2. Abstract formulation of the finite element method for elliptic problems

2.1 Introduction. The continuous problem

We shall now give an abstract formulation of the finite element method for elliptic problems of the type that we have studied in Chapter 1. This is not a goal in itself, but makes it possible to give a unified treatment of many problems in mechanics and physics so that we do not have to repeat in principle the same argument in different concrete cases. Further the abstract formulation is very easy to grasp and helps us to understand the basic structure of the finite element method.

Thus, let V be a Hilbert space with scalar product $(\cdot,\cdot)_V$ and corresponding norm $\|\cdot\|_V$ (the V-norm). Suppose that (cf Section 1.5) $a(\cdot,\cdot)$ is a bilinear form on $V\times V$ and L a linear form on V such that

(i) $a(\cdot,\cdot)$ is symmetric,

(ii) $a(\cdot,\cdot)$ is *continuous*, i e, there is a constant $\gamma>0$ such that

(2.1) $$|a(v, w)|\leq\gamma\|v\|_V\|w\|_V \qquad \forall v, w\in V,$$

(iii) $a(\cdot,\cdot)$ is V-*elliptic*, i e, there is a constant $\alpha>0$ such that

(2.2) $$a(v, v)\geq\alpha\|v\|_V^2 \qquad \forall v\in V.$$

(iv) L is *continuous*, i e, there is a constant $\Lambda>0$ such that

(2.3) $$|L(v)|\leq\Lambda\|v\|_V \qquad \forall v\in V.$$

Let us now consider the following abstract minimization problem (*M*): Find $u\in V$ such that

(2.4) $$F(u)=\min_{v\in V} F(v),$$

where

$$F(v)=\frac{1}{2}a(v,v)-L(v),$$

and consider also the following abstract variational problem (*V*): Find $u\in V$ such that

(2.5) $$a(u,v)=L(v) \qquad \forall v\in V.$$

Let us now first prove:

Theorem 2.1 The problems (2.4) and (2.5) are equivalent, i e, $u\in V$ satisfies (2.4) if and only if u satisfies (2.5). Moreover, there exists a unique solution $u\in V$ of these problems and the following stability estimate holds

(2.6) $$\|u\|_V\leq\frac{\Lambda}{\alpha}.$$

Proof Existence of a solution follows from the Lax-Milgram theorem which is variant of the Riesz' representation theorem in Hilbert space theory (see e g [Ne], [Ci], cf also Theorem 13.1 below). The reader unfamiliar with these concepts may simply bypass this remark. To prove that (2.4) and (2.5) are equivalent, we argue exactly as in Section 1.1. We first show that if $u\in V$ satisfies (2.4), then also (2.5) holds, and we leave the proof of the reverse implication to the reader. Thus, let $v\in V$ and $\varepsilon\in R$ be arbitrary. Then $(u+\varepsilon v)\in V$ so that since u is a minimum,

$$F(u)\leq F(u+\varepsilon v) \qquad \forall\varepsilon\in R.$$

Using the notation $g(\varepsilon)=F(u+\varepsilon v)$, $\varepsilon\in R$, we thus have

$$g(0)\leq g(\varepsilon) \qquad \forall\varepsilon\in R,$$

so that g has a minimum at $\varepsilon=0$. Hence $g'(0)=0$ if the derivative $g'(\varepsilon)$ exists at $\varepsilon=0$. But

$$\begin{aligned} g(\varepsilon)&=\frac{1}{2}a(u+\varepsilon v, u+\varepsilon v)-L(u+\varepsilon v)\\ &=\frac{1}{2}a(u, u)+\frac{\varepsilon}{2}a(u, v)+\frac{\varepsilon}{2}a(v, u)+\frac{\varepsilon^2}{2}a(v, v)-L(u)-\varepsilon L(v)\\ &=\frac{1}{2}a(u, u)-L(u)+\varepsilon a(u, v)-\varepsilon L(v)+\frac{\varepsilon^2}{2}a(v, v), \end{aligned}$$

