Discontinuous Galerkin approximations for the convection-diffusion equation

Ramon Codina

Outline

Introduction

Hybrid finite element approximation

Approximation of fluxes and traces

The resulting dG approximation

Stabilized dG approximation

Outline

Introduction

Hybrid finite element approximation

Approximation of fluxes and traces

The resulting dG approximation

Stabilized dG approximation

Objectives

The objectives of this work are:

- To present a new framework to design dG methods for convection-diffusion problems.
- To explain why the inherent stabilization might not be sufficient.
- To show that simple stabilization methods used in cG methods enhance stability.

Main ideas

The main ideas are:

- To use a three field hybrid formulation, taking as unknowns:
  - The main field in the interior of the elements.
  - Its fluxes and traces on the element boundaries.
- To propose a closed form expression for the fluxes and traces based on:
  - Finite difference expressions to compute the fluxes in terms of the element unknown and its traces.
  - Imposition of flux continuity to compute the traces.

Applications

Possible applications of the framework described include:

- The design of discontinuous Galerkin methods (this work).
- Stabilized finite element formulations including boundary subscales.
- New transmission conditions in domain interaction problems (with applications for example to domain decomposition and fluid-structure interaction).

Convection-diffusion-reaction (CDR) BVP (I)

Find $u : \Omega \longrightarrow \mathbb{R}$ such that

$$\begin{aligned} \mathcal{L}(u) &:= -\nabla \cdot (k \nabla u) + \boldsymbol{a} \cdot \nabla u + s u = f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_D, \\ \mathcal{T}(u) &:= k \partial_n u = q && \text{on } \Gamma_N. \end{aligned}$$

In the following we will make use of the operators

$$\begin{aligned} \mathcal{L}^*(v) &:= -\nabla \cdot (k \nabla v) - \boldsymbol{a} \cdot \nabla v + s v, \\ \mathcal{T}^*(v) &:= k \partial_n v + \boldsymbol{n} \cdot \boldsymbol{a}\, v. \end{aligned}$$

Convection-diffusion-reaction (CDR) BVP (II)

Weak form: find $u \in V = \{v \in H^1(\Omega) \mid v|_{\Gamma_D} = 0\}$ such that

$$\begin{aligned} B(u,v) &= L(v) \qquad \forall v \in V, \\ B(u,v) &:= (k\nabla u, \nabla v) + (\boldsymbol{a} \cdot \nabla u, v) + (su, v), \\ L(v) &:= \langle f, v\rangle + \langle q, v\rangle_{\Gamma_N}. \end{aligned}$$

There holds:

$$\begin{aligned} B(u,v) &= \langle \mathcal{L}(u), v\rangle + \langle \mathcal{T}(u), v\rangle_{\partial\Omega} \\ &= \langle u, \mathcal{L}^*(v)\rangle + \langle u, \mathcal{T}^*(v)\rangle_{\partial\Omega}, \end{aligned}$$

in the sense of distributions.

Outline

Introduction

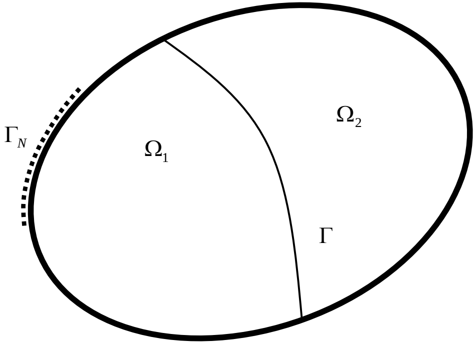
Hybrid finite element approximation

Approximation of fluxes and traces

The resulting dG approximation

Stabilized dG approximation

Notation



Hybrid formulation of an abstract problem

Abstract variational problem: find $u \in X$ such that

$$a(u, v) = l(v) \quad \forall v \in X.$$

Hybrid formulation: find $u_i \in X_i$, $\lambda_i \in F_i$ $(i = 1, 2)$ and $\gamma \in T$ such that

