element methods of so-called *equilibrium* type (for such a method the equilibrium condition $\operatorname{div} q+f=0$ will be satisfied exactly in the discrete model). Methods of this type may in certain cases have advantages as compared to the conventional finite element methods, so-called *displacement methods*, that we have studied above (in a displacement method for (2.26) the compatibility relation (2.26a) is satisfied exactly). Hint: First show that $p\in H_f$ is a solution of (2.32) if and only if

$$\int_\Omega p\cdot q\, dx=0 \qquad \forall q\in H_0,$$

where $H_0=\{q\in H, \operatorname{div} q=0 \text{ in } \Omega\}$.

2.10 Solve Problem 2.3 with the following alternative boundary conditions:

$$u(0)=-u''(0)+\gamma u'(0)=0,\ u(1)=u''(1)+\gamma u'(1)=0,$$

where γ is a positive constant. Also give a mechanical interpretation of the boundary conditions.

2.11 Consider the Neumann problem

(2.33a) $-\Delta u=f$ in Ω,

(2.33b) $\dfrac{\partial u}{\partial n}=g$ on Γ,

(2.33c) $\int_\Omega u dx=0$.

Note that if u satisfies (2.33a, b), then so does $u+c$ for any constant c, and that the condition (2.33c) is added to give uniqueness. Give a variational formulation of (2.33) using the space

$$V=\{v\in H^1(\Omega):\ \int_\Omega v dx=0\},$$

and prove that the conditions (i)−(iv) are satisfied.

3. Some finite element spaces

3.1 Introduction. Regularity requirements

We shall now present some commonly used finite element spaces V_h. These spaces will consist of piecewise polynomial functions on subdivisions or "triangulations" $T_h=\{K\}$ of a bounded domain $\Omega\subset R^d$, $d=1, 2, 3$, into elements K. For $d=1$, the elements K will be intervals, for $d=2$, triangles or quadrilaterals and for $d=3$ tetrahedrons for instance.

We will need to satisfy either $V_h\subset H^1(\Omega)$ or $V_h\subset H^2(\Omega)$, corresponding to second order or fourth order boundary value problems, respectively. Since the space V_h consists of piecewise polynomials, we have

(3.1) $V_h\subset H^1(\Omega) \Leftrightarrow V_h\subset C^0(\bar\Omega)$,

(3.2) $V_h\subset H^2(\Omega) \Leftrightarrow V_h\subset C^1(\bar\Omega)$,

where $\bar\Omega=\Omega\cap\Gamma$ and

$$C^0(\bar\Omega)=\{v: v \text{ is a continuous function defined on } \bar\Omega\},$$

$$C^1(\bar\Omega)=\{v\in C^0(\bar\Omega):\ D^\alpha v\in C^0(\bar\Omega),\quad |\alpha|=1\}.$$

Thus, $V_h\subset H^1(\Omega)$ if and only if the functions $v\in V_h$ are continuous, and $V_h\subset H^2(\Omega)$ if an only if the functions $v\in V_h$ *and* their first derivatives are continuous. The equivalence (3.1) depends on the fact that the functions v in V_h are polynomials on each element K so that if v is continuous across the common boundary of adjoining elements, then the first derivatives $D^\alpha v$, $|\alpha|=1$, exist and are piecewise continuous so that $v\in H^1(\Omega)$. On the other hand, if v is not continuous across a certain inter-element boundary, i e $v\notin C^0(\bar\Omega)$, then the derivatives $D^\alpha v$, $|\alpha|=1$, do not exist as functions in $L_2(\Omega)$ and thus $v\notin H^1(\Omega)$ (if v is discontinuous across an element side S, then $D^\alpha v$, $|\alpha|=1$, would be a δ-function supported by S which is not a square-integrable function). In a similar way we realize that (3.2) holds.

To define a finite element space V_h we will have to specify:

(a) the triangulation $T_h=\{K\}$ of the domain Ω,

(b) the nature of the functions v in V_h on each element K (e g linear, quadratic, cubic, etc),

(c) the parameters to be used to describe the functions in V_h.

3.2 Some examples of finite elements

Let us now consider some examples. We first consider the case when Ω is a domain in the plane R^2 with polygonal boundary Γ. Let $T_h=\{K\}$ be a given triangulation of Ω according to Section 1.4 into triangles K. We shall use the following notation for r=0, 1, 2, . . .,

$$P_r(K)=\{v: v \text{ is a polynomial of degree} \leq r \text{ on } K\}.$$

Thus, $P_1(K)$ is the space of linear functions defined on K, i e, functions of the form

$$v(x)=a_{00}+a_{10}x_1+a_{01}x_2,\ x\in K,$$

where the $a_{ij}\in R$. We see that $\{\psi_1, \psi_2, \psi_3\}$, where

$$\psi_1(x)\equiv 1,\ \psi_2(x)=x_1,\ \psi_3(x)=x_2,$$

is a basis for $P_1(K)$, and that $\dim P_1(K)=3$, where dim W denotes the dimension of the linear space W.