where we used the symmetry of $a(\cdot,\cdot)$. It follows that

$$0=g'(0)=a(u, v)-L(v),$$

which proves (2.5). To prove the stability result we choose $v=u$ in (2.5) and use (2.2) and (2.3) to obtain

$$\alpha\|u\|_V^2 \leq a(u, u)=L(u)\leq \Lambda\|u\|_V,$$

which proves (2.6) upon division by $\|u\|_V\neq 0$. Finally, the uniqueness follows from the stability estimate (2.6) since if u_1 and u_2 are two solutions so that $u_i \in V$ and

$$a(u_i,v)=L(v) \qquad \forall v\in V,\ i=1, 2,$$

then by subtraction we see that $u_1-u_2\in V$ satisfies

$$a(u_1-u_2,v)=0 \qquad \forall v\in V.$$

Applying the stability estimate to this situation (with $L\equiv 0$, i e, $\Lambda=0$) we conclude that $\|u_1-u_2\|_V=0$, i e, $u_1=u_2$. □

Remark 2.1 Even without the symmetry condition (i) and with only (ii)−(iv) satisfied, one can prove that there exists a unique $u\in V$ such that

$$a(u,v)=L(v) \qquad \forall v\in V,$$

and the stability estimate (2.6) of course holds (cf Example 2.6 below). In this case there is however no associated minimization problem. □

2.2 Discretization. An error estimate

Now let V_h be a finite-dimensional subspace of V of dimension M. Let $\{\varphi_1,\dots,\varphi_M\}$ be a basis for V_h, so that $\varphi_i\in V_h$ and any $v\in V_h$ has the unique representation

(2.7) $$v=\sum_{i=1}^{M}\eta_i\varphi_i, \quad \text{where } \eta_i\in R.$$

We can now formulate the following discrete analogues of the problems (*M*) and (*V*): Find $u_h\in V_h$ such that

(2.8) $$F(u_h)\leq F(v) \qquad \forall v\in V_h,$$

or equivalently: Find $u_h\in V_h$ such that

(2.9) $$a(u_h, v)=L(v) \qquad \forall v\in V_h.$$

As in Section 1.2 we see that (2.9) is equivalent to

$$a(u_h, \varphi_j)=L(\varphi_j),\ j=1,\dots, M.$$

Using the representation

(2.10) $$u_h=\sum_{i=1}^{M}\xi_i\varphi_i, \quad \xi_i\in R,$$

(2.9) can be written as

$$\sum_{i=1}^{M} a(\varphi_i,\varphi_j)\xi_i=L(\varphi_j),\ j=1,\dots, M,$$

or, in matrix form,

(2.11) $$A\xi=b,$$

where $\xi=(\xi_i)\in R^M$, $b=(b_i)\in R^M$ with $b_i=L(\varphi_i)$, and $A=(a_{ij})$ is an $M\times M$ matrix with elements $a_{ij}=a(\varphi_i,\varphi_j)$. From the representation (2.7), we have

$$a(v,v)=a(\sum_{i=1}^{M}\eta_i\varphi_i, \sum_{j=1}^{M}\eta_j\varphi_j)=\sum_{i,j=1}^{M}\eta_i a(\varphi_i,\varphi_j)\eta_j=\eta\cdot A\eta,$$

$$L(v)=L(\sum_{j=1}^{M}\eta_j\varphi_j)=\sum_{j=1}^{M}\eta_j L(\varphi_j)=b\cdot\eta,$$

where the dot denotes the usual scalar product in R^M:

$$\xi\cdot\eta=\sum_{i=1}^{M}\xi_i\eta_i.$$

It follows that (2.8) may be formulated as

(2.12) $$\frac{1}{2}\xi\cdot A\xi-b\cdot\xi=\min_{\eta\in R^M}\left[\frac{1}{2}\eta\cdot A\eta-b\cdot\eta\right].$$

We also have, recalling (2.2),

$$\eta\cdot A\eta=a(v,v)\geq\alpha\|v\|_V^2>0,$$

if $v\neq 0$, i e, if $\eta\neq 0$. Since also $a(\varphi_i,\varphi_j)=a(\varphi_j,\varphi_i)$, this proves the following result.