$$\begin{aligned}
&a_1(u_1, v_1) - \langle \lambda_1, v_1 \rangle_\Gamma - \langle \lambda_N, v_1 \rangle_{\Gamma_N} = l_1(v_1) && \forall v_1 \in X_1, \\
&a_2(u_2, v_2) - \langle \lambda_2, v_2 \rangle_\Gamma = l_2(v_2) && \forall v_2 \in X_2, \\
&\langle \mu_1, u_1 - \gamma \rangle_\Gamma + \langle \mu_1, u_1 - \gamma \rangle_{\Gamma_N} = 0 && \forall \mu_1 \in F_1, \\
&\langle \mu_2, u_2 - \gamma \rangle_\Gamma = 0 && \forall \mu_2 \in F_2, \\
&\langle \kappa, \lambda_1 + \lambda_2 \rangle_\Gamma + \langle \kappa, \lambda_N \rangle_{\Gamma_N} = \langle \kappa, q \rangle_{\Gamma_N} && \forall \kappa \in T,
\end{aligned}$$

Finite element spaces

Let $\mathcal{P}_h = \{K\}$ be a regular finite element partition, $\mathcal{E}_h$ the collection of edges and consider:

$$V_h = \left\{ v_h : \bigcup K \longrightarrow \mathbb{R} \mid v_h|_K \in P_{p_d}(K),\ K \in \mathcal{P}_h \right\},$$

$$F_h = \left\{ \mu_h : \bigcup \partial K \longrightarrow \mathbb{R} \mid \mu_h|_{\partial K} \in P_{p_f}(\partial K),\ K \in \mathcal{P}_h \right\},$$

$$T_h = \left\{ \kappa_h : \bigcup E \longrightarrow \mathbb{R} \mid \kappa_h|_E \in P_{p_t}(E),\ E \in \mathcal{E}_h,\ \kappa_h|_E = 0 \text{ if } E \subset \Gamma_D \right\}.$$

Hybrid finite element problem

Find $(u_h, \lambda_h, \gamma_h) \in V_h \times F_h \times T_h$ such that

$$\sum_K B_K(u_h, v_h) - \sum_K \langle \lambda_h, v_h \rangle_{\partial K} = \sum_K L_K(v_h),$$
$$\sum_K \langle \kappa_h, \lambda_h \rangle_{\partial K} = \langle \kappa_h, q \rangle_{\Gamma_N},$$
$$\sum_K \langle \mu_h, u_h - \gamma_h \rangle_{\partial K} = 0,$$

for all $(v_h, \mu_h, \kappa_h) \in V_h \times F_h \times T_h$.
In all what follows we will assume that the Dirichlet condition $u = 0$ on Γ_D is enforced in an essential manner.

Outline

Introduction

Hybrid finite element approximation

Approximation of fluxes and traces

The resulting dG approximation

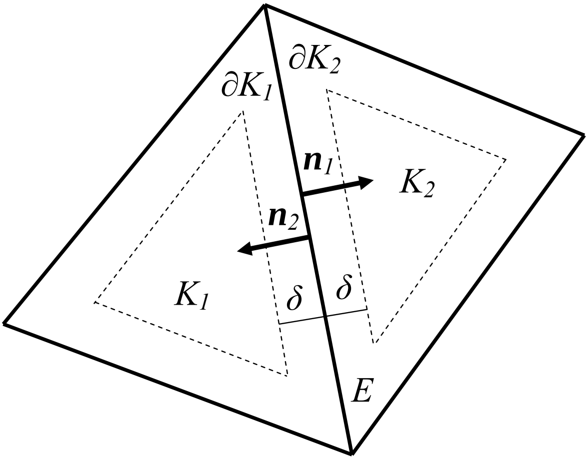
Stabilized dG approximation

Problem reduction

The key point to step to the method we propose is to design expressions to compute the fluxes λ_h and the traces γ_h from finite difference-like approximations. The steps we propose are:

1. Compute the fluxes λ_h from values of u_h, γ_h using finite difference-like expressions.
2. Compute the traces γ_h by imposing continuity of total fluxes.
3. Go to the expression obtained in 1 with the expression of γ_h just obtained to compute λ_h in terms of u_h.

Notation


∂K_2
∂K_1
$\boldsymbol{n}_1$
K_2
$\boldsymbol{n}_2$
K_1
δ
δ
E

Approximation of fluxes

We assume that the approximation u_h is meaningful to compute the fluxes λ_h up to a distance δ to edge E, on which the trace takes a value γ_h. The distance δ will be a parameter of the formulation.

