall\boldsymbol{v}\in V$$

$$b(q,\boldsymbol{u}^\epsilon) + \epsilon(p^\epsilon,q) = 0 \quad \forall q\in Q$$

$$\text{PDE: } -\nu\Delta\boldsymbol{u}^\epsilon + \nabla p^\epsilon = \boldsymbol{f}, \quad \epsilon p^\epsilon + \nabla\cdot\boldsymbol{u}^\epsilon = 0$$

Convergence

Theorem. Given $\epsilon > 0$, let $[\boldsymbol{u}^\epsilon, p^\epsilon] \in V \times Q$ the solution of the variational problem

$$a(\boldsymbol{u}^\epsilon, \boldsymbol{v}) - b(p^\epsilon, \boldsymbol{v}) = l(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in V$$
$$b(q, \boldsymbol{u}^\epsilon) + \epsilon(p^\epsilon, q) = 0 \quad \forall q \in Q$$

and let $[\boldsymbol{u}, p] \in V \times Q$ the solution for $\epsilon = 0$. Then

$$\|\boldsymbol{u} - \boldsymbol{u}^\epsilon\|_V \leq C\epsilon, \quad \|p - p^\epsilon\|_Q \leq C'\epsilon$$

for certain constants C and C'. Likewise, if $[\boldsymbol{u}_h^\epsilon, p_h^\epsilon]$ is a stable finite element approximation to $[\boldsymbol{u}^\epsilon, p^\epsilon]$ and $[\boldsymbol{u}_h, p_h]$ to $[\boldsymbol{u}, p]$,

$$\|\boldsymbol{u}_h - \boldsymbol{u}_h^\epsilon\|_V \leq C\epsilon, \quad \|p_h - p_h^\epsilon\|_Q \leq C'\epsilon$$

PROOF: It is enough to prove the result in the continuous case:

$$a(\boldsymbol{u} - \boldsymbol{u}^\epsilon, \boldsymbol{v}) - b(p - p^\epsilon, \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in V$$
$$0 \leq (p - p^\epsilon, p - p^\epsilon) \Rightarrow (p^\epsilon, p - p^\epsilon) \leq (p, p - p^\epsilon)$$

From these two expressions:

$$\begin{aligned} K_a \|\boldsymbol{u} - \boldsymbol{u}^\epsilon\|_V^2 &\leq a(\boldsymbol{u} - \boldsymbol{u}^\epsilon, \boldsymbol{u} - \boldsymbol{u}^\epsilon) = b(p - p^\epsilon, \boldsymbol{u} - \boldsymbol{u}^\epsilon) \\ &= -b(p - p^\epsilon, \boldsymbol{u}^\epsilon) = \epsilon(p^\epsilon, p - p^\epsilon) \\ &\leq \epsilon(p, p - p^\epsilon) \leq \epsilon \|p\|_Q \|p - p^\epsilon\|_Q \end{aligned}$$

From the inf-sup condition, there exists $\boldsymbol{v} \in V$ such that

$$\begin{aligned} K_b \|p - p^\epsilon\|_Q \|\boldsymbol{v}\|_V &\leq b(p - p^\epsilon, \boldsymbol{v}) = a(\boldsymbol{u} - \boldsymbol{u}^\epsilon, \boldsymbol{v}) \\ &\leq N_a \|\boldsymbol{u} - \boldsymbol{u}^\epsilon\|_V \|\boldsymbol{v}\|_V \end{aligned}$$

$$\Rightarrow \quad K_a \|\boldsymbol{u} - \boldsymbol{u}^\epsilon\|_V^2 \leq \epsilon \|p\|_Q \frac{N_a}{K_b} \|\boldsymbol{u} - \boldsymbol{u}^\epsilon\|_V$$

Discretization

Let $Q_h \subset Q$, $V_h \subset V$ be such that the discrete inf-sup condition holds. If functions in Q_h are discontinuous, we may take as test function for the continuity equation

$$q_h = N_p^a \begin{cases} \neq 0 & \text{in } K \text{ if node } a \in K \\ = 0 & \text{outside } K \end{cases}$$

The continuity equation becomes:

$$\epsilon \sum_b \left(\int_K N_p^a N_p^b \right) \mathrm{P}^b + \sum_c \left(\int_K N_p^a \partial_i N_u^c \right) \mathrm{U}_i^c = 0$$

which has the form

$$\epsilon \mathrm{M}_{p,K} \mathrm{P}_K + \mathrm{D}_K \mathrm{U}_K = 0$$

where subindex K denotes that all matrices and arrays refer only to element K.

Since $\mathsf{M}_{p,K}$ is easily inverted:

$$\mathsf{P}_K = -\epsilon^{-1}\mathsf{M}_{p,K}^{-1}\mathsf{D}_K\mathsf{U}_K$$

The momentum equation becomes:

$$\begin{aligned}\mathsf{KU} - \sum_K \mathsf{D}_K^T\mathsf{P}_K &= \mathsf{KU} + \epsilon^{-1}\sum_K \mathsf{D}_K^T\mathsf{M}_{p,K}^{-1}\mathsf{D}_K\mathsf{U}_K \\ &= \left[\mathsf{K} + \epsilon^{-1}\mathbb{A}_K\left(\mathsf{D}_K^T\mathsf{M}_{p,K}^{-1}\mathsf{D}_K\right)\right]\mathsf{U} = \mathsf{F}\end{aligned}$$

which can be solved for U without relying on P. It has to be noted that

- Matrix $\mathsf{K}_\epsilon := \mathsf{K} + \epsilon^{-1}\mathbb{A}_K\left(\mathsf{D}_K^T\mathsf{M}_{p,K}^{-1}\mathsf{D}_K\right)$ is symmetric and positive definite. No pivoting will be needed.
- The condition number of K_ϵ is $\chi = \mathcal{O}(\epsilon^{-1})$. If ϵ is small, iterative solvers are unfeasible.