Theorem 2.2 The stiffness matrix A is symmetric and positive definite.

We can now prove the following basic result where the equivalence follows as above.

Theorem 2.3 There exists a unique solution $\xi \in R^M$ to the equivalent problems (2.11) and (2.12), i e, there exists a unique solution $u_h \in V_h$ to the equivalent problems (2.8) and (2.9). Further, the following stability estimate holds:

$$\|u_h\|_V \leq \frac{\Lambda}{\alpha}. \tag{2.13}$$

Proof Since A is positive definite, A is non-singular, which proves existence and uniqueness. The stability estimate follows by choosing $v=u_h$ in (2.9) which gives, using (2.2) and (2.3),

$$\alpha\|u_h\|_V^2 \leq a(u_h,u_h)=L(u_h)\leq \Lambda\|u_h\|_V,$$

from which (2.13) follows upon division by $\|u_h\|_V \neq 0$.

Remark The stability estimate (2.13) for the finite element solution, which is an analogue of the stability estimate (2.6) for the continuous problem, reflects a very important property of the finite element method. In a certain sense it can be viewed as the theoretical basis for the success of the method. □

Let us now prove the following error estimate:

Theorem 2.4 Let $u \in V$ be the solution of (2.5) and $u_h \in V_h$ that of (2.9) where $V_h \subset V$. Then

$$\|u-u_h\|_V \leq \frac{\gamma}{\alpha}\|u-v\|_V \quad \forall v \in V_h.$$

Proof Since $V_h \subset V$ we have from (2.5) in particular

$$a(u,w)=L(w) \qquad \forall w \in V_h,$$

so that after subtracting (2.9),

$$a(u-u_h,w)=0 \qquad \forall w \in V_h. \tag{2.14}$$

For an arbitrary $v \in V_h$, define $w=u_h-v$. Then $w \in V_h$, $v=u_h-w$ and by (2.2) and (2.14), we have

$$\alpha\|u-u_h\|_V^2 \leq a(u-u_h, u-u_h)=a(u-u_h, u-u_h)+a(u-u_h, w)$$

$$=a(u-u_h, u-u_h+w)=a(u-u_h, u-v)\leq \gamma\|u-u_h\|_V\|u-v\|_V,$$

where the last inequality follows from (2.1). Dividing by $\|u-u_h\|_V$ we obtain the desired estimate. □

From the abstract qualitative estimate of Theorem 2.4 we may obtain a quantitative estimate by choosing a suitable function $v \in V_h$ and estimating $\|u-v\|_V$. Usually one then chooses $v=\pi_h u$ where $\pi_h u \in V_h$ is a suitable interpolant of u (e g $\pi_h u$ may be the piecewise linear interpolant $\tilde{u}_h$ of Section 1.3). In Chapter 4 we give estimates for the interpolation error $\|u-\pi_h u\|_V$ in a variety of situations.

2.3 The energy norm

By (2.1) and (2.2) it follows that we may introduce a new norm $\|\cdot\|_a$ on V defined by

$$\|v\|_a^2=a(v, v), \quad v \in V.$$

This norm is *equivalent* to the norm $\|\cdot\|_V$, i e, there are positive constants c and C such that

$$c\|v\|_V \leq \|v\|_a \leq C\|v\|_V \qquad \forall v \in V. \tag{2.15}$$

More precisely, we may choose $c=\sqrt{\alpha}$ and $C=\sqrt{\gamma}$. The scalar product $(\cdot,\cdot)_a$ corresponding to $\|\cdot\|_a$ is given by

$$(v,w)_a=a(v,w).$$

The norm $\|\cdot\|_a$ is referred to as the *energy norm*. The error equation (2.14) may now be written

$$(u-u_h, v)_a=0 \qquad \forall v \in V_h,$$

from which follows as in Section 1.3 or by the proof of Theorem 2.4, that

$$\|u-u_h\|_a \leq \|u-v\|_a \qquad \forall v \in V_h, \tag{2.16}$$

or equivalently that u_h is the projection of u onto V_h with respect to the scalar product $(\cdot,\cdot)_a$ (cf Section 1.6). Clearly (2.16) shows that u_h is a best approximation of u in the energy norm.