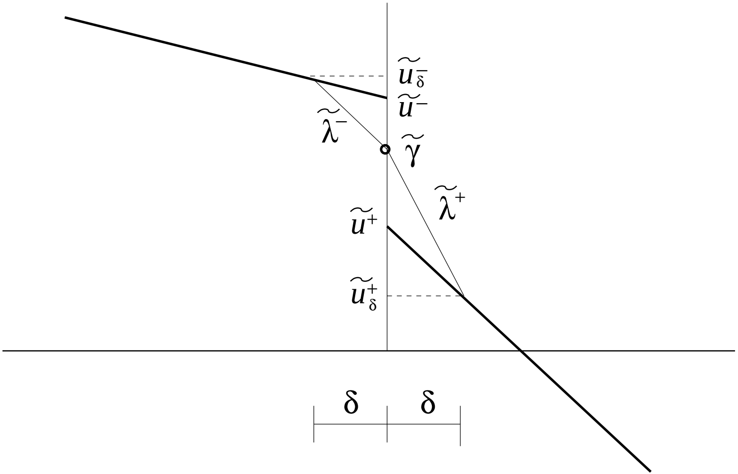
The fluxes can be approximated as

$$\lambda_h^{\pm} \approx \frac{k^{\pm}}{\delta}(\gamma_h - u_{h,\delta}^{\pm}),$$

$$u_{h,\delta}^{\pm} = u_h^{\pm} - \delta \partial_{n^{\pm}} u_h^{\pm} + \mathcal{O}(\delta^2),$$

$$\lambda_h^{\pm} \approx \frac{k^{\pm}}{\delta}(\gamma_h - u_h^{\pm}) + k^{\pm} \partial_{n^{\pm}} u_h^{\pm}.$$

Notation


$\widetilde{u}_\delta^-$
$\widetilde{u}^-$
$\widetilde{\lambda}^-$
$\widetilde{\gamma}$
$\widetilde{\lambda}^+$
$\widetilde{u}^+$
$\widetilde{u}_\delta^+$
δ
δ

Approximation of traces

The second step consists of imposing continuity of total fluxes. This leads to

$$
\begin{aligned}
0 &= [\![\mathcal{T}(u)]\!] + \alpha [\![\boldsymbol{n} \cdot \boldsymbol{a}\, u]\!] \\
&\approx (\lambda_{h,1} + \lambda_{h,2}) + \alpha [\![\boldsymbol{n} \cdot \boldsymbol{a}\, u_h]\!] \\
&\approx \frac{1}{\delta}[(k_1 + k_2)\gamma_h - k_1 u_{h,1} - k_2 u_{h,2}] + [\![\mathcal{T}(u_h) + \alpha\, \boldsymbol{n} \cdot \boldsymbol{a}\, u_h]\!] \\
&\approx [\![\mathcal{T}(u_h) + \alpha\, \boldsymbol{n} \cdot \boldsymbol{a}\, u_h]\!] + \frac{2}{\delta}(\{k\}\gamma_h - \{k u_h\}).
\end{aligned}
$$

Therefore:

$$
\boxed{\gamma_h = \frac{\{k u_h\}}{\{k\}} - \frac{\delta}{2\{k\}}([\![\mathcal{T}(u_h) + \alpha\, \boldsymbol{n} \cdot \boldsymbol{a}\, u_h]\!]).}
$$

Remarks

1. If the method has to work as $k \to 0$, α must depend on k.
2. The continuity of fluxes imposed could be considered too stringent. An weaker condition could be:

$$0 = [\![\mathcal{T}(u)]\!] - \alpha \left[(\min(0, \boldsymbol{a} \cdot \boldsymbol{n}_1)(\gamma - u_1) + \min(0, \boldsymbol{a} \cdot \boldsymbol{n}_2)(\gamma - u_2) \right].$$

3. Another possible approach would be to integrate the convective term by parts, using as bilinear form of the problem

$$B_\alpha(u, v) := (k\nabla u, \nabla v) - \alpha(u, \nabla \cdot (\boldsymbol{a}v)) + (1 - \alpha)(\boldsymbol{a} \cdot \nabla u, v) + (su, v).$$

   Enforcing the continuity of the resulting flux would amount to a continuity condition similar to the one imposed.
4. In our numerical experiments, best results have been obtained for $\alpha \sim 3$.