Further, $P_2(K)$ is the space of *quadratic* functions on K, i e, functions of the form

$$v(x)=a_{00}+a_{10}x_1+a_{01}x_2+a_{20}x_1^2+a_{11}x_1x_2+a_{02}x_2^2,\ x\in K,$$

where the $a_{ij}\in R$. We see that $\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$ is a basis for $P_2(K)$ and that $\dim P_2(K)=6$. In general we have

$$P_r(K)=\{v : v(x)=\sum_{0\leq i+j\leq r} a_{ij}x_1^i x_2^j \text{ for } x\in K, \text{ where } a_{ij}\in R\},$$

and

$$\dim P_r(K)=\frac{(r+1)(r+2)}{2}.$$

Example 3.1 Let

(3.3) $\quad V_h=\{v\in C^0(\bar{\Omega}): v|_K\in P_1(K),\ \forall K\in T_h\},$

i e, V_h is the space of continuous piecewise linear functions that we have met in Section 1.7. As parameters, or *global degrees of freedom*, to describe the functions in V_h, we choose

(3.4) $\quad$ the values at the node points of T_h,

(including the node points on Γ). Let us now convince ourselves that this is a legitimate choice and show that a function $v\in V_h$ is uniquely determined by the values (3.4). This is of course intuitively quite obvious but let us anyway carry out the argument in detail here, since it will be a model to be used in more complicated situations below. We then first notice that if $K\in T_h$ is a triangle with vertices a^i, i=1, 2, 3, then the degrees of freedom for K corresponding to (3.4), i e, *the element degrees of freedom,* are

(3.5) $\quad$ the values at the vertices a^i, i=1, 2, 3.

To show that a function $v\in V_h$ is uniquely determined by the degrees of freedom (3.4) it is sufficient to show:

Theorem 3.1 Let $K\in T_h$ be a triangle with vertices $a^i=(a_1^i, a_2^i)$, i=1, 2, 3. A function $v\in P_1(K)$ is uniquely determined by the degrees of freedom (3.5), i e, given the values α_i, i=1, 2, 3, there is a uniquely determined function $v\in P_1(K)$ such that

(3.6) $\quad v(a^i)=\alpha_i \qquad i=1, 2, 3.$

Proof Since $v(x)=c_1x_1+c_2x_2+c_3$ for some constants $c_i\in R$, (3.6) is equivalent to the linear system of equations

(3.7) $\quad c_1a_1^i+c_2a_2^i+c_3=\alpha_i, \qquad i=1, 2, 3,$

in the unknowns c_i. This system has a unique solution for given α_i if and only if the determinant detB of the coefficient matrix

$$B=\begin{bmatrix} a_1^1 & a_2^1 & 1 \\ a_1^2 & a_2^2 & 1 \\ a_1^3 & a_2^3 & 1 \end{bmatrix}$$

is different from zero. However by basic linear algebra

(3.8) $\quad \det B/2=\text{area of } K,$

and thus $\det B\neq 0$. Hence B is non-singular, which proves the desired result. Since this argument will be used below, we also give a somewhat different version of this proof. We notice first that

$$\dim P_1(K) = \text{number of degrees of freedom } (=3),$$

i e, (3.7) has the same number of unknowns as equations. In this case it follows, again by basic linear algebra, that $\det B \neq 0$ if and only if solutions of (3.7) are unique, or in other words if the only solution of (3.7) with $\alpha_i=0$, $i=1, 2, 3$, is given by $c_i=0$, $i=1, 2, 3$, or formally:

(3.9) If $v \in P_1(K)$ and $v(a^i)=0$, $i=1, 2, 3$, then $v \equiv 0$.

In fact it is easy to prove (3.9) directly without using (3.8), which shows that we do not have to be able to compute $\det B$ in order to prove that $\det B \neq 0$. As we shall see below, this latter method of proof makes it possible to easily prove analogues of Theorem 3.1 for higher order polynomials in which case a direct computation of the determinant of the corresponding coefficient matrix could be very complicated. □

We can now determine the (nodal) basis functions for $P_1(K)$ associated with the degrees of freedom (3.5), i e, the functions $\lambda_i \in P_1(K)$, $i=1, 2, 3$, such that (see Fig 3.1):

$$\lambda_i(a^j)=\delta_{ij}=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases} \qquad i, j=1, 2, 3.$$

A function $v(x) \in P_1(K)$ then has the representation

$$(3.10) \qquad v(x)=\sum_{i=1}^{3} v(a_i)\lambda_i(x) \quad x \in K.$$

To determine the basis functions λ_i, we have to solve the system of equations (3.7) for three special choices of right hand side, namely, (1, 0, 0), (0, 1, 0) and (0, 0, 1).