2.4 Some examples

Let us now consider some concrete examples of the form (2.5). In Chapter 5 further examples from mechanics and physics will be presented. Let Ω be a bounded domain in R^2 or R^3 with boundary Γ. The coordinates in R^2 and R^3 are denoted by $x=(x_1, x_2)$ and $x=(x_1, x_2, x_3)$.

Example 2.1 Let $V=H^1(\Omega)$, $\Omega\subset R^2$,

$$a(v, w)=\int_\Omega[\nabla v\cdot\nabla w+vw]dx,$$

$$L(v)=\int_\Omega fv dx,$$

where $f\in L_2(\Omega)$ in which case (2.5) is a variational formulation of the Neumann problem (1.37) with $g=0$. Let us verify that the conditions (i)–(iv) above are satisfied. Clearly $a(.,.)$ is a symmetric bilinear form on $V\times V$ and L is a linear form. Further,

$$a(v, v)=\|v\|^2_{H^1(\Omega)}$$

and by Cauchy's inequality

$$a(v,w)\leq a(v,v)^{1/2}a(w,w)^{1/2}=\|v\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)},$$

which proves (2.1) and (2.2) with $\alpha=\gamma=1$. Finally

$$|L(v)|\leq\left|\int_\Omega fv\,dx\right|\leq\|f\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)},$$

which proves (2.3) with $\Lambda=\|f\|_{L_2(\Omega)}$. □

Example 2.2 Let $V=H_0^1(I)$, $(I=0, 1)$,

$$a(v, w)=\int_I v'w'dx,\quad L(v)=\int_I fv\,dx,$$

where $f\in L_2(I)$ is given, which corresponds to our introductory boundary value problem (1.30). To verify that (i)–(iv) are satisfied, we first note that $a(.,.)$ is obviously symmetric and bilinear and L is linear and since

$$|a(v,w)|\leq\|v'\|_{L_2(I)}\|w'\|_{L_2(I)}\leq\|v\|_{H^1(I)}\|w\|_{H^1(I)},$$

we have that $a(.,.)$ is continuous. The continuity of L follows as in Example 2.1 and it thus remains to prove the V-ellipticity (2.2), i e, the inequality

$$\int_I(v')^2dx\geq\alpha\left(\int_I v^2dx+\int_I(v')^2dx\right)\qquad\forall v\in H_0^1(I),\tag{2.17}$$

for some positive constant α. We shall prove that

$$\int_I v^2dx\leq\int_I(v')^2dx\qquad\forall v\in H_0^1(I),\tag{2.18}$$

from which (2.17) follows with $\alpha=\frac{1}{2}$. Since $v(0)=0$ for $v\in H_0^1(I)$, we have

$$v(x)=v(0)+\int_0^x v'(y)dy=\int_0^x v'(y)dy,$$

so that by Cauchy's inequality

$$|v(x)|\leq\int_0^1|v'|dy\leq\left(\int_0^1 dy\right)^{1/2}\left(\int_0^1(v')^2dy\right)^{1/2}=\left(\int_0^1(v')^2dy\right)^{1/2}.$$

Squaring this inequality and then integrating over I we obtain (2.18). We note that the inequality (2.18) does not hold for $v(x)\equiv1$, in which case the left hand side is 1 and the right hand side 0. Thus we need e g a boundary condition of the form $v(0)=0$ for (2.18) to hold in order to control the norm of the function v by the norm of the derivative v', i e, we need a "fixed point" to start from.