Problem formulation

Variational formulation: Find $\boldsymbol{u}_h^\epsilon \in V_h$ such that

$$a(\boldsymbol{u}_h^\epsilon, \boldsymbol{v}_h) + \frac{1}{\epsilon}(\nabla \cdot \boldsymbol{u}_h^\epsilon, \nabla \cdot \boldsymbol{v}_h) = l(\boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in V_h$$

Associated pressure space:

$$p_h^\epsilon = -\frac{1}{\epsilon}\nabla \cdot \boldsymbol{u}_h^\epsilon \iff Q_h = \nabla \cdot V_h$$

Matrix form:

$$\left(\mathrm{K} + \frac{1}{\epsilon}\mathrm{K}_v\right)\mathrm{U}^\epsilon = \mathrm{F}, \quad {\mathrm{K}_v}_{ij}^{ab} = (\partial_i N_u^a, \partial_j N_u^b)$$

Locking: if matrix K_v is nonsingular (after prescribing BC's)

$$\lim_{\epsilon \longrightarrow 0} \|\mathrm{U}^\epsilon\| = 0$$

Underintegration

Original idea: integrate K_v with a low order rule that yields a rank-defficient matrix (Zienkiewicz, Taylor, Too (1973) in plates, Fried (1974) in elasticity, formalization by Oden (1982)).

Reduced-integration-penality (RIP) method: Find $\boldsymbol{u}_h^\epsilon \in V_h$ such that

$$a(\boldsymbol{u}_h^\epsilon, \boldsymbol{v}_h) + \frac{1}{\epsilon}(\nabla \cdot \boldsymbol{u}_h^\epsilon, \nabla \cdot \boldsymbol{v}_h)_* = l(\boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in V_h$$

where

$$(f, g)_* = \sum_K \sum_{i=1}^{n_{\text{int}}} w_K^i f(\xi_K^i) g(\xi_K^i)$$

where n_{int} is the number of integration points per element.

Equivalence with weak penalization

Theorem. Let Q_h the space of piecewise polynomials that are n_{int}-unisolvent. If

$$(q_h, q_h')_* = (q_h, q_h') \quad \forall q_h, q_h' \in Q_h$$
$$(q_h, \nabla \cdot \boldsymbol{v}_h)_* = (q_h, \nabla \cdot \boldsymbol{v}_h) \quad \forall q_h \in Q_h, \forall \boldsymbol{v}_h \in V_h$$

then RIP method $\iff$ weak penalization in $V_h \times Q_h$

PROOF:

$$0 = \epsilon(q_h, p_h^\epsilon) + (q_h, \nabla \cdot \boldsymbol{u}_h^\epsilon) = (q_h, \epsilon p_h^\epsilon + \nabla \cdot \boldsymbol{u}_h^\epsilon)_*$$
$$= \sum_K \sum_{i=1}^{n_{\text{int}}} w_K^i q_h(\xi_K^i) \left(\epsilon p_h^\epsilon(\xi_K^i) + \nabla \cdot \boldsymbol{u}_h^\epsilon(\xi_K^i) \right)$$

Since Q_h is n_{int}-unisolvent, $\epsilon p_h^\epsilon(\xi_K^i) = -\nabla \cdot \boldsymbol{u}_h^\epsilon(\xi_K^i)$. It can be checked that the starting weak penalty method reduces to the RIP method.

Example: Q_1 element

Consider:

- Method A: Weak penalty method with Q_1/P_0 interpolation
- Method B: RIP method, with 1 point integration for $\epsilon^{-1}(\nabla \cdot \boldsymbol{u}_h^\epsilon, \nabla \cdot \boldsymbol{v}_h)$

Then

$$\text{Method A} \iff \text{Method B}$$

since 1 point integration on quads is exact for functions of the form

(q_h, q_h')	product of two constants
$(q_h, \nabla \cdot \boldsymbol{v}_h)$	product of constant and linear function

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   5.2 Stabilization through bubble functions
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Motivation of stabilized finite element methods

- In general:
  - Singular perturbation problems
  - Non-definite bilinear forms with non-trivial inf-sup conditions
- For the Stokes problem:
  - Equal velocity-pressure interpolation
  - Stability of convenient elements such as Q_1/P_0 and P_1/P_0
- Other examples that “need” stabilization:
  - Convection-diffusion, reaction-diffusion
  - Plate problem
  - Darcy’s problem
  - Waves...

