4. Approximation theory for FEM. Error estimates for elliptic problems

4.1 Introduction

For a typical elliptic problem satisfying the conditions (i)−(iv) of Section 2.1, we have by Theorem 2.4

$$\|u-u_h\|_V \leq \frac{\gamma}{\alpha}\|u-v\|_V \qquad \forall v\in V_h.$$

Choosing $v=\pi_h u\in V_h$ to be a suitable interpolant of u and estimating the interpolation error $\|u-\pi_h u\|_V$ we obtain an estimate of the error $\|u-u_h\|_V$. In this chapter we study the problem of estimating the interpolation error $\|u-\pi_h u\|_V$. The interpolant $\pi_h u\in V_h$ is usually chosen so that the degrees of freedom for V_h agree for u and $\pi_h u$. In this case the problem of estimating $\|u-\pi_h u\|_V$ is reduced to the problem of estimating $u-\pi_h u$ individually on each element $K\in T_h$.

4.2 Interpolation with piecewise linear functions in two dimensions

We shall first consider the case where $V=H^1(\Omega)$ and $V_h=\{v\in V: v|_K\in P_1(K), \forall K\in T_h\}$ where $T_h=\{K\}$ is a triangulation of $\Omega\subset R^2$, i e, V_h is the standard finite element space of piecewise linear functions on triangles K (cf Section 1.7). For $K\in T_h$ we define (see Fig 4.1)

h_K=the diameter of K=the longest side of K,
ϱ_K=the diameter of the circle inscribed in K,
$h=\max_{K\in T_h} h_K$.

To be more precise, we will subsequently be concerned with not only one triangulation T_h but a family of triangulations $\{T_h\}$ that are indexed by the parameter h. We shall below assume that there is a positive constant β independent of the triangulation $T_h\in\{T_h\}$, i e, independent of h, such that

$$(4.1)\qquad \frac{\varrho_K}{h_K}\geq\beta \qquad \forall K\in T_h.$$

This condition means that the triangles $K\in T_h$ are not allowed to be arbitrarily thin, or equivalently, the angles of the triangles K are not allowed to be arbitrarily small; the constant β is a measure of the smallest angle in any $K\in T_h$ for any $T_h\in\{T_h\}$.

Let N_i, i=1, . . ., M, be the nodes of T_h. Given $u\in C^0(\bar{\Omega})$ we define the interpolant $\pi_h u\in V_h$ by

$$\pi_h u(N_i)=u(N_i) \qquad i=1, \ldots, M.$$

Thus $\pi_h u$ is the piecewise linear function agreeing with u at the nodes of T_h. We will start by estimating the interpolation error $u-\pi_h u$ on each triangle K. We have the following result.

Theorem 4.1 Let $K\in T_h$ be a triangle with vertices a^i, i=1, 2, 3. Given $v\in C^0(K)$ let the interpolant $\pi v\in P_1(K)$ be defined by

$$(4.2)\qquad \pi v(a^i)=v(a^i),\ i=1, 2, 3.$$

Then

$$(4.3)\qquad \|v-\pi v\|_{L_\infty(K)}\leq 2h_K^2 \max_{|\alpha|=2}\|D^\alpha v\|_{L_\infty(K)},$$

$$(4.4)\qquad \max_{|\alpha|=1}\|D^\alpha(v-\pi v)\|_{L_\infty(K)}\leq 6\,\frac{h_K^2}{\varrho_K}\max_{|\alpha|=2}\|D^\alpha v\|_{L_\infty(K)},$$

where

$$\|v\|_{L_\infty(K)}=\max_{x\in K}|v(x)|.$$

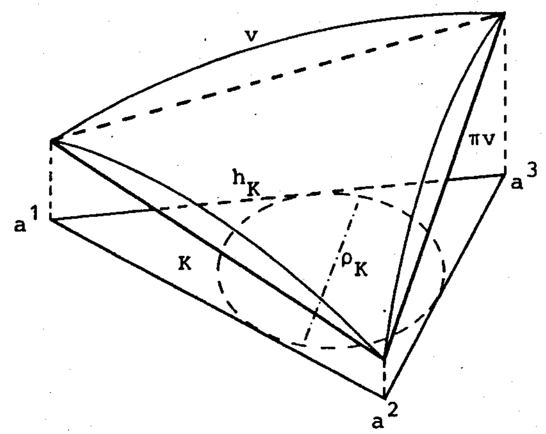

Fig 4.1

Before giving a proof of Theorem 4.1 let us comment on the estimates (4.3) and (4.4). We note that the size of the errors $v-\pi v$ and $D^\alpha(v-\pi v)$ depend on the second partial derivatives of v; the larger these derivatives are, the more "curved" is the surface representing the function v and thus the larger is the deviation $v-\pi v$ from the plane representing πv (see Fig 4.1). Also note that the assumption (4.1) will be used in the estimate (4.4) to bound the quantity h_K/ϱ_K.