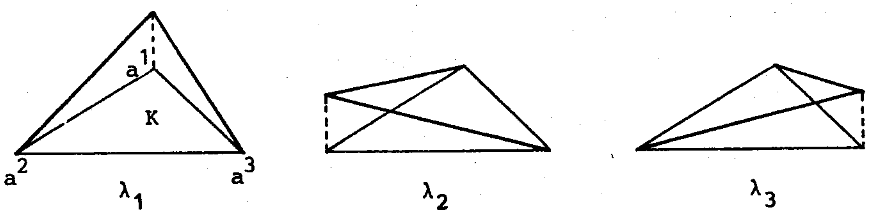


Fig 3.1

The basis function λ_1, say, can also be determined as follows. Let

$$d_1x_1+d_2x_2+d_3=0,$$

be the equation for the straight line through the vertices a^2 and a^3. Then

$$\lambda_1(x)=\gamma(d_1x_1+d_2x_2+d_3),$$

where the constant γ is chosen so that $\lambda_1(a^1)=1$. In the same way we may determine λ_2 and λ_3. If the triangle K has vertices at (1, 0), (0, 1) and (0, 0), then $\lambda_1=x_1$, $\lambda_2=x_2$ and $\lambda_3=1-x_1-x_2$. The notation λ_1, λ_2 and λ_3 for the nodal basis functions for $P_1(K)$ will be kept below.

Given the choice of global degrees of freedom in (3.4), it is natural to describe the space V_h given by (3.3) alternatively as

$$(3.11) \qquad V_h=\{v: v|_K \in P_1(K),\ \forall K \in T_h, \text{ and } v \text{ is continuous at the nodes}\}.$$

We then view a function $v \in V_h$ as a piecewise linear function taking on certain values at the nodes of T_h. Let us be careful and check that (3.11) defines the same space as (3.3) above. We need to check if a function $v \in V_h$ according to (3.11) is continuous, i e, if $v \in C^0(\bar{\Omega})$. Clearly, it is sufficient to check that v is continuous across all interelement sides. Thus, let K_1 and K_2 be two triangles in T_h having the common side S with the end points N_1 and N_2, say. Suppose now $v \in V_h$ according to (3.11) and let $v_i=v|_{K_i} \in P_1(K_i)$, $i=1, 2$, be the restrictions of v to the K_i. Then the function $w=v_1-v_2$ defined on S vanishes at the end points N_1 and N_2 and since w is linear on S it follows that in fact w vanishes on S. Hence, v is continuous across S and we obtain the desired conclusion that $v \in C^0(\bar{\Omega})$.

Example 3.2 Let us now show how to construct a space V_h using piecewise quadratic functions v, i e, $v|_K \in P_2(K)$. Let us first specify the element degrees of freedom. Let $K \in T_h$ be a triangle with vertices a^i, $i=1, 2, 3$, and denote the midpoints of the sides of K by a^{ij}, $i<j$, $i, j=1, 2, 3$, see Fig 3.2.

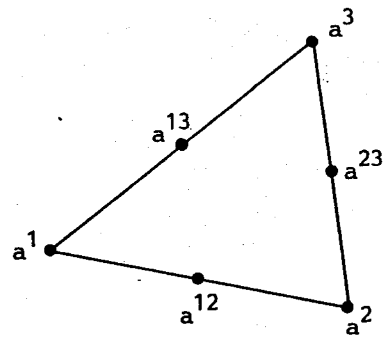


Fig 3.2

We shall prove

Theorem 3.2 A function $v \in P_2(K)$ is uniquely determined by the following degrees of freedom:

$$(3.12) \qquad \begin{array}{ll} v(a^i), & i=1,2,3, \\ v(a^{ij}), & i<j,\ i,j=1,2,3. \end{array}$$