The Stokes BVP reformulated

The Stokes problem can be written in vector form as

$$\mathcal{L}(\boldsymbol{u}, p) = \begin{bmatrix} \mathcal{L}_1(\boldsymbol{u}, p) \\ \mathcal{L}_2(\boldsymbol{u}, p) \end{bmatrix} := \begin{bmatrix} -\nu \Delta \boldsymbol{u} + \nabla p \\ \nabla \cdot \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \end{bmatrix} =: \mathcal{F}$$

Matrix form of the “convective” (first order) terms in 2D:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial x_1} \begin{bmatrix} u_1 \\ u_2 \\ p \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \frac{\partial}{\partial x_2} \begin{bmatrix} u_1 \\ u_2 \\ p \end{bmatrix} = \begin{bmatrix} \nabla p \\ \nabla \cdot \boldsymbol{u} \end{bmatrix}$$

Original idea: extension of SUPG

Since the first order terms can be understood as "convective", the idea is to use a SUPG-like strategy (diffusion is 0 for the pressure). This means to multiply the original problem by $[\boldsymbol{v}_h, q_h]$ to obtain the Galerkin terms and by

$$\boldsymbol{\tau} \begin{bmatrix} \nabla q_h \\ \nabla \cdot \boldsymbol{v}_h \end{bmatrix}$$

to obtain the stabilizing terms. Here $\boldsymbol{\tau}$ is a matrix that can be taken in 2D as

$$\boldsymbol{\tau} = \mathrm{diag}(\tau_1, \tau_1, \tau_2)$$

Original method

The stabilized finite element problem based on SUPG is: Find $\boldsymbol{U}_h := [\boldsymbol{u}_h, p_h]$ such that:

$$\nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - (p_h, \nabla \cdot \boldsymbol{v}_h) + (q_h, \nabla \cdot \boldsymbol{u}_h) - \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle + \sum_K \int_K \tau_1 \nabla q_h \cdot (-\nu \Delta \boldsymbol{u}_h + \nabla p_h - \boldsymbol{f}) + \sum_K \int_K \tau_2 (\nabla \cdot \boldsymbol{v}_h)(\nabla \cdot \boldsymbol{u}_h) = 0$$

for all $\boldsymbol{V}_h := [\boldsymbol{v}_h, q_h]$. The stabilizing term is of the form

$$\sum_K \int_K [\boldsymbol{\tau} \mathcal{P}(\boldsymbol{V}_h)] \cdot [\mathcal{L}(\boldsymbol{U}_h) - \mathcal{F}]$$

where $\mathcal{P}(\boldsymbol{V}_h)$ is part of the Stokes operator applied to $\boldsymbol{V}_h$.

The GLS method

The next idea is to apply all the differential operator to the test functions to obtain the stabilization term. This yields the term

$$
\begin{aligned}
&\sum_K \int_K [\boldsymbol{\tau}\mathcal{L}(\boldsymbol{V}_h)] \cdot [\mathcal{L}(\boldsymbol{U}_h) - \mathcal{F}] \\
&= \sum_K \int_K \tau_1 \mathcal{L}_1(\boldsymbol{v}_h, q_h)[\mathcal{L}_1(\boldsymbol{u}_h, p_h) - \boldsymbol{f}] \\
&\quad + \sum_K \int_K \tau_2 \mathcal{L}_2(\boldsymbol{v}_h, q_h)\mathcal{L}_2(\boldsymbol{u}_h, p_h) \\
&= \sum_K \int_K \tau_1 \left(-\nu\Delta\boldsymbol{v}_h + \nabla q_h\right) \cdot \left(-\nu\Delta\boldsymbol{u}_h + \nabla p_h - \boldsymbol{f}\right) \\
&\quad + \sum_K \int_K \tau_2 (\nabla \cdot \boldsymbol{v}_h)(\nabla \cdot \boldsymbol{u}_h)
\end{aligned}
$$

The reason for the name

If A is a matrix and x and b are vectors, we have:

$$\|Ax - b\|^2 = (Ax - b, Ax - b)$$
$$\text{is minimum} \iff$$
$$\frac{\partial}{\partial x}\|Ax - b\|^2 = 0 \iff (Ay, Ax - b) = 0 \quad \forall y$$

Thus, the stabilizing term

$$\sum_K \int_K [\boldsymbol{\tau}\mathcal{L}(\boldsymbol{V}_h)] \cdot [\mathcal{L}(\boldsymbol{U}_h) - \mathcal{F}]$$

can be thought of as a least square form (weighted by $\boldsymbol{\tau}$) of the elementwise finite element residual.

System of discrete variational equations

Find $[\boldsymbol{u}_h, p_h]$ such that:

First (momentum) equation:

$$\nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - (p_h, \nabla \cdot \boldsymbol{v}_h) - \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle + \sum_K \int_K \tau_1 \, (-\nu \Delta \boldsymbol{v}_h) \cdot (-\nu \Delta \boldsymbol{u}_h + \nabla p_h - \boldsymbol{f}) + \sum_K \int_K \tau_2 (\nabla \cdot \boldsymbol{v}_h)(\nabla \cdot \boldsymbol{u}_h) = 0$$

Second (continuity) equation:

$$(q_h, \nabla \cdot \boldsymbol{u}_h) + \sum_K \int_K \tau_1 \nabla q_h \cdot (-\nu \Delta \boldsymbol{u}_h + \nabla p_h - \boldsymbol{f}) = 0$$

for all $[\boldsymbol{v}_h, q_h]$.