Proof of Theorem 4.1 Let λ_i, $i=1, 2, 3$, be the basis functions for $P_1(K)$ described in Example 3.1. A general function $w\in P_1(K)$ then has the representation

$$w(x)=\sum_{i=1}^{3} w(a^i)\lambda_i(x), \qquad x\in K,$$

so that in particular

$$\pi v(x)=\sum_{i=1}^{3} v(a^i)\lambda_i(x), \qquad x\in K, \tag{4.5}$$

since by (4.2) $\pi v(a^i)=v(a^i)$. We now derive representation formulas for the errors $v-\pi v$ and $D^\alpha(v-\pi v)$, $|\alpha|=1$, using the following Taylor expansion at $x\in K$:

$$v(y)=v(x)+\sum_{j=1}^{2}\frac{\partial v}{\partial x_j}(x)\,(y_j-x_j)+R(x, y),$$

where

$$R(x, y)=\frac{1}{2}\sum_{i,j=1}^{2}\frac{\partial^2 v}{\partial x_i\partial x_j}(\xi)\,(y_i-x_i)\,(y_j-x_j),$$

is the remainder term of order 2 and ξ is a point on the line segment between x and y. In particular by choosing $y=a^i$, we have

$$v(a^i)=v(x)+p_i(x)+R_i(x), \tag{4.6}$$

where

$$p_i(x)=\sum_{j=1}^{2}\frac{\partial v}{\partial x_j}(x)\,(a^i_j-x_j), \quad a^i=(a^i_1, a^i_2),$$

$$R_i(x)=R(x, a^i).$$

Since

$$|a^i_j-x_j|\le h_K, \quad i=1, 2, 3, \quad j=1, 2,$$

we have the following estimate of the remainder term $R_i(x)$: 013

$$R_i(x)\le 2h_K^2 \max_{|\alpha|=2}\|D^\alpha v\|_{L_\infty(K)}, \quad i=1, 2, 3. \tag{4.7}$$

Now (4.5) and (4.6) combine to give

$$\pi v(x)=v(x)\sum_{i=1}^{3}\lambda_i(x)+\sum_{i=1}^{3}p_i(x)\lambda_i(x)+\sum_{i=1}^{3}R_i(x)\lambda_i(x) \qquad x\in K. \tag{4.8}$$

We now need the following lemma whose simple proof is given below.

Lemma 4.1 For $j=1, 2$ and $x\in K$ we have 014

$$\sum_{i=1}^{3}\lambda_i(x)=1, \tag{4.9}$$

$$\sum_{i=1}^{3}p_i(x)\lambda_i(x)=0, \tag{4.10}$$

$$\sum_{i=1}^{3}\frac{\partial}{\partial x_j}\lambda_i(x)=\frac{\partial}{\partial x_j}\sum_{i=1}^{3}\lambda_i(x)=0, \tag{4.11}$$

01

$$\sum_{i=1}^{3}p_i(x)\frac{\partial\lambda_i}{\partial x_j}(x)=\frac{\partial v}{\partial x_j}(x). \tag{4.12}$$

By (4.9), (4.10) and (4.8) we have

$$\pi v(x)=v(x)+\sum_{i=1}^{3}R_i(x)\lambda_i(x),$$

01

which gives us the following representation of the interpolation error:

$$v(x)-\pi v(x)=-\sum_{i=1}^{3}R_i(x)\lambda_i(x).$$

Since $0\le\lambda_i(x)\le 1$, if $x\in K$, $i=1, 2, 3$, we can use the previous estimate (4.7) of the remainder term R_i to get

$$|v(x)-\pi v(x)|\le\sum_{i=1}^{3}|R_i(x)|\;|\lambda_i(x)|$$

01

$$\le\max_i|R_i(x)|\sum_{i=1}^{3}\lambda_i(x)\le 2h_K^2\max_{|\alpha|=2}\|D^\alpha v\|_{L_\infty(K)}, \qquad x\in K,$$

which proves (4.3).

To prove (4.4) we differentiate (4.5) with respect to x_1 to get 01

$$\frac{\partial \pi v}{\partial x_1}(x)=\sum_{i=1}^{3} v(a^i)\frac{\partial \lambda_i}{\partial x_1}(x),$$

which together with (4.6) shows that

$$(4.13) \qquad \frac{\partial \pi v}{\partial x_1}(x)=v(x)\sum_{i=1}^{3}\frac{\partial \lambda_i}{\partial x_1}(x)+\sum_{i=1}^{3} p_i(x)\frac{\partial \lambda_i}{\partial x_1}(x)+\sum_{i=1}^{3} R_i(x)\frac{\partial \lambda_i}{\partial x_1}(x).$$