Proof Since dim $P_2(K)$ is equal to the number of degrees of freedom (=6), it is (see the proof of Theorem 3.1) sufficient to prove that if $v \in P_2(K)$ and

$$(3.13) \qquad v(a^i)=0, \quad v(a^{ij})=0, \quad i<j, \quad i,j=1,2,3,$$

then $v \equiv 0$. To this end, consider the side a^2a^3. Along this side the function v has a quadratic variation and v vanishes at the three distinct points a^2, a^{23} and a^3. Thus, (cf Problem 3.1) v vanishes identically on a^{23} which means (cf Problem 3.3) that we can "factor out" the function λ_1 and write

$$v(x)=\lambda_1(x)w_1(x), \qquad x \in K,$$

where $w_1 \in P_1(K)$ and λ_i, $i=1,2,3$, are the basis functions for $P_1(K)$ according to Example 3.1. In the same way we see that v also vanishes along the side a^1a^3 which means that we may also factor out the function λ_2, so that

$$v(x)=\lambda_1(x)\lambda_2(x)w_0, \qquad x \in K,$$

where now w_0 has degree zero, i e, $w_0=\gamma$=constant. If we now finally take $x=a^{12}$, we see that

$$0=v(a^{12})=\gamma\lambda_1(a^{12})\lambda_2(a^{12})=\gamma\,\frac{1}{2}\cdot\frac{1}{2},$$

so that $\gamma=0$ and hence $v \equiv 0$ and the proof is complete. □

A function $v \in P_2(K)$ has the representation

$$(3.14) \qquad v=\sum_{i=1}^{3} v(a^i)\lambda_i(2\lambda_i-1)+\sum_{\substack{i,j=1\\ i<j}}^{3} v(a^{ij})4\lambda_i\lambda_j.$$

To see this, by Theorem 3.2 it is sufficient to check that the right hand side, RH, and left hand side, LH, of (3.14) take the same values at the node points a^i and a^{ij}, since the difference LH−RH$\in P_2(K)$. From (3.14) it is clear what the nodal basis functions for $P_2(K)$ corresponding to the degrees of freedom (3.12) are: the basis function corresponding to a particular degree of freedom, the value at the vertex a^i for instance, is of course the function $\psi \in P_2(K)$ such that $\psi(a^i)=1$ and ψ vanishes at the other five points a^j, a^{ij} (see Fig 3.3).

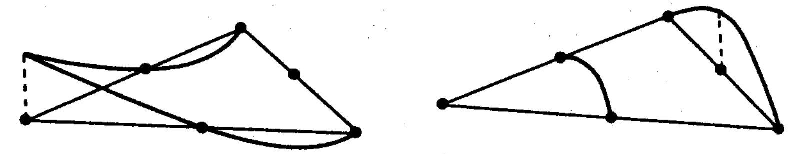


Fig 3.3 Different basis functions for $P_2(K)$

Let us also show that if $v_i \in P_2(K_i)$, $i=1,2$, where K_1 and K_2 are two triangles with the common side S, and v_1 and v_2 take the same values at the end points and the mid point of S, then v_1 and v_2 agree on S. But this follows immediately from the fact that $w=v_1-v_2$ varies quadratically along S and w vanishes at three distinct points on S so that $w \equiv 0$ on S.

Defining now

$$V_h=\{v \in C^0(\bar{\Omega}):\ v|_K \in P_2(K),\ \forall K \in T_h\},$$

we have seen that the global degrees of freedom of the functions $v \in V_h$ can be chosen as follows:

(i) the values of v at the nodes of T_h,
(ii) the values of v at the mid points of all the sides of the triangles in T_h.

The corresponding global basis functions have the following form:

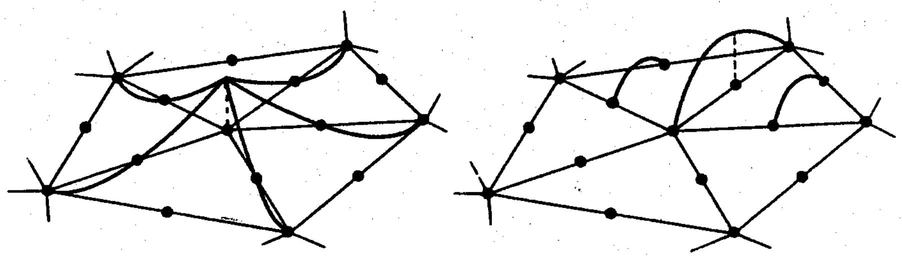


Fig 3.4

Example 3.3 We now define a space V_h using piecewise cubic functions, i e, functions v such that $v|_K \in P_3(K)$, $\forall K \in T_h$. Let K be a triangle with vertices a^i, $i=1,2,3$, and define (see Fig 3.5):

$$a^{iij}=\frac{1}{3}(2a^i+a^j), \ i,j=1,2,3, \quad i \neq j,$$

$$a^{123}=\frac{1}{3}(a^1+a^2+a^3).$$

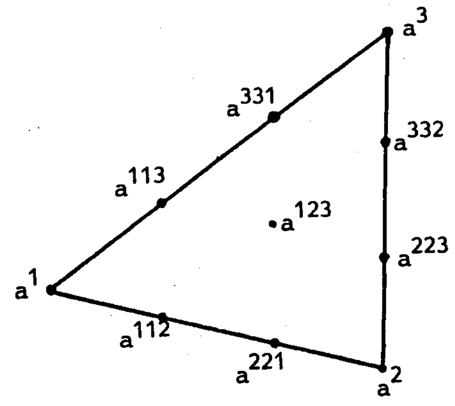


Fig 3.5

We have

Theorem 3.3 A function $v \in P_3(K)$ is uniquely determined by the following degrees of freedom:

(3.15) $\quad v(a^i),\ v(a^{iij}),\ i, j=1, 2, 3, \quad i \neq j,$
$v(a^{123})$.