Matrix form

The matrix structure is the following:

$$\begin{bmatrix} \mathrm{K} + \bar{\mathrm{K}} & -\mathrm{D}^t - \bar{\mathrm{D}}^t \\ \mathrm{D} - \bar{\mathrm{D}} & \mathrm{L} \end{bmatrix} \begin{bmatrix} \mathrm{U} \\ \mathrm{P} \end{bmatrix} = \begin{bmatrix} \mathrm{F} + \bar{\mathrm{F}} \\ \bar{\bar{\mathrm{F}}} \end{bmatrix}$$

where the arrays with an overbar come from the stabilization terms and the rest from the Galerkin contribution:

$$\mathrm{K} \longleftarrow \nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h)$$

$$\bar{\mathrm{K}} \longleftarrow \sum_K \int_K \tau_1 \left(-\nu \Delta \boldsymbol{v}_h\right) \cdot \left(-\nu \Delta \boldsymbol{u}_h\right) + \sum_K \int_K \tau_2 (\nabla \cdot \boldsymbol{v}_h)(\nabla \cdot \boldsymbol{u}_h)$$

$$\mathrm{D} \longleftarrow (q_h, \nabla \cdot \boldsymbol{u}_h)$$

$$\bar{\mathrm{D}} \longleftarrow \sum_K \int_K \tau_1 \nabla q_h \cdot (-\nu \Delta \boldsymbol{u}_h)$$

$$\mathsf{L} \longleftarrow \sum_K \int_K \tau_1 \nabla q_h \cdot \nabla p_h$$

$$\bar{\mathsf{F}} \longleftarrow \sum_K \int_K \tau_1 \left(-\nu \Delta \boldsymbol{v}_h\right) \cdot \boldsymbol{f}$$

$$\bar{\bar{\mathsf{F}}} \longleftarrow \sum_K \int_K \tau_1 \nabla q_h \cdot \boldsymbol{f}$$

Observe that the final algebraic system is not symmetric, but it is positive-definite.

The stabilizing effect of the GLS method on the pressure comes from the term involving L, which provides a pressure Laplacian.

Main convergence theorem

Theorem. Suppose that the Stokes problem is solved using the GLS method with

$$\tau_1 = C_\tau \frac{h^2}{\nu}, \quad \tau_2 = 0$$

Let B_{GLS} the bilinear form associated to this method and define the mesh dependent norm $|||\cdot|||$ by

$$|||\boldsymbol{V}_h|||^2 = \nu \|\boldsymbol{v}_h\|^2 + \sum_K \int_K \tau_1 \|\nabla q_h\|^2$$

Then, for a small enough value of the constant C_τ,

$$B_{\mathrm{GLS}}(\boldsymbol{U}_h, \boldsymbol{U}_h) \geq C|||\boldsymbol{U}_h|||^2 \quad \text{(Stability)}$$

$$|||\boldsymbol{U} - \boldsymbol{U}_h||| \leq \inf_{\boldsymbol{v}_h \in \mathcal{W}_0} |||\boldsymbol{U} - \boldsymbol{V}_h||| \quad \text{(Convergence)}$$

Remarks

- The method presented is not what was originally called GLS in the case of the Stokes problem. This name was given to a method yielding a symmetric problem which we will recover from a different point of view later on.
- It is also possible to obtain the same result with the norm

$$|||\boldsymbol{V}_h|||'^2 = \nu\|\boldsymbol{v}_h\|^2 + \sum_K \int_K \tau_1 \| -\nu\Delta\boldsymbol{v}_h + \nabla q_h\|^2$$

- If $\tau_2 > 0$, the norm in which stability and convergence is proven is

$$|||\boldsymbol{V}_h|||^2 + \sum_K \int_K \tau_2 \|\nabla \cdot \boldsymbol{v}_h\|^2$$

Inclusion of jumps for discontinuous pressures

The Q_1/P_0 element (for example) does not satisfy the BB condition:

- There are spurious pressure modes
- $K_b = K_b(h)$ in the inf-sup condition, at least for regular meshes.

Let us consider a partition of Ω into patches of 2×2 Q_1/P_0 elements (in 2D). Let Γ_p be the collection of interior edges of patch number p and $[\![\cdot]\!]$ is the jump operator.

The Silvester-Kechkar stabilization consists of replacing the incompressibility condition $(q_h, \nabla \cdot \boldsymbol{u}_h) = 0$ by

$$(q_h, \nabla \cdot \boldsymbol{u}_h) + \sum_p \alpha h \int_{\Gamma_p} [\![q_h]\!] [\![p_h]\!] = 0$$

Generalized convergence result

Theorem. Consider the GLS method with the associated bilinear form enlarged with the term $\sum_E \alpha h \int_E [\![q_h]\!][\![p_h]\!]$, where the sum extends to all the internal edges (2D) or faces (3D) of the finite element partition. Define the mesh dependent norm $|||\boldsymbol{V}_h|||^2 = \nu\|\boldsymbol{v}_h\|^2 + \sum_K \int_K \tau_1 \|\nabla q_h\|^2 + \sum_E \int_E \alpha h [\![q_h]\!]^2$. Then, for a small enough value of the constant C_τ, the method is stable and optimally convergent in this norm if at least one of the following conditions is verified:

- Pressures are continuous (the jump disappears)
- $\alpha > 0$
- The order of velocity interpolation is $k \geq d$

GLS with P_1 elements

Let us consider the GLS method using P_1 finite element interpolations for both the velocity and the pressure.

If $\tau_2 = 0$ and $\Delta \boldsymbol{u}_h = \Delta \boldsymbol{v}_h = \boldsymbol{0}$ within each element, the GLS method reduces to

$$\nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - (p_h, \nabla \cdot \boldsymbol{v}_h) = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle$$

$$(q_h, \nabla \cdot \boldsymbol{u}_h) + \sum_K \int_K \tau_1 \nabla q_h \cdot (\nabla p_h - \boldsymbol{f}) = 0$$

for all $[\boldsymbol{v}_h, q_h]$.