Hence, by (4.11) and (4.12) we have

$$\frac{\partial \pi v(x)}{\partial x_1}(x)=\frac{\partial v}{\partial x_1}(x)+\sum_{i=1}^{3} R_i(x)\frac{\partial \lambda_i}{\partial x_1}(x),$$

which gives the following representation of the error $\frac{\partial v}{\partial x_1}-\frac{\partial \pi v}{\partial x_1}$:

$$\frac{\partial v}{\partial x_1}(x)-\frac{\partial \pi v}{\partial x_1}(x)=-\sum_{i=1}^{3} R_i(x)\frac{\partial \lambda_i}{\partial x_1}(x), \qquad x\in K.$$

It is now easy to see (cf Problem 4.2) that

$$(4.14) \qquad \max_{x\in K}\left|\frac{\partial \lambda_i}{\partial x_1}(x)\right|\le\frac{1}{\varrho_K},$$

which together with (4.7) finally gives

$$\left|\frac{\partial v}{\partial x_1}(x)-\frac{\partial \pi v}{\partial x_1}(x)\right|\le 6\frac{h_K^2}{\varrho_K}\max_{|\alpha|=2}\|D^\alpha v\|_{L_\infty(K)}.$$

In the same way we estimate $\frac{\partial v}{\partial x_2}-\frac{\partial \pi v}{\partial x_2}$ and thus (4.4) follows. The proof of the theorem is now complete once the lemma is established.

Proof of Lemma 4.1 The proof is based on the following observation:

$$(4.15) \qquad \pi v=v \text{ if } v\in P_1(K),$$

which of course follows from the fact there is a unique function $v\in P_1(K)$ assuming given values at the vertices of K. If we now choose $v(x)\equiv 1$ in (4.8), in which case clearly $v=\pi v$, we get

$$1=\sum_{i=1}^{3}\lambda_i(x), \quad x\in K,$$

since in this case $p_1\equiv R_1\equiv 0$. This proves (4.9) and (4.11) follows directly.

To prove (4.10) we choose $v(x)=d_1x_1+d_2x_2$ in (4.8) with $d_i\in R$. Again $v=\pi v$ and further

$$p_i(x)=d_1(a_1^i-x_1)+d_2(a_2^i-x_2),$$

and $R_i\equiv 0$ so that by (4.8)

$$v(x)=v(x)+\sum_{i=1}^{3}[d_1(a_1^i-x_1)+d_2(a_2^i-x_2)]\lambda_i(x), \quad x\in K.$$

and so for all $d_i\in R$ we have

$$\sum_{i=1}^{3}[d_1(a_1^i-x_1)+d_2(a_2^i-x_2)]\,\lambda_i(x)=0 \quad x\in K.$$

This proves (4.10) by choosing $d_i=\frac{\partial v}{\partial x_i}(x)$, $i=1, 2$. Finally, (4.12) follows in a similar way by choosing $v=d_1x_1+d_2x_2$ in (4.13). This finishes the proof of the lemma and the proof of Theorem 4.1 is complete. □

Since Theorem 4.1 states estimates of the interpolation error using the $L_\infty(K)$-norm, it is not ideally suited to give estimates for $\|u-\pi_h u\|_{H^1(\Omega)}$ involving the L_2-norm. For this purpose we will use instead the following analogue of Theorem 4.1. Here we use the following notation for r=0, 1, 2, . . .,

$$|v|_{H^r(\Omega)}=\Big(\sum_{|\alpha|=r}\int_\Omega |D^\alpha v|^2dx\Big)^{1/2}.$$

Note that $|v|_{H^r(\Omega)}$ measures the $L_2(\Omega)$-norm of the partial derivatives of v of order exactly equal to r, whereas derivatives of order less than r are not included. We say that $|\cdot|_{H^r(\Omega)}$ is a *seminorm*. Since we may have $|v|_{H^r(\Omega)}=0$ even if $v\neq 0$ (e g if $v\equiv 1$ and $r\geq 1$), it is not a norm.

Theorem 4.2 Under the assumptions of Theorem 4.1 there is an absolute constant C such that

$$\|v-\pi v\|_{L_2(K)}\le Ch_K^2|v|_{H^2(K)},$$

$$|v-\pi v|_{H^1(K)}\le C\frac{h_K^2}{\varrho_K}|v|_{H^2(K)}.$$

We see that Theorem 4.1 and 4.2 have exactly the same structure, the only difference being the norm involved, either the L_∞ or the L_2-norm. For simplicity we have chosen to present a proof in the L_∞-case since we then avoid some technical complications (for a proof of Theorem 4.2, see [DS]).