If we choose V_h to consist of piecewise linear functions on I as in Section 1.2, we obtain in this case

$$\|u-u_h\|_{H^1(\Omega)}\leq Ch,$$

if u is smooth enough. □

Example 2.3 Let $V=H_0^1(\Omega)$, $\Omega\subset R^2$,

$$a(v,w)=\int_\Omega\nabla v\cdot\nabla w\,dx,\ L(v)=\int_\Omega fv dx,$$

where $f\in L_2(\Omega)$, in which case (2.5) is a variational formulation of the Dirichlet problem (1.16) for the Poisson equation. We directly see that (i), (ii) and (iv) are satisfied in this case. Thus, only the V-ellipticity, i e, the inequality

$$\int_\Omega|\nabla v|^2dx\equiv a(v,v)\geq\alpha\|v\|^2_{H^1(\Omega)}\equiv\alpha\left(\int_\Omega(v^2+|\nabla v|^2)dx\right)\tag{2.19}$$

requires comment. To prove (2.19), it is sufficient to prove that there is a constant C such that

$$\int_\Omega v^2dx\leq C\int_\Omega|\nabla v|^2dx\qquad\forall v\in H_0^1(\Omega),\tag{2.20}$$

since then (2.19) follows with $\alpha=\frac{1}{C+1}$. The proof of (2.20), which is often referred to as Poincare's inequality, is analogous to the proof of (2.18) (cf Problem 2.1 below). With the V_h of Section 1.4 we obtain the error estimate

$$\|u-u_h\|_{H^1(\Omega)}\leq Ch,$$

if u is sufficiently smooth. □

Example 2.4 Consider the following boundary value problem

$$\frac{d^4u}{dx^4}=f\qquad\text{for } x\in I=(0, 1),\tag{2.21a}$$

(2.21b) $\quad u(0)=u'(0)=u(1)=u'(1)=0,$

where $f\in L_2(I)$ (cf Problem 1.5). We introduce the space

$$H^2(I)=\{v\in L_2(I):\ v',\ v''\in L_2(I)\},$$

with norm

$$\|v\|_{H^2(I)}=\Big(\int_I [v^2+(v')^2+(v'')^2]dx\Big)^{1/2},$$

and the space

$$H_0^2(I)=\{v\in H^2(I):\ v(0)=v'(0)=v(1)=v'(1)=0\}$$

with the same norm. The problem (2.21) can now be given the variational formulation: Find $u\in V$ such that

$$a(u,v)=L(v) \qquad \forall v\in V,$$

where $V=H_0^2(\Omega)$,

$$a(v,w)=\int_I v''w''dx,\ \ L(v)=\int_I fv\ dx.$$

We see that the conditions (i), (ii) and (iv) are satisfied. By (2.18) we have for $v\in H_0^2(I)$

$$\int_I v^2dx\leqslant\int_I (v')^2dx\leqslant\int_I (v'')^2dx,$$

since $v(0)=v'(0)=0$, which proves that

$$\|v\|^2_{H^2(I)}\leqslant 3\int_I (v'')^2dx\equiv 3\ a(v,v),$$

and (iii) holds with $\alpha=\frac{1}{3}$. □

We now introduce some notation that will be used below. We define

$$D^\alpha v=\frac{\partial^{|\alpha|}v}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}},$$

where here $\alpha=(\alpha_1,\alpha_2)$, α_i is a non-negative natural number and $|\alpha|=\alpha_1+\alpha_2$. As an example, a partial derivative of order 2 can then be written as $D^\alpha v$ with $\alpha=(2,0)$, $\alpha=(1,1)$ or $\alpha=(0,2)$, which are the α with $|\alpha|=2$. We now define for $k=1,2,\ldots$,

$$H^k(\Omega)=\{v\in L_2(\Omega):\ D^\alpha v\in L_2(\Omega),\ |\alpha|\leqslant k\},$$

with norm

$$\|v\|_{H^k(\Omega)}=\Big(\sum_{|\alpha|\leqslant k}\int_\Omega |D^\alpha v|^2dx\Big)^{1/2}.$$

Thus the space $H^k(\Omega)$ consists of all functions v on Ω that, together with the partial derivatives $D^\alpha v$ of order $|\alpha|$ at most k, belong to $L_2(\Omega)$. The space $H^k(\Omega)$ is a Hilbert space with the indicated norm and corresponding scalar product. The spaces $H^k(\Omega)$ are examples of so called *Sobolev spaces* named after the Russian mathematician S. L. Sobolev 1908–, cf [Ad].