Proof Since dim $P_3(K)$ is equal to the number of degrees of freedom (=10), it is sufficient to show that if $v \in P_3(K)$ and

(3.16) $\quad v(a^i)=v(a^{iij})=v(a^{123})=0,\ i, j=1, 2, 3, \quad i \neq j,$

then $v \equiv 0$. Observe that if v has a cubic variation along the side a^2a^3 then $v \equiv 0$ on a^2a^3. In the same way it follows that v vanishes on the sides a^1a^3 and a^1a^2 and hence

$$v(x)=\gamma\lambda_1(x)\lambda_2(x)\lambda_3(x),$$

where γ is a constant. If we now choose $x=a^{123}$, we get from (3.16)

$$0=v(a^{123})=\gamma\,\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3},$$

so that $\gamma=0$ and thus $v \equiv 0$. □

Now let $v_i \in P_3(K_i)$, $i=1, 2$, where K_1 and K_2 are two triangles with common side S and suppose that v_1 and v_2 take the same values at the end points and the two points a^{iij} of S. Since v_1-v_2 varies cubically on S it follows that $v_1=v_2$ on S (see Fig 3.6).

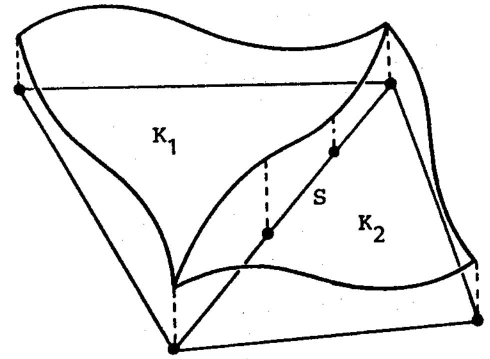


Fig 3.6

We can now introduce the space

$$V_h=\{v \in C^0(\bar{\Omega}):\ v|_K \in P_3(K),\ \forall K \in T_h\},$$

with the following degrees of freedom:

(i) the values of v at the nodes of T_h.
(ii) the values of v at the points a^{iij} on the sides of T_h,
(iii) the values of v at the center of gravity for all $K \in T_h$.

Example 3.4 There is another way of choosing the degrees of freedom for $P_3(K)$, where K is a triangle with vertices a^i, $i=1, 2, 3$, and center of gravity a^{123}. We have

Theorem 3.4 A function $v \in P_3(K)$ is uniquely determined by the following degrees of freedom:

$v(a^i), \quad i=1, 2, 3,$

(3.17) $\quad \dfrac{\partial v}{\partial x_j}(a^i), \quad i=1, 2, 3,\ j=1, 2,$

$v(a^{123})$.

Proof Since again dim $P_3(K)$ is equal to the number of degrees of freedom, it suffices to prove that if $v \in P_3(K)$ and

(3.18) $\quad v(a^i)=\dfrac{\partial v}{\partial x_j}(a^i)=v(a^{123})=0,\ i=1, 2, 3,\ j=1, 2,$

then $v \equiv 0$. It follows from (3.18) that

$$\frac{\partial v}{\partial s}(a^i)=\frac{\partial v}{\partial x_1}(a^i)s_1+\frac{\partial v}{\partial x_2}(a^i)s_2=0,\ i=1, 2, 3,$$

where $\frac{\partial v}{\partial s}$ is the derivative in a direction $s=(s_1, s_2)$. In particular we then have

$$\frac{\partial v}{\partial s}(a^2)=\frac{\partial v}{\partial s}(a^3)=0,$$

where s is the direction from a^2 to a^3. Together with the fact that $v(a^2)=v(a^3)$ this shows that v vanishes along the side a^2a^3 since v varies as a cubic polynomial along this side. In the same way see that v vanishes on a^1a^2 and a^1a^3 and the argument is then completed as in the proof of Theorem 3.3. □

We further note that if $v_i \in P_3(K_i)$, $i=1, 2$, where K_1 and K_2 are two triangles with the common side S with endpoints $N_j=1, 2$, and v_1 and v_2 agree together with the first derivatives $\frac{\partial v_1}{\partial x_i}(N_j)$ and $\frac{\partial v_2}{\partial x_i}(N_j)$, i, j=1, 2, then $v_1 \equiv v_2$ on S.