Let us now apply Theorem 4.2 to estimate the global interpolation errors $\|u-\pi_h u\|_{L_2(\Omega)}$ and $|u-\pi_h u|_{H^1(\Omega)}$. We have by summing over $K\in T_h$,

$$\|u-\pi_h u\|^2_{L_2(\Omega)} = \sum_{K\in T_h} \|u-\pi_h u\|^2_{L_2(K)} \leq \sum_{K\in T_h} C^2 h_K^4 |u|^2_{H^2(K)}$$

$$\leq C^2 h^4 \sum_{K\in T_h} |u|^2_{H^2(K)} = C^2 h^4 |u|^2_{H^2(\Omega)},$$

and similarly using (4.1), i e, $\frac{h_K}{\varrho_K} \leq \frac{1}{\beta}$,

(4.16) $$|u-\pi_h u|^2_{H^1(\Omega)} \leq \sum_{K\in T_h} C^2 \frac{h_K^4}{\varrho_K^2} |u|^2_{H^2(K)} \leq \sum_{K\in T_h} \frac{C^2 h_K^2}{\beta^2} |u|^2_{H^2(K)}$$

$$\leq \frac{C^2 h^2}{\beta^2} |u|^2_{H^2(\Omega)}$$

so that

(4.17) $$|u-\pi_h u|_{H^1(\Omega)} \leq \frac{Ch}{\beta} |u|_{H^2(\Omega)} = Ch|u|_{H^2(\Omega)},$$

if the constant β is included in the constant C, and

(4.18) $$\|u-\pi_h u\|_{L_2(\Omega)} \leq Ch^2 |u|_{H^2(\Omega)}.$$

4.3 Interpolation with polynomials of higher degree

The estimates (4.17) and (4.18) are typical examples of estimates for the interpolation error $u-\pi_h u$, in this case for interpolation with piecewise linear functions. If we work with piecewise polynomials of degree $r \geq 1$ on triangulations T_h satisfying (4.1), we have in the typical case the following estimates:

(4.19) $$\|u-\pi_h u\|_{L_2(\Omega)} \leq Ch^{r+1} |u|_{H^{r+1}(\Omega)},$$

(4.20) $$|u-\pi_h u|_{H^1(\Omega)} \leq Ch^r |u|_{H^{r+1}(\Omega)},$$

where the constant β is absorbed in the constant C in (4.20). If $V_h \subset H^2(\Omega)$, then we also have

(4.21) $$|u-\pi_h u|_{H^2(\Omega)} \leq Ch^{r-1} |u|_{H^{r+1}(\Omega)}.$$

Note that for each derivative of the error $u-\pi_h u$, the power of h on the right hand side drops by one. Note that the constant C in (4.19)–(4.21) only depends on the constant β in (4.1) and the degree r, but not on the mesh parameter h or the function u.

Remark 4.1 If u does not have the regularity required in (4.19) or (4.20), we get the corresponding reduction in the power of h: For $1 \leq s \leq r+1$, we have

(4.22) $$\|u-\pi_h u\|_{L_2(\Omega)} \leq Ch^s |u|_{H^s(\Omega)},$$

(4.23) $$\|u-\pi_h u\|_{H^1(\Omega)} \leq Ch^{s-1} |u|_{H^s(\Omega)}.$$ □

Example 4.1 Let $\{T_h\}$ be a family of triangulations $T_h=\{K\}$ of $\Omega \subset R^2$ satisfying (4.1) and let $V_h=\{v\in C^0(\bar{\Omega}): v|_K \in P_2(K), \forall K\in T_h\}$. For the finite element of Example 3.2 we may for $v\in C^0(\bar{\Omega})$ define the interpolant $\pi_h v \in V_h$ by

$\pi_h v = v$ at the nodes of T_h,
$\pi_h v = v$ at the midpoints of the sides of T_h.

In this case (4.19) and (4.20) hold with r=2. □

Example 4.2 With $T_h=\{K\}$ as in Example 4.1 define $V_h=\{v\in C^1(\bar{\Omega}): v|_K \in P_5(K), \forall K\in T_h\}$ and for $v\in C^2(\bar{\Omega})$ specify the interpolant $\pi_h v\in V_h$ by

$D^\alpha \pi_h v = D^\alpha v$ at the nodes of T_h, $|\alpha| \leq 2$,

$\frac{\partial}{\partial n}\pi_h v = \frac{\partial v}{\partial n}$ at the midpoints of each side S of T_h,

where $\frac{\partial}{\partial n}$ denotes differentiation in the normal direction to S. In this case (4.19)–(4.21) hold with r=5. □

4.4 Error estimates for FEM for elliptic problems

Recalling again the typical abstract error estimate for an elliptic problem

$$\|u-u_h\|_V \leq C\|u-v\|_V \quad \forall v\in V_h,$$

and choosing here $v=\pi_h u$ with $\pi_h u\in V_h$ and interpolant of u, we have

(4.24) $$\|u-u_h\|_V \leq C\|u-\pi_h u\|_V \quad \forall v\in V_h.$$

Using estimates for the interpolation error $\|u-\pi_h u\|_V$ we then obtain estimates for the finite element error $\|u-u_h\|_V$. Using the interpolation estimates of Sections 4.2 and 4.3 we have for example the following error estimates:

Example 4.3 With $V=H_0^1(\Omega)$ and (cf Examples 3.1–3.3)

$$V_h=\{v\in V\colon v|_K\in P_r(K),\ \forall K\in T_h\},\ r=1, 2, 3,$$

we obtain from (4.20) and (4.24)

$$\|u-u_h\|_{H^1(\Omega)}\leq Ch^r|u|_{H^{r+1}(\Omega)}$$

for the finite element method for the Dirichlet problem (1.16). We obtain a similar result for the Neumann problem (1.36). □