Galerkin with the minielement

Consider the P_1^+/P_1 interpolation (div-stable) and the standard Galerkin method.

Within each element, the velocity (which we denote $\boldsymbol{u}_h^b(\boldsymbol{x})$) can be expanded as:

$$\boldsymbol{u}_h^b(\boldsymbol{x})|_K = \boldsymbol{u}_h(\boldsymbol{x})|_K + \sum_j \boldsymbol{B}_j^K(\boldsymbol{x}) u_j^{K,b}$$

$\boldsymbol{u}_h(\boldsymbol{x})$: linear part of the velocity.
$\boldsymbol{B}_j^K(\boldsymbol{x})$: vector that has the bubble function $B^K(\boldsymbol{x})$ in the jth component and zero the rest.
$u_j^{K,b}$: j-th component of the velocity nodal value associated to the bubble of element K.

Condensation of the bubble velocity

Using the first equation to be solved and taking $\boldsymbol{v}_h = \boldsymbol{B}_j^K$ we have, for each element,

$$\nu(\nabla \boldsymbol{u}_h^b, \nabla \boldsymbol{B}_j^K) - (p_h, \nabla \cdot \boldsymbol{B}_j^K) = \langle \boldsymbol{f}, \boldsymbol{B}_j^K \rangle$$

$$u_j^{K,b} \nu(\nabla \boldsymbol{B}_j^K, \nabla \boldsymbol{B}_j^K) = \langle \boldsymbol{f}, \boldsymbol{B}_j^K \rangle + (p_h, \nabla \cdot \boldsymbol{B}_j^K) - \nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{B}_j^K)$$

$$= \int_K \boldsymbol{B}_j^K \cdot (\boldsymbol{f} - \nabla p_h + \nu \Delta \boldsymbol{u}_h) + \int_{\partial K} \boldsymbol{B}_j^K \cdot (p\boldsymbol{n} - \nu \boldsymbol{n} \cdot \nabla \boldsymbol{u}_h)$$

$$= \int_K \boldsymbol{B}_j^K \cdot (\boldsymbol{f} - \nabla p_h),$$

$$u_j^{K,b} = \frac{1}{\nu(\nabla B^K, \nabla B^K)} \int_K \boldsymbol{B}_j^K \cdot (\boldsymbol{f} - \nabla p_h)$$

Modified continuity equation

For the second equation to be solved:

$$(q_h, \nabla \cdot \boldsymbol{u}_h^b) = (q_h, \nabla \cdot \boldsymbol{u}_h) + \sum_K \sum_j u_j^{K,b} (q_h, \nabla \cdot \boldsymbol{B}_j^K)$$

$$= (q_h, \nabla \cdot \boldsymbol{u}_h)$$

$$- \sum_K \sum_j \frac{1}{\nu(\nabla B^K, \nabla B^K)} \left[\int_K \boldsymbol{B}_j^K \cdot (\boldsymbol{f} - \nabla p_h) \right] \left[\int_K \boldsymbol{B}_j^K \cdot \nabla q_h \right]$$

$$= (q_h, \nabla \cdot \boldsymbol{u}_h) - \sum_K \int_K \nabla q_h \cdot \hat{\tau} (\boldsymbol{f} - \nabla p_h) = 0$$

where we have used the fact that ∇q_h and ∇p_h are piecewise constants and we have defined

$$\hat{\tau} := \left[\nu(\nabla B^K, \nabla B^K) \right]^{-1} (\text{meas} K)^{-1} \left(\int_K B^K \right)^2$$

Comparison of GLS with P_1 and Galerkin with P_1^+/P_1

Momentum equation: If $\boldsymbol{B}_j^K$ is a bubble function, it follows that

$$\int_K \nabla \boldsymbol{B}_j^K : \nabla \boldsymbol{v}_h = -\int_K \boldsymbol{B}_j^K \cdot \Delta \boldsymbol{v}_h + \int_{\partial K} \boldsymbol{B}_j^K \cdot (\boldsymbol{n} \cdot \nabla \boldsymbol{v}_h) = 0$$

so the momentum equation for Galerkin with P_1^+/P_1 is

$$\nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - (p_h, \nabla \cdot \boldsymbol{v}_h) = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle$$

This equation is the same for GLS (and for Galerkin) with P_1

Continuity equation: It can be written as

$$(q_h, \nabla \cdot \boldsymbol{u}_h) + \sum_K \int_K \nabla q_h \cdot \hat{\tau}(\nabla p_h - \boldsymbol{f}) = 0$$

This equation is the same for GLS with P_1, with an appropriate definition of τ, which in both cases behave as $\frac{h^2}{\nu}$.

Remarks

- For elements of higher order, the analogy GLS = Galerkin with bubble functions does not hold exactly.
- However, the previous development motivated the design of stabilized methods by introducing bubble functions.
- The problem of estimating the stabilization parameters is transformed into the problem of selecting the bubble. The classical cubic bubble is only a possibility.
- This framework was particularly exploited in the context of "residual free bubbles" introduced later.