The corresponding finite element space $V_h \subset C^0(\bar{\Omega})$ is given by

$$V_h=\{v: v|_K \in P_3(K), \quad \forall K \in T_h, \text{ and } v \text{ and } \frac{\partial v}{\partial x_i}, i=1, 2, \text{ are continuous at the nodes}\},$$

with the following degrees of freedom:

(i) the values of v and $\frac{\partial v}{\partial x_i}$, i=1, 2, at the nodes of T_h,

(ii) the values of v at the center of gravity of each $K \in T_h$. □

Example 3.5 Let us now consider a finite element space V_h satisfying the condition $V_h \subset C^1(\bar{\Omega})$. We will then work with functions that are polynomials of degree five on each triangle; with polynomials of lower degree, special constructions are required to satisfy the C^1-condition.

Theorem 3.5 Let K be a triangle with vertices a^i, i=1, 2, 3 and let a^{ij} be the midpoint on the side a^ia^j, i, j=1, 2, 3, i<j (see Fig 3.2). A function $v \in P_5(K)$ is uniquely determined by the following degrees of freedom:

$$D^\alpha v(a^i),\ i=1, 2, 3,\ |\alpha| \leq 2,$$

$$\frac{\partial v}{\partial n}(a^{ij}),\ i, j=1, 2, 3,\ i<j, \tag{3.19}$$

where $\frac{\partial}{\partial n}$ denotes differentiation in the outward normal direction to the boundary of K.

Proof Since dim $P_5(K)$ is equal to the number of degrees of freedom (=21), it is sufficient as usual to prove that if all the degrees of freedom according

to (3.19) are zero, then $v \equiv 0$. To see this, we first note that if s denotes the direction of the side a^2a^3, then

$$v(a^i)=\frac{\partial v}{\partial s}(a^i)=\frac{\partial^2 v}{\partial s^2}(a^i)=0 \qquad i=2, 3. \tag{3.20}$$

Since v is a polynomial on the side a^2a^3 of degree at most 5, it follows that v vanishes on a^2a^3. Further, $\frac{\partial v}{\partial n}$ is a polynomial of degree at most 4 on a^2a^3 and

$$\frac{\partial v}{\partial n}(a^{23})=\frac{\partial v}{\partial n}(a^i)=\frac{\partial}{\partial s}\left(\frac{\partial v}{\partial n}\right)(a^i)=0, \qquad i=2, 3, \tag{3.21}$$

which is only possible if $\frac{\partial v}{\partial n} \equiv 0$ on a^2a^3. Thus, both v and $\frac{\partial v}{\partial n}$ vanish on a^2a^3 which means that we may factor $(\lambda_1(x))^2$ out of v(x) (check this in the special case when a^2a^3 lies on the x_2-axis). Therefore

$$v(x)=(\lambda_1(x))^2 p_3(x), \qquad x \in K,$$

where $p_3 \in P_3(K)$.. In the same way we see that we may also factor out $(\lambda_i(x))^2$, i=2, 3, and thus

$$v=\gamma \lambda_1^2 \lambda_2^2 \lambda_3^2,$$

where $\gamma \in R$. But $v \in P_5(K)$ and the only possibility then is that $\gamma=0$ so that $v \equiv 0$ on K. □

Now let $v_i \in P_5(K_i)$, i=1, 2, where K_1 and K_2 are two triangles with common side S and suppose that

$$D^\alpha v_1 = D^\alpha v_2 \quad \text{at the endpoints of S}, |\alpha| \leq 2,$$

$$\frac{\partial v_1}{\partial n}=\frac{\partial v_2}{\partial n} \quad \text{at the midpoint of S},$$

where $\frac{\partial}{\partial n}$ denotes differentiation in the normal direction to S. Then we have the relations (3.20) and (3.21) for the difference $w=v_1-v_2$ and it follows that

$$w=\frac{\partial w}{\partial n}=0 \text{ on S}. \tag{3.22}$$

But if w=0 on S we also have that

$$\frac{\partial w}{\partial s}=0 \text{ on S}, \tag{3.23}$$

where $\frac{\partial}{\partial s}$ denotes differentiation in the direction tangential to S. By (3.22) and (3.23) we see the function v defined by $v|_{K_i}=v_i$ varies continuously across S as do its first derivatives.

We may now define the space $V_h \subset C^1(\bar{\Omega})$ as follows

$$V_h=\{v: v|_K \in P_5(K),\ \forall K \in T_h,\ D^\alpha v \text{ is continuous at the nodes for } |\alpha| \leq 2 \text{ and } \frac{\partial v}{\partial n} \text{ is continuous at the mid points of each side}\},$$

with the degrees of freedom of (3.19).