Example 4.4 With V_h as in Example 4.2 we have for the biharmonic problem of Example 2.5 the following estimate

$$\|u-u_h\|_{H^2(\Omega)}\leq Ch^4|u|_{H^6(\Omega)}.\quad □$$

Remark 4.2 It is possible to prove analogues of (4.24) in norms other than that given by the space V. For example one can prove for the finite element method of Section 1.4 that (see [RS])

$$\|\nabla u-\nabla u_h\|_{L_\infty(\Omega)}\leq C\|\nabla u-\nabla\pi_h u\|_{L_\infty(\Omega)},$$

which together with Theorem 4.1 gives

$$(4.25)\qquad \|\nabla u-\nabla u_h\|_{L_\infty(\Omega)}\leq C\max_K[h_K\max_{|\alpha|=2}\|D^\alpha u\|_{L_\infty(K)}].\quad □$$

4.5 On the regularity of the exact solution

We have seen that the regularity of the exact solution u is involved in estimating the error $\|u-u_h\|_V$ in the finite element method. Let us now give a typical result that shows how the regularity of the exact solution u depends on the regularity of the given data. Let us then consider the Poisson equation:

$$(4.26)\qquad \begin{aligned} -\Delta u&=f \quad \text{in } \Omega,\\ u&=0 \quad \text{on } \Gamma,\end{aligned}$$

where Ω is bounded domain in R^2 with boundary Γ and f is a given function. Let us first assume that Γ is smooth, i e, Γ is a smooth curve in particular without corners or cups. In this case there is for $s=0, 1, \ldots,$ a constant C independent of f such that

$$(4.27)\qquad \|u\|_{H^{s+2}(\Omega)}\leq C\|f\|_{H^s(\Omega)},$$

i e, if $f\in H^s(\Omega)$ then $u\in H^{s+2}(\Omega)$, or loosely speaking, we "gain two derivatives" in (4.26).

If Γ is not smooth, then (4.27) may not hold, not even for $s=0$. If Γ has a corner, then the solution u or derivatives of u will in general have singularities at the corner even if f is very smooth ($f\in H^s(\Omega)$ for s large). More precisely, the solution u of (4.26) with f smooth basically has the following form close to a corner with angle ω (cf Problem 4.6):

$$(4.28)\qquad u(r,\theta)=r^\gamma\alpha(\theta)+\beta(r,\theta),\quad \gamma=\frac{\pi}{\omega},$$

where α and β are smooth functions (here we use polar coordinates (r, θ) with the pole at the corner). It is easy to see that if $\omega>\pi$ then a function u of the form (4.28) does not belong to $H^2(\Omega)$ if $\alpha\neq 0$. On the other hand, one can show that (4.27) holds with $s=0$ if Ω is a convex polygonal domain (in which case the corner angles satisfy $\omega<\pi$).

For the biharmonic problem (2.22) we have if the boundary Γ is smooth, for $s=0, 1, \ldots,$

$$\|u\|_{H^{s+4}(\Omega)}\leq C\|f\|_{H^s(\Omega)}.$$

If Γ has corners there are results analogous to those just stated for the Poisson equation (4.26).

Example 4.5 For a solution u of the form (4.28) we have formally that $u\in H^s(\Omega)\Leftrightarrow$ derivatives $D^s u$ of order s belong to $L_2(\Omega)\Leftrightarrow$

$$\int_\Omega |D^s u|^2dx\sim C\int_0^R[r^{\gamma-s}]^2rdr<\infty.$$

Hence $u\in H^s(\Omega)$ if and only if $s<\gamma+1$. By Remark 4.1 we thus have for the standard finite element method of Section 1.4 for the Poisson equation in a polygonal domain that for any $\varepsilon>0$

$$(4.29)\qquad \|u-u_h\|_{H^1(\Omega)}\leq Ch^{\gamma-\varepsilon}\|u\|_{H^{\gamma+1-\varepsilon}(\Omega)}=Ch^{\gamma-\varepsilon},$$

where $\gamma=\pi/\omega$ and ω is the maximal angle of a corner of Γ. For example if $\gamma=2/3$, which corresponds to a concave corner of angle $3\pi/2$, then

$$\|u-u_h\|_{H^1(\Omega)} \le Ch^{\frac{2}{3}-\varepsilon}.$$

We see that in this case we do not obtain the full rate of convergence which is $0(h)$. □

4.6 Adaptive methods

If the exact solution u has e g a corner singularity, then it is natural to refine the triangulation close to the corner to increase the accuracy. Recalling that for the method of Section 1.4 (cf (4.16))

(4.30) $$|u-u_h|_{H^1(\Omega)} \le |u-\pi_h u|_{H^1(\Omega)} \le C[\Sigma(h_K|u|_{H^2(K)})^2]^{1/2},$$

it is clear that we somehow would like to balance the size of h_K with that of $|u|_{H^2(K)}$ and in particular choose h_K small where $|u|_{H^2(K)}$ is large. If u has the form (4.28) with $0<\gamma<1$, then one possible refinement is given by (cf Problem 4.4.)