Final expression of the fluxes

Using the expression of the traces:

$$\begin{aligned}\lambda_h &= \frac{k}{\delta}(\gamma_h - u_h) + \mathcal{T}(u_h) \\ &= \frac{k}{\delta}\left(\frac{\{k u_h\}}{\{k\}} - u_h\right) - \frac{k}{2\{k\}}([\![\mathcal{T}(u_h) + \alpha\,\boldsymbol{n}\cdot\boldsymbol{a}\,u_h]\!]) + \mathcal{T}(u_h).\end{aligned}$$

Consider the harmonic average:

$$\langle k\rangle := \frac{k_1 k_2}{\{k\}} = \frac{2k_1 k_2}{k_1 + k_2}.$$

If we define $\{\mathcal{T}(u_h)\} = \langle k\rangle \boldsymbol{n}\cdot\{\nabla u_h\}$, we finally obtain:

$$\lambda_h = -\frac{\langle k\rangle}{2\delta}\boldsymbol{n}\cdot[\![\boldsymbol{n}u_h]\!] + \{\mathcal{T}(u_h)\} - \frac{k\alpha}{2\{k\}}[\![\boldsymbol{n}\cdot\boldsymbol{a}\,u_h]\!].$$

Neumann edges

Proceeding similarly for Neumann edges:

$$\begin{aligned} q &= \lambda_h \\ &\approx \mathcal{T}(u_h) + \frac{k}{\delta}(\gamma_h - u_h). \end{aligned}$$

Traces:

$$\gamma_h = u_h - \frac{\delta}{k}(\mathcal{T}(u_h) - q).$$

Fluxes:

$$\lambda_h = q.$$

Outline

Introduction

Hybrid finite element approximation

Approximation of fluxes and traces

The resulting dG approximation

Stabilized dG approximation

Reformulation of the problem

The problem can be written as a single variational equation as: find $(u_h, \lambda_h, \gamma_h) \in V_h \times F_h \times T_h$ such that

$$\sum_K B_K(u_h, v_h) - \sum_K \langle \lambda_h, v_h \rangle_{\partial K} + \sum_K \langle \kappa_h, \lambda_h \rangle_{\partial K} - \sum_K \langle \mu_h, u_h - \gamma_h \rangle_{\partial K} = \sum_K L_K(v_h) + \langle \kappa_h, q \rangle_{\Gamma_N},$$

for all $(v_h, \mu_h, \kappa_h) \in V_h \times F_h \times T_h$.

Fluxes, traces and their test functions

According to what we have obtained, we may take:

$$\gamma_h = \frac{\{ku_h\}}{\{k\}} - \frac{\delta}{2\{k\}}(\llbracket k\boldsymbol{n}\cdot\nabla u_h + \alpha\,\boldsymbol{n}\cdot\boldsymbol{a}\,u_h\rrbracket),$$

$$\lambda_h = -\frac{\langle k\rangle}{2\delta}\boldsymbol{n}\cdot\llbracket\boldsymbol{n}u_h\rrbracket + \langle k\rangle\boldsymbol{n}\cdot\{\nabla u_h\} - \frac{k\alpha}{2\{k\}}\llbracket\boldsymbol{n}\cdot\boldsymbol{a}\,u_h\rrbracket,$$

$$\kappa_h = \frac{\{kv_h\}}{\{k\}} - \frac{\delta}{2\{k\}}(\llbracket k\boldsymbol{n}\cdot\nabla v_h + \hat{\alpha}\,\boldsymbol{n}\cdot\boldsymbol{a}\,v_h\rrbracket),$$

$$\mu_h = -\frac{\langle k\rangle}{2\delta}\boldsymbol{n}\cdot\llbracket\boldsymbol{n}v_h\rrbracket + \langle k\rangle\boldsymbol{n}\cdot\{\nabla v_h\} - \frac{k\hat{\alpha}}{2\{k\}}\llbracket\boldsymbol{n}\cdot\boldsymbol{a}\,v_h\rrbracket,$$

with $v_h \in V_h$. In principle, the parameter $\hat{\alpha}$ may be taken different from α, for generality.