Model problem

System of convection-diffusion-reaction equations:

$$\mathcal{L}(\boldsymbol{U}) = \boldsymbol{F} \quad \text{in} \quad \Omega$$
$$\mathcal{L}(\boldsymbol{U}) := \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(\boldsymbol{A}_i \boldsymbol{U}) - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(\boldsymbol{K}_{ij}\frac{\partial \boldsymbol{U}}{\partial x_j}\right) + \boldsymbol{S}\boldsymbol{U}$$
$$\boldsymbol{U} = \boldsymbol{0} \quad \text{on } \partial\Omega$$

$\boldsymbol{U}$ and $\boldsymbol{F}$ are vectors of D unknowns and $\boldsymbol{A}_i$, $\boldsymbol{K}_{ij}$ and $\boldsymbol{S}$ are $D \times D$ matrices ($i, j = 1, ..., n_{\rm sd}$).

Let $\mathcal{W}$ be the space where the problem is posed. The weak form is: Find $\boldsymbol{U} \in \mathcal{W}$ such that

$$B(\boldsymbol{U}, \boldsymbol{V}) = L(\boldsymbol{V}) \qquad \forall \boldsymbol{V} \in \mathcal{W}$$

Integrating by parts the second-order terms and assuming that the boundary contributions vanish:

$$B(\boldsymbol{U},\boldsymbol{V}) := \int_\Omega \boldsymbol{V}^{\mathrm{t}} \frac{\partial}{\partial x_i}(\boldsymbol{A}_i\boldsymbol{U}) + \int_\Omega \frac{\partial \boldsymbol{V}^{\mathrm{t}}}{\partial x_i}\boldsymbol{K}_{ij}\frac{\partial \boldsymbol{U}}{\partial x_j} + \int_\Omega \boldsymbol{V}^{\mathrm{t}}\boldsymbol{S}\boldsymbol{U}$$

$$L(\boldsymbol{V}) := \int_\Omega \boldsymbol{V}^{\mathrm{t}}\boldsymbol{F}$$

If $\mathcal{W}_h \subset \mathcal{W}$ is a finite element approximation to $\mathcal{W}$, the Galerkin finite element problem is: find $\boldsymbol{U}_h \in \mathcal{W}_h$ such that

$$B(\boldsymbol{U}_h,\boldsymbol{V}_h) = L(\boldsymbol{V}_h) \qquad \forall \boldsymbol{V}_h \in \mathcal{W}_h$$

Description of the method

Let $\mathcal{W} = \mathcal{W}_h \oplus \tilde{\mathcal{W}}$. The continuous problem is equivalent to: find $\boldsymbol{U}_h \in \mathcal{W}_h$ and $\tilde{\boldsymbol{U}} \in \tilde{\mathcal{W}}$ such that

$$
\begin{aligned}
B(\boldsymbol{U}_h, \boldsymbol{V}_h) + B(\tilde{\boldsymbol{U}}, \boldsymbol{V}_h) &= L(\boldsymbol{V}_h) \qquad && \forall \boldsymbol{V}_h \in \mathcal{W}_h \\
B(\boldsymbol{U}_h, \tilde{\boldsymbol{V}}) + B(\tilde{\boldsymbol{U}}, \tilde{\boldsymbol{V}}) &= L(\tilde{\boldsymbol{V}}) \qquad && \forall \tilde{\boldsymbol{V}} \in \tilde{\mathcal{W}}
\end{aligned}
$$

Function $\tilde{\boldsymbol{U}} \in \tilde{\mathcal{W}}$ is called the subgrid-scale or subscale, and $\tilde{\mathcal{W}}$ the is thus the space of subscales.

Let us introduce the notation:

$$
\int_{\Omega'} := \sum_K \int_K, \quad \int_{\partial\Omega'} := \sum_{\partial K} \int_{\partial K}
$$

Integrating by parts within each element:

$$B(\boldsymbol{U}_h, \boldsymbol{V}_h) + \int_{\partial\Omega'} \tilde{\boldsymbol{U}}^{\mathrm{t}} n_i \boldsymbol{K}_{ij} \frac{\partial \boldsymbol{V}_h}{\partial x_j} \, \mathrm{d}\Gamma + \int_{\Omega'} \tilde{\boldsymbol{U}}^{\mathrm{t}} \mathcal{L}^*(\boldsymbol{V}_h) = L(\boldsymbol{V}_h)$$

$$\int_{\partial\Omega'} \tilde{\boldsymbol{V}}^{\mathrm{t}} n_i \boldsymbol{K}_{ij} \frac{\partial}{\partial x_j} (\boldsymbol{U}_h + \tilde{\boldsymbol{U}}) \, \mathrm{d}\Gamma + \int_{\Omega'} \tilde{\boldsymbol{V}}^{\mathrm{t}} \mathcal{L}(\tilde{\boldsymbol{U}}) = \int_{\Omega'} \tilde{\boldsymbol{V}}^{\mathrm{t}} [\boldsymbol{F} - \mathcal{L}(\boldsymbol{U}_h)]$$

Adjoint of the convection-diffusion-reaction operator:

$$\mathcal{L}^*(\boldsymbol{V}_h) = -\boldsymbol{A}_i^{\mathrm{t}} \frac{\partial \boldsymbol{V}_h}{\partial x_i} - \frac{\partial}{\partial x_i} \left(\boldsymbol{K}_{ij}^{\mathrm{t}} \frac{\partial \boldsymbol{V}_h}{\partial x_j} \right) + \boldsymbol{S}^{\mathrm{t}} \boldsymbol{V}_h$$

Problem for the subscales

Since diffusive fluxes must be continuous across inter-element boundaries, the second equation is equivalent to:

$$\mathcal{L}(\tilde{\boldsymbol{U}}) = \boldsymbol{F} - \mathcal{L}(\boldsymbol{U}_h) + \boldsymbol{V}_{h,\mathrm{ort}} \quad \text{in } K$$
$$\tilde{\boldsymbol{U}} = \tilde{\boldsymbol{U}}_{\mathrm{ske}} \quad \text{on } \partial K$$

for all K. We call $\tilde{\boldsymbol{U}}_{\mathrm{ske}}$ the *skeleton* of $\boldsymbol{U}$.