(4.31) $$h_K = Chd_K^{1-\gamma},$$

if $h_K \le d_K$, where d_K is the distance from K to the corner and h is the mesh size away from the corner. With this refinement we have, disregarding the ε in (4.29),

(4.32) $$|u-u_h|_{H^1(\Omega)} \le Ch.$$

Notice that the total number of elements with a refinement of the form (4.31) is of the order $0(h^{-2})$, i.e., the same as with a uniform mesh of size h. Thus, in this case the refinement does not increase the total number of unknowns significantly but significantly increases the precision (from (4.29) to (4.32)).

In general the nature of the exact solution u is not known beforehand and then it is not clear how to locally refine the finite element mesh. Recently methods for automatic mesh refinement, so-called *adaptive methods,* have been developed which do not require the user to supply information on the smoothness of the exact solution. In these methods this information is instead obtained through a sequence of computed solutions on successively refined meshes.

To very briefly describe some of the basic ideas underlying adaptive methods, suppose $\delta>0$ is a given *tolerance* and suppose we want to obtain a finite element approximation u_h such that

(4.33) $$|u-u_h|_{H^1(\Omega)} \lesssim \delta.$$

Relying on the error estimate (4.30) we see that (4.33) will be satisfied if the corresponding finite element mesh $T_h=\{K\}$ is chosen so that

(4.34) $$\sum_{K\in T_h} (h_K|u|_{H^2(K)})^2 \sim \left(\frac{\delta}{C}\right)^2.$$

To determine a mesh satisfying (4.34) we may proceed as follows: Choose a first mesh $\bar{T}_h=\{\bar{K}\}$ and compute a corresponding finite element solution $\bar{u}_h$. Using $\bar{u}_h$ compute approximations to $|u|_{H^2(\bar{K})}$ denoted by $|\bar{u}_h|_{H^2(\bar{K})}$ for $\bar{K}\in\bar{T}_h$. The quantity $|\bar{u}_h|_{H^2(\bar{K})}$ may be obtained using difference quotients based on the values of $\nabla\bar{u}_h$ at the centers of gravity of $\bar{K}$ and neighbouring triangles in $\bar{T}_h$. Next, construct a new mesh $T_h=\{K\}$ by subdividing into four equal triangles each $\bar{K}\in\bar{T}_h$ for which

$$(h_{\bar{K}}|\bar{u}_h|_{H^2(\bar{K})})^2 > \frac{\delta^2}{\bar{N}C^2},$$

where $\bar{N}$ is the number of triangles in $\bar{T}_h$. Next, compute the finite element solution u_h on the new mesh T_h and repeat the process until

(4.35) $$\sum_{K\in T_h} (h_K|u_h|_{H^2(K)})^2 \le \left(\frac{\delta}{C}\right)^2.$$

Note that by the construction if follows (if δ is small enough) that for the final mesh T_h satisfying (4.35), all the terms in the sum will be approximately equal. Note also that after refinement of certain triangles, the resulting mesh is completed into a triangulation as in Fig 1.15.

It is also possible to control the error in other norms than the $H^1(\Omega)$-norm used in (4.33), for instance we may want to control the gradient error in the maximum norm. In this case we base the adaptive method on the error estimate (4.25) and seek to find a mesh $T_h=\{K\}$ such that

(4.36) $$Ch_K \max_{|\alpha|=2} \|D^\alpha u_h\|_{L_\infty(K)} \sim \delta \quad \forall K\in T_h,$$

where as above $\|D^\alpha u_h\|_{L_\infty(K)}$ is a computed approximation of $\|D^\alpha u\|_{L_\infty(K)}$. Again the final mesh satisfying (4.36) is constructed through a sequence of successively refined meshes where triangles K for which the left hand side of (4.36) is larger than δ are refined. In Fig. 4.2 we give the sequence of meshes (with a zoom at the origin for the final mesh) obtained by applying an adaptive method of this form with $\delta=0.1$ and $C=1$ to the problem

$$\begin{aligned} \Delta u &= 0 \text{ in } \Omega, \\ u &= u_0 \text{ on } \Gamma, \end{aligned}$$

where $\Omega=\{x=r(\cos\theta, \sin\theta): 0<r<1, 0<\theta<3\pi/4\}$ with exact solution $u(r, \theta)=r^\gamma \sin(\gamma\theta)$, $\gamma=4/3$. In Fig. 4.3 we give the actual gradient error $|\nabla e(x)|$ as a function of the distance $|x|$ to the origin along the radius $\theta=\pi/2$. We observe that the gradient error is roughly equal to the tolerance and thus we see that the adaptive method is able to find a good mesh in this case. This example is taken from [EJ2], where theoretical and computational results for adaptive methods of the indicated type are given, see also [E]. The problem of computationally estimating the constant C in (4.32) and (4.34) is discussed in [EJ2].