Final dG method

Replacing and operating the method we propose is: find $u_h \in V_h$ such that

$$\sum_K k\langle \nabla v_h, \nabla u_h\rangle_K + \sum_K \langle v_h, \boldsymbol{a}\cdot\nabla u_h\rangle_K + \sum_K s\langle v_h, u_h\rangle_K$$

$$+\sum_E \langle k\rangle\langle \frac{1}{\delta}[\![\boldsymbol{n}v_h]\!] - \{\nabla v_h\}, [\![\boldsymbol{n}u_h]\!] - \delta\{\nabla u_h\}\rangle_E - \sum_E \langle k\rangle \delta\langle\{\nabla v_h\}, \{\nabla u_h\}\rangle_E$$

$$+\sum_E \frac{\delta}{2\{k\}}\langle [\![k\boldsymbol{n}\cdot\nabla v_h + \hat{\alpha}\boldsymbol{n}\cdot\boldsymbol{a}\,v_h]\!], [\![k\boldsymbol{n}\cdot\nabla u_h + \alpha\boldsymbol{n}\cdot\boldsymbol{a}\,u_h]\!]\rangle_E$$

$$-\sum_E \frac{\delta}{2\{k\}}\langle [\![k\boldsymbol{n}\cdot\nabla v_h]\!], [\![k\boldsymbol{n}\cdot\nabla u_h]\!]\rangle_E$$

$$= \sum_K L_K(v_h) + \langle v_h, q\rangle_{\Gamma_N},$$

for all $v_h \in V_h$.

Remarks

- For $\boldsymbol{a} = \mathbf{0}$ we recover the interior penalty method.
- Adjoint consistency is ensured.
- There is a built-in convective flux continuity whose effect is similar (the same in 1D) as the more common $\sum_E \langle \{v_h\}, [\![\, \boldsymbol{n} \cdot \boldsymbol{a} \, u_h \,]\!] \rangle_E$ or $\sum_E \langle v_h^+, [\![\, \boldsymbol{n} \cdot \boldsymbol{a} \, u_h \,]\!] \rangle_E$ terms.
- Alternative methods could be designed by changing:
  - The bilinear form of the problem, integrating part of the convective term by parts.
  - Imposition of the continuity of convective fluxes.
  - Approximation of the diffusive fluxes using finite-difference-like expressions.
  - Expression of the test functions.

Outline

Introduction

Hybrid finite element approximation

Approximation of fluxes and traces

The resulting dG approximation

Stabilized dG approximation

Stability I

The following terms are easily controlled:

- Interior element terms: $\sum_K k\|\nabla u_h\|_K^2$, $\sum_K s\|u_h\|_K^2$.
- Interior penalty term: $\sum_E \frac{\langle k\rangle}{\delta}\|[\![\boldsymbol{n}u_h]\!]\|_E^2$.
- Jump of total flux (for example for $\hat{\alpha} = \alpha$): $\sum_E \frac{\delta}{2\{k\}}\|[\![k\boldsymbol{n}\cdot\nabla u_h + \alpha\boldsymbol{n}\cdot\boldsymbol{a}\,u_h]\!]\|_E^2$.

The following terms need to be controlled by the diffusive term:

- $-\sum_E \langle k\rangle\delta\|\{\nabla u_h\}\|_E^2$
- $-\sum_E \frac{\delta}{2\{k\}}\|[\![k\boldsymbol{n}\cdot\nabla u_h]\!]\|_E^2$.

Stability II

Key point for the stability of dG methods in convection-dominated problems:

$$\boldsymbol{a} \cdot \nabla u_h \in V_h \quad \text{(or for a projected } \boldsymbol{a}\text{)}$$

so that $v_h = \tau \boldsymbol{a} \cdot \nabla u_h$ can be taken as test function. This allows one to obtain stability for $\tau \|\boldsymbol{a} \cdot \nabla u_h\|^2$. However,

The scaling parameter τ must be proportional to $1/\alpha$

and so stability is lost as α grows! In this case, problems similar to those of cG methods are found.

Stabilized dG method

To overcome the problem described, we propose to use stabilized finite element methods. For example, one can add

$$\sum_K \tau_K \langle -\mathcal{L}^*(v_h), \mathcal{L}(u_h) - f \rangle_K ,$$

with

$$\tau_K^{-1} = \frac{4k}{p^4 h^2} + \frac{2|\boldsymbol{a}|}{ph}$$

to the discrete variational problem.