Example 3.6 Let us now construct a three-dimensional finite element. We then assume that Ω is the union of a collection $T_h=\{K\}$ of non-overlapping tetrahedrons K such that no vertex of one tetrahedron lies on a side of another tetrahedron. As above, for r=1, 2,. . ., and $K \in T_h$, we define

$$P_r(K)=\{v: v \text{ is a polynomial to degree} \leq r \text{ on } K, \text{ i e } v \text{ has the form } v(x)=\sum_{i+j+m \leq r} a_{ijm} x_1^i x_2^j x_3^m,\ a_{ijm} \in R\}.$$

For r=1 a function $v \in P_1(K)$ is uniquely determined by the values $v(a^i)$, i=1,. . ., 4, where the a^i are the vertices of K. We can then introduce the space

$$V_h=\{v \in C^0(\bar{\Omega}): v|_K \in P_1(K),\ \forall K \in T_h\},$$

and as global degress of freedom we may take the values at the nodes of T_h points. □

Example 3.7 Let us also consider some rectangular finite elements that can be used for example if $\Omega \subset R^2$ is a square. Let then K be a rectangle with vertices a^i, i=1,. . ., 4, and with sides parallel to the coordinate axis in R^2. Define

$$Q_1(K)=\{v: v \text{ is } \textit{bilinear} \text{ on } K, \text{ i e}, v(x)=a_{00}+a_{10}x_1+a_{01}x_2+a_{11}x_1x_2,\ x \in K, \text{ where the } a_{ij} \in R\}.$$

It is easy to see (prove this!) that a function $v \in Q_1(K)$ is uniquely determined by the values $v(a^i)$, i=1,. . ., 4. Further, if K_1 and K_2 are two rectangles with the common side S and the functions $v_i \in Q_1(K_i)$ agree at the endpoints of S then $v_1-v_2 \equiv 0$ on S since v_1-v_2 varies linearly on S. We may now define

$$V_h=\{v \in C^0(\bar{\Omega}): v|_K \in Q_1(K),\ \forall K \in T_h\}$$

assuming that $T_h=\{K\}$ is a subdivision of Ω into non-overlapping rectangles such that no vertex of any rectangle lies on a side of another rectangle. The values at the nodes may be used as global degrees of freedom.

We can also use polynomials of higher degree on each rectangle. For example we may choose

$$V_h=\{v \in C^0(\bar{\Omega}): v|_K \in Q_2(K),\ \forall K \in T_h\},$$

where $Q_2(K)$ is the set of *biquadratic* functions on K, i e,

$$Q_2(K)=\{v: v(x)=\sum_{i,j=0}^{2} a_{ij} x_1^i x_2^j,\ x \in K, \text{ where the } a_{ij} \in R\},$$

and use as global degrees of freedom

(i) the values at the nodes of T_h,
(ii) the values at the midpoints of the sides of T_h,
(iii) the values at the midpoint of each rectangle $K \in T_h$.

Since the use of rectangular elements requires very special geometry of Ω it is of interest to also consider more general quadrilateral elements. The simplest such element is presented in Problem 12.3 below in connection with so-called isoparametric finite elements.

3.3 Summary

We have not yet given a formal definition of what we mean by a "finite element". To fill this gap define a *finite element* to mean a triple (K, P_K, Σ), where

- K is a geometric object, for example a triangle,
- P_K is a finite-dimensional linear space of functions defined on K,
- Σ is a set of degrees of freedom,

such that a function $v \in P_K$ is uniquely determined by the degrees of freedom Σ. From Example 3.1 we have that (K, P_K, Σ), where

- K is a triangle,
- $P_K=P_1(K)$,
- Σ is the values at the vertices of K,

is a finite element. In Fig 3.7 below we have collected some of the most common finite elements (cf [Ci]). The various degrees of freedom are denoted as follows:

. function values
o values of the first derivatives,
O values of the second derivatives,
/ value of the normal derivative,
↗ value of the mixed derivative $\frac{\partial^2 v}{\partial x_1 \partial x_2}$.

Finally, Fig 3.8 indicates in the case of two dimensions the *support* of certain basis function $v \in V_h$, i e, the points x such that $v(x) \neq 0$. The different cases correspond to a value at a node, the midpoint of a side or a point in the interior of an element. Clearly the support is always small and if φ and ψ are two basis functions associated with the nodes N_1 and N_2, then the supports of the functions φ and ψ overlap only if N_1 and N_2 belong to the same element.