For adaptive methods for parabolic problems we refer to Section 8.4.4. For another approach to adaptivity, see [BR], [BM].

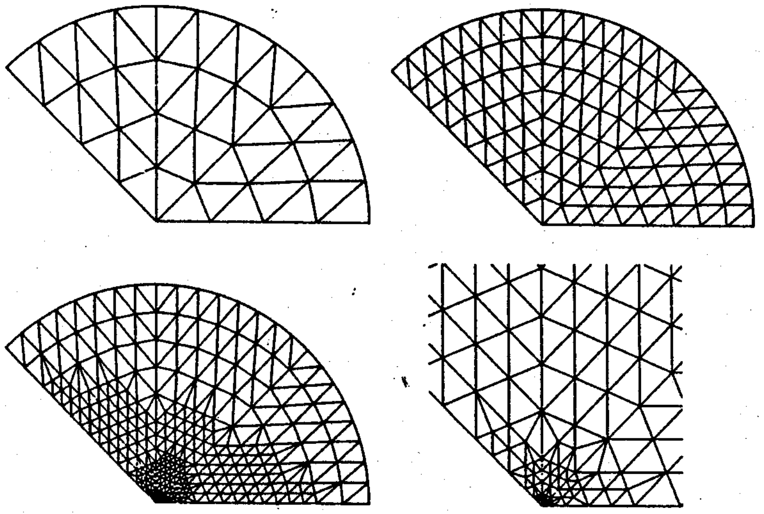

Fig 4.2 Sequence of meshes obtained by adaptive FEM

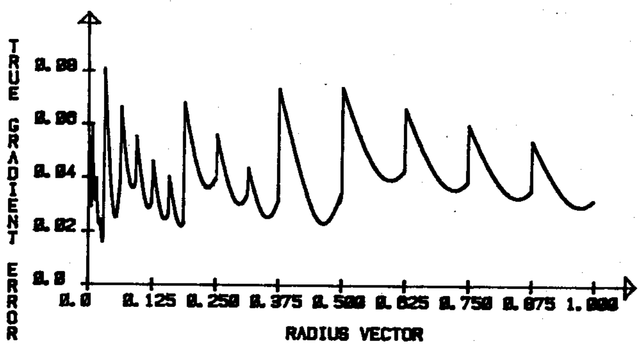

Fig 4.3 Gradient error

4.7 An error estimate in the $L_2(\Omega)$-norm

We have seen that if we apply the finite element method with the space $V_h=\{v\in H_0^1(\Omega): v|_K\in P_1(K), \forall K\in T_h\}$ to the Poisson equation (1.16) with Ω a polygonal domain, then we have the following estimate for the error $u-u_h$ in the $H^1(\Omega)$-norm:

(4.37) $\qquad \|u-u_h\|_{H^1(\Omega)}\leq Ch|u|_{H^2(\Omega)}.$

This trivially gives the following $L_2(\Omega)$-estimate:

(4.38) $\qquad \|u-u_h\|_{L_2(\Omega)}\leq Ch|u|_{H^2(\Omega)}.$

On the other hand by (4.18) the interpolation error, $u-\pi_h u$, satisfies the second order estimate:

$$\|u-\pi_h u\|_{L_2(\Omega)}\leq Ch^2|u|_{H^2(\Omega)}.$$

We shall now prove that we have a similar estimate for $\|u-u_h\|_{L_2(\Omega)}$ so that this quantity in fact converges at the optimal rate. We shall then assume that the polygonal domain Ω is convex (if Ω has a smooth boundary, then convexity is not required).

Theorem 4.3. If Ω is a convex polygonal domain and u_h is the finite element solution of the Poisson equation (1.16) with piecewise linear functions, i e u_h satisfies (1.20), then there is a constant independent of u and h such that

$$\|u-u_h\|_{L_2(\Omega)}\leq Ch^2|u|_{H^2(\Omega)}.$$

Proof. Subtracting (1.19) and (1.20) we obtain the error equation

(4.39) $\qquad a(e, v)=0 \quad \forall v\in V_h,$

where $e=u-u_h$ and the notation of (1.19) is used. We shall now estimate $(e, e)=\|e\|^2_{L_2(\Omega)}$ using a so-called *duality* argument which is often used in finite element analysis (see also Chapter 8). Let φ be the solution of the following auxiliary dual problem:

$$-\Delta\varphi=e \quad \text{in } \Omega,$$
$$\varphi=0 \quad \text{on } \Gamma.$$