Main problems:

- How to choose $\boldsymbol{V}_{h,\mathrm{ort}}$?
- How to approximate $\tilde{\boldsymbol{U}}_{\mathrm{ske}}$?
- How to solve for $\tilde{\boldsymbol{U}}$?

Approximation of the subscales

In the following, we will make the approximations:

- $\boldsymbol{V}_{h,\mathrm{ort}} = \boldsymbol{0}$ (this defines the space for $\tilde{\boldsymbol{U}}$).
- $\tilde{\boldsymbol{U}}_{\mathrm{ske}} = \boldsymbol{0}$

It remains to define how to solve for $\tilde{\boldsymbol{U}}$. Different approaches, yielding different stabilization methods, are possible. In any case, the problem to be solved is:

$$B(\boldsymbol{U}_h, \boldsymbol{V}_h) + \int_{\Omega'} \tilde{\boldsymbol{U}}^{\mathrm{t}} \mathcal{L}^*(\boldsymbol{V}_h) = L(\boldsymbol{V}_h) \quad \forall \boldsymbol{V}_h$$

$$\mathcal{L}(\tilde{\boldsymbol{U}}) = \boldsymbol{F} - \mathcal{L}(\boldsymbol{U}_h) \quad \text{in } K$$

$$\tilde{\boldsymbol{U}} = \boldsymbol{0} \quad \text{on } \partial K$$

Approximation to the Green's function

The problem for the subscales is

$$\mathcal{L}(\tilde{\boldsymbol{U}}) = \boldsymbol{F} - \mathcal{L}(\boldsymbol{U}_h) \quad \text{in } K$$
$$\tilde{\boldsymbol{U}} = \boldsymbol{0} \quad \text{on } \partial K$$

The solution can be written using the formula:

$$\tilde{\boldsymbol{U}}(\boldsymbol{x}) = \int_K \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) \left[\boldsymbol{F} - \mathcal{L}(\boldsymbol{U}_h) \right] (\boldsymbol{y}) d\boldsymbol{y}$$

where $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y})$ is the Green's function of the problem (which now is a matrix) and the integration arguments have been explicitly indicated.

The problem now is to approximate $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y})$

Algebraic approximation to the Green's function

Let us consider the approximation

$$\boldsymbol{g}(\boldsymbol{x},\boldsymbol{y}) \approx \boldsymbol{\tau}(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y})$$

where $\delta(\boldsymbol{x}-\boldsymbol{y})$ is the Dirac delta function. Using this approximation:

$$\begin{aligned}\tilde{\boldsymbol{U}}(\boldsymbol{x}) &\approx \int_K \delta(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{\tau}(\boldsymbol{x})\left[\boldsymbol{F}-\mathcal{L}(\boldsymbol{U}_h)\right](\boldsymbol{y})d\boldsymbol{y} \\ &= \boldsymbol{\tau}(\boldsymbol{x})\left[\boldsymbol{F}-\mathcal{L}(\boldsymbol{U}_h)\right](\boldsymbol{x})\end{aligned}$$

A possible way to determine $\boldsymbol{\tau}$ is to impose that moments of $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{y})$ and $\boldsymbol{\tau}(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y})$ coincide. In particular, if $\boldsymbol{\tau}(\boldsymbol{x})$ is taken constant,

$$\boldsymbol{\tau} = \frac{1}{\mathrm{meas}(K)}\int_K\int_K \boldsymbol{g}(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y}$$

In general, $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y})$ is unknown and $\boldsymbol{\tau}$ must be approximated by other means.

If $\boldsymbol{\tau}$ is taken constant, we call it algebraic approximation to the subscales. In this case, the stabilized method is:

$$B(\boldsymbol{U}_h, \boldsymbol{V}_h) + \int_{\Omega'} [-\mathcal{L}^*(\boldsymbol{V}_h)]^{\mathrm{t}} \boldsymbol{\tau} [\mathcal{L}(\boldsymbol{U}_h) - \boldsymbol{F}] = L(\boldsymbol{V}_h) \quad \forall \boldsymbol{V}_h$$

Equivalently, the stabilized problem can be written:

$$B_{\mathrm{asgs}}(\boldsymbol{U}_h, \boldsymbol{V}_h) = L_{\mathrm{asgs}}(\boldsymbol{V}_h) \quad \forall \boldsymbol{V}_h$$

where

$$B_{\mathrm{asgs}}(\boldsymbol{U}_h, \boldsymbol{V}_h) = B(\boldsymbol{U}_h, \boldsymbol{V}_h) + \int_{\Omega'} [-\mathcal{L}^*(\boldsymbol{V}_h)]^{\mathrm{t}} \boldsymbol{\tau} [\mathcal{L}(\boldsymbol{U}_h)]$$

$$L_{\mathrm{asgs}}(\boldsymbol{V}_h) = L(\boldsymbol{V}_h) + \int_{\Omega'} [-\mathcal{L}^*(\boldsymbol{V}_h)]^{\mathrm{t}} \boldsymbol{\tau} [\boldsymbol{F}]$$