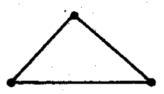
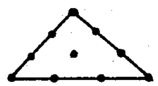
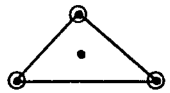
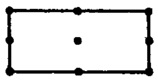
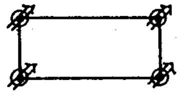
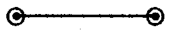
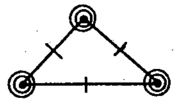
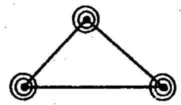
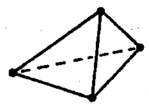
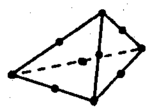
Degrees of freedom Σ Geometry		Function space P_K	Degree of continuity of corresponding FEM-space V_h
	3	$P_1(K)$	C^0
	6	$P_2(K)$	C^0
	10	$P_3(K)$	C^0
	10	$P_3(K)$	C^0
	4	$Q_1(K)$	C^0
	9	$Q_2(K)$	C^0
	16	$Q_3(K)$	C^1
	2	$P_1(K)$	C^0
	3	$P_2(K)$	C^0
	4	$P_3(K)$	C^1
	21	$P_5(K)$	C^1
	18	$P_5'(K)$ (see Problem 3.7)	C^1
	4	$P_1(K)$	C^0
	10	$P_2(K)$ (See Problem 3.4)	C^0

Fig 3.7 Some common finite elements.

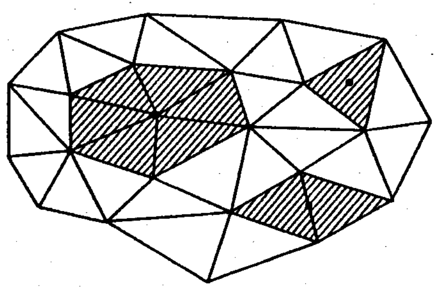


Fig 3.8 The support of different basis functions.

Problems

3.1 Show that if $v \in P_r(I)=\{v: v(x)=\sum_{i=0}^{r} a_i x^i, x \in I, \text{ where } a_i \in R\}$, the set of polynomials of degree at most r on the interval I, and if v vanishes at r+1 distinct points on I, then $v \equiv 0$. Recall that if $v \in P_r(I)$ and $v(b)=0$ for some $b \in I$, then $v(x)=(x-b)w(x)$ where $w \in P_{r-1}(I)$.

3.2 Prove that if $v \in P_r(K)$ where K is a triangle, then $v \in P_r(S)$ for any side S of K.

3.3 Let K be a triangle with vertices a^i, i=1, 2, 3. Suppose that $v \in P_r(K)$ and that v vanishes on the side a^2a^3. Prove that v has the form

$$v(x)=\lambda_1(x)w_{r-1}(x), \quad x \in K,$$

where $w_{r-1} \in P_{r-1}(K)$ and λ_1 is defined in Example 3.1.

3.4 Let K be a tetrahedron with vertices a^i, i=1, . . ., 4, and let a^{ij} denote the midpoint on the straight line a^ia^j, $i<j$. Show that a function $v \in P_2(K)$ is uniquely determined by the degrees of freedom: $v(a^i)$, $v(a^{ij})$, i, j=1, . . ., 4, $i<j$. Show that the corresponding finite element space V_h satisfies $V_h \subset C^0(\Omega)$.

3.5 Determine the stiffness matrix corresponding to the Poisson equation (1.16) when Ω is a square with side 1 and we use the bilinear element of Example 3.7 with $h=\frac{1}{4}$.

3.6 Let K be a triangle with vertices a^i and let a^{ij}, $i<j$, denote the midpoints of the sides of K. Show that a function $v \in P_1(K)$ is uniquely determined by the degrees of freedom $v(a^{ij})$, $i<j$. Consider the corresponding finite element space V_h. Is it true that $V_h \subset H^1(\Omega)$? Can we apply the theory of Chapter 2 in this case?

3.7 Show that a function $v \in P'_5(K)=\{v \in P_5(K): \frac{\partial v}{\partial n}$ is a polynomial of degree at most 3 on each side of K$\}$ is uniquely determined by the degrees of freedom $D^\alpha v(a^i)$, $|\alpha| \leq 2$, i=1, 2, 3, where the a^i are the vertices of the triangle K.

3.8 Let K be the triangle of Problem 3.6 and let a^{123} denote the center of gravity of K. Prove that $v \in P_4(K)$ is uniquely determined by the following degrees of freedom

$$v(a^i), \ \frac{\partial v}{\partial x_j}(a^i), \quad i=1, 2, 3, \ j=1, 2,$$

$$v(a^{ij}), \ i, j=1, 2, 3, \ i<j, \ v(a^{123}).$$

Also show that the functions in the corresponding finite element V_h are continuous.