Since Ω is convex we have from (4.27) with s=0,

(4.40) $\qquad \|\varphi\|_{H^2(\Omega)}\leq C\|e\|_{L_2(\Omega)},$

where the constant C does not depend on e. Using Green's formula and the fact that $e=0$ on Γ,

$$(e, e)=-(e, \Delta\varphi)=a(e, \varphi)=a(e, \varphi-\pi_h\varphi),$$

where the last inequality follows from (4.39) since $\pi_h\varphi\in V_h$ so that $a(e, \pi_h\varphi)=0$. Applying now the interpolation estimate (4.18) to φ and using also (4.40), we find

$$\|e\|^2_{L_2(\Omega)}\leq\|e\|_{H^1(\Omega)}\|\varphi-\pi_h\varphi\|_{H^1(\Omega)}\leq C\|e\|_{H^1(\Omega)}h|\varphi|_{H^2(\Omega)}$$
$$\leq Ch\|e\|_{H^1(\Omega)}\|e\|_{L_2(\Omega)}.$$

Dividing by $\|e\|_{L_2(\Omega)}$ and recalling (4.37) we finally get

$$\|e\|_{L_2(\Omega)}\leq Ch\|e\|_{H^1(\Omega)}\leq Ch^2|u|_{H^2(\Omega)}$$

and the proof is complete. □

Remark 4.3 The basic stability inequality (2.6) for (4.26) states that

$$\|u\|_{H^1(\Omega)}\leq\frac{\Lambda}{\alpha}, \tag{4.41}$$

where Λ is any constant such that

$$|L(v)|=|(f, v)|\leq\Lambda\|v\|_{H^1(\Omega)} \qquad \forall v\in H^1_0(\Omega).$$

The smallest possible choice of Λ is given by

$$\Lambda=\sup_{\substack{v\in H^1_0(\Omega)\\ v\neq0}}\frac{|(f, v)|}{\|v\|_{H^1(\Omega)}}. \tag{4.42}$$

Clearly the quantity Λ defined by (4.42) measures the size of f in a certain sense and in fact we may define a norm $\|\cdot\|_{H^{-1}(\Omega)}$ by

$$\|f\|_{H^{-1}(\Omega)}=\sup_{\substack{v\in H^1_0(\Omega)\\ v\neq0}}\frac{|(f, v)|}{\|v\|_{H^1(\Omega)}}. \tag{4.43}$$

This is the norm in the so-called *dual space* $H^{-1}(\Omega)$ of $H^1_0(\Omega)$. Note that

$$\|f\|_{L_2(\Omega)}=\sup_{\substack{v\in L_2(\Omega)\\ v\neq0}}\frac{|(f, v)|}{\|v\|_{L_2(\Omega)}},$$

and since we take sup over a larger set, we clearly have that $\|\cdot\|_{L_2(\Omega)}$ is a stronger norm than $\|\cdot\|_{H^{-1}(\Omega)}$, i e,

$$\|f\|_{H^{-1}(\Omega)}\leq\|f\|_{L_2(\Omega)}.$$

By (4.42) and (4.43) it follows that the basic stability inequality (4.41) may be written as

$$\|u\|_{H^1(\Omega)}\leq\frac{1}{\alpha}\|f\|_{H^{-1}(\Omega)},$$

which formally corresponds to (4.27) with $s=-1$. □

Problems

4.1 Let $I=[0, h]$ and let $\pi v\in P_1(I)$ be the linear interpolant that agrees with $v\in C^0(I)$ at the end points of I. Using the technique of the proof of Theorem 4.1 prove estimates for $\|v-\pi v\|_{L_\infty(I)}$ and $\|v'-(\pi v)'\|_{L_\infty(I)}$, cf (1.12) and (1.13).

4.2 Prove (4.14).

4.3 Estimate the error $\|u-u_h\|_{H^2(I)}$ for Problem 1.5 and Example 2.4.

4.4 Prove that the total number of elements with a corner refinement of the form (4.31) is $0(h^{-2})$.

4.5 Determine a suitable refinement in case the exact solution has a singularity of the form (4.28) with $1<\gamma<2$ and we want to control $\|\nabla u-\nabla u^h\|_{L_\infty(\Omega)}$ via the estimate (4.25), cf [EJ2].

4.6 Using polar coordinates (r, θ), let $\Omega=\{(r, \theta): 0<r<1, 0<\theta<\omega\}$ be a pie-shaped domain of angle ω. Prove that the function $u(r, \theta)=r^\gamma\sin(\gamma\theta)$, $\gamma=\frac{\pi}{\omega}$, satisfies: $\Delta u=0$ in Ω, $u=0$ on the straight parts of the boundary of Ω.

4.7 Prove, by modifying the proof of Theorem 4.3, the following L_2-estimate for the standard finite element method of Section 1.4 for Poisson's equation on an L-shaped domain (cf Example 4.5):

$$\|u-u_h\|_{L_2(\Omega)}\leq Ch^{4/3-\varepsilon}.$$

4.8 Let V_h be a finite element space on a triangulation T_h of the domain $\Omega\subset R^d$ satisfying (4.19). Given $u\in L_2(\Omega)$ let $u_h\in V_h$ be the $L_2(\Omega)$-*projection of* u *onto* V_h, i e,