Approximation to the subscales using residual free bubbles

To simplify the notation, let us consider an abstract problem of the form $a(u, v) = l(v)$. The problem for the subscales is

$$\mathcal{L}(\tilde{u}) = f - \mathcal{L}(u_h) \quad \text{in } K$$
$$\tilde{u} = 0 \quad \text{on } \partial K$$

If $\tilde{W}_K$ is the space of subscales within each element, the problem can also be written as

$$a(\tilde{u}, \tilde{v}) = l(\tilde{v}) - a(u_h, \tilde{v}) \quad \forall \tilde{\boldsymbol{v}} \in \tilde{W}_K$$

Let us consider the finite element subspace of $\tilde{W}_K$:

$$B_h := \mathrm{Span}\{B_1(\boldsymbol{x}), ..., B_{n_{\mathrm{bub}}}(\boldsymbol{x})\} \subset \tilde{W}_K$$

We approximate:

$$\tilde{u} \approx \tilde{u}_h = \sum_{i=1}^{n_{\text{bub}}} B_i(\boldsymbol{x})\tilde{u}_i \quad \text{in } K$$

Then, for each element:

$$\sum_{i=1}^{n_{\text{bub}}} a(B_j, B_i)\tilde{u}_i = l(B_j) - a(u_h, B_j)$$

$$= \int_K B_j(\boldsymbol{x})\left[f - \mathcal{L}(u_h)\right] \quad \forall j = 1, \ldots, n_{\text{bub}}$$

If we call $a_{ij} = a(B_j, B_i)$ and a_{ij}^{-1} are the components of the inverse matrix, we have, in each element K:

$$\tilde{u}_h(\boldsymbol{x}) = \sum_{i,j=1}^{n_{\text{bub}}} \int_K B_i(\boldsymbol{x}) a_{ij}^{-1} B_j(\boldsymbol{y})\left[\mathcal{L}(u_h) - f\right](\boldsymbol{y})d\boldsymbol{y}$$

We note that:

$$\boldsymbol{g}(\boldsymbol{x},\boldsymbol{y}) \approx \boldsymbol{g}_h(\boldsymbol{x},\boldsymbol{y}) = \sum_{i,j=1}^{n_{\text{bub}}} B_i(\boldsymbol{x}) a_{ij}^{-1} B_j(\boldsymbol{y})$$

It remains to specify which are the bubble functions that span B_h within each element. Ideally:

$$\mathcal{L}(\tilde{u}) = f - \mathcal{L}(u_h) \iff \sum_{i=1}^{n_{\text{bub}}} \mathcal{L}(B_i)\tilde{u}_i = f - \sum_{j=1}^{n_{\text{nod}}} \mathcal{L}(N_j) u_j$$

where n_{nod} is the number of nodes in the element under consideration, the nodal shape functions are N_j and the nodal values of the finite element unknown are u_j.

Let $n_{\text{bub}} = n_{\text{nod}} + 1$ and let us construct the B_i's such that

$$\mathcal{L}(B_i) = -\mathcal{L}(N_i), \quad i = 1, ..., n_{\text{nod}}$$
$$\mathcal{L}(B_{n_{\text{bub}}}) = f$$

Then

$$\tilde{u}_i = u_i, \quad i = 1, ..., n_{\text{nod}}, \quad \text{and} \quad \tilde{u}_{n_{\text{bub}}} = 1 \iff$$
$$\tilde{u}(\boldsymbol{x}) = \sum_{i=1}^{n_{\text{nod}}} B_i(\boldsymbol{x})u_i + B_{n_{\text{bub}}}(\boldsymbol{x})$$

The drawback is that one has to solve $n_{\text{nod}} + 1$ problems of the form

$$\mathcal{L}(w) = \phi \quad \text{in} \quad K$$
$$w = 0 \quad \text{on} \quad \partial K$$

The bubble constructed this way are called residual free.

Application to the Stokes problem

$$\mathcal{L}(\boldsymbol{u}, p) = \begin{bmatrix} -\nu\Delta\boldsymbol{u} + \nabla p \\ \nabla\cdot\boldsymbol{u} \end{bmatrix}, \quad \mathcal{L}^*(\boldsymbol{v}, q) = \begin{bmatrix} -\nu\Delta\boldsymbol{v} - \nabla q \\ \nabla\cdot\boldsymbol{v} \end{bmatrix}$$

Taking $\boldsymbol{\tau} = \mathrm{diag}(\tau_1, \tau_1, \tau_2)$ (in 2D) it follows that the stabilizing terms are

$$\begin{aligned} &\sum_K \int_K [-\boldsymbol{\tau}\mathcal{L}^*(\boldsymbol{V}_h)] \cdot [\mathcal{L}(\boldsymbol{U}_h) - \mathcal{F}] \\ &= \sum_K \int_K \tau_1 \left(\nu\Delta\boldsymbol{v}_h + \nabla q_h\right) \cdot \left(-\nu\Delta\boldsymbol{u}_h + \nabla p_h - \boldsymbol{f}\right) \\ &\quad + \sum_K \int_K \tau_2 (\nabla\cdot\boldsymbol{v}_h)(\nabla\cdot\boldsymbol{u}_h) \end{aligned}$$

These terms are the same as for GLS, except for the sign of the viscous term. However, the same convergence results apply.