Example 2.5 Let us now consider a fourth-order problem in a two-dimensional domain Ω, namely the *biharmonic problem:*

(2.22a) $\quad \Delta\Delta u=f \quad \text{in } \Omega,$

(2.22b) $\quad u=\frac{\partial u}{\partial n}=0 \quad \text{on } \Gamma,$

where $\frac{\partial}{\partial n}$ denotes differentiation in the outward normal direction to the boundary Γ. This problem gives a formulation of the Stokes equations in fluid mechanics (cf Problem 5.3) and also models the displacement of a thin elastic plate, clamped at its boundary, under a transversal load (cf Problem 5.4). To give a variational formulation of (2.22), we introduce the space

$$H_0^2(\Omega)=\{v\in H^2(\Omega):\ v=\frac{\partial v}{\partial n}=0 \text{ on } \Gamma\}.$$

Now we multiply (2.22a) with $v\in H_0^2(\Omega)$ and integrate over Ω. By Green's formula as $v=\frac{\partial v}{\partial n}=0$ on Γ, we have

$$\int_\Omega fv dx=\int_\Omega \Delta\Delta u\ v\ dx=$$

$$=\int_\Gamma \frac{\partial}{\partial n}(\Delta u)v\ ds-\int_\Omega \nabla(\Delta u)\cdot\nabla v\ dx=$$

$$=-\int_\Omega \nabla(\Delta u)\nabla v dx=-\int_\Gamma \Delta u\frac{\partial v}{\partial n}ds+\int_\Omega \Delta u\Delta v dx=\int_\Omega \Delta u\Delta v\ dx.$$

We are thus led to the following variational formulation of the biharmonic problem (2.22): Find $u\in V$ such that

$$a(u,v)=L(v) \qquad \forall v\in V,$$

where $v=H_0^2(\Omega)$ and

$$a(u, v)=\int_\Omega \Delta u\ \Delta v\ dx,\ L(v)=\int_\Omega fv\ dx.$$

Again we see directly that (i), (ii) and (iv) are satisfied in this case and the V-ellipticity (iii) can easily be proved using the hints of Problem 2.2 below. In Chapter 3 below we shall construct finite element spaces $V_h \subset H_0^2(\Omega)$. □

Example 2.6 Consider the following problem in a domain $\Omega \subset R^2$:

$$\text{(2.23a)} \qquad -\mu\Delta u+\beta_1 \frac{\partial u}{\partial x_1}+\beta_2 \frac{\partial u}{\partial x_2}+u=f \qquad \text{in } \Omega,$$

$$\text{(2.23b)} \qquad u=0 \text{ on } \Gamma,$$

where μ and the β_i are constants with $\mu>0$. This is an example of a stationary *convection-diffusion problem;* the Laplace term corresponds to diffusion with diffusion coefficient μ and the first order derivatives correspond to convection in the direction $\beta=(\beta_1, \beta_2)$. Let us here assume that $\mu=1$ and that the size of $|\beta|$ is moderate (for convection-diffusion problems with $|\beta|/\mu$ large, see Chapter 9). By multiplying (2.23a) by a test function $v \in V=H_0^1(\Omega)$, integrating over Ω and using Green's formula for the Laplace-term as usual, we are led to the following variational formulation of (2.23): Find $u \in V$ such that

$$\text{(2.24)} \qquad a(u, v)=L(v) \qquad \forall v \in V,$$

where

$$a(v, w)=\int_\Omega (\nabla v \cdot \nabla w+(\beta_1 \frac{\partial v}{\partial x_1}+\beta_2 \frac{\partial v}{\partial x_2}+v)w)dx, \quad L(v)=\int_\Omega fv\ dx.$$

It is clear that $a(.,.)$ is V-elliptic since if $v \in V$, we have by Green's formula:

$$\int_\Omega (\beta_1 \frac{\partial v}{\partial x_1} v+\beta_2 \frac{\partial v}{\partial x_2} v)dx=\int_\Gamma v^2(\beta_1 n_1+\beta_2 n_2)ds-$$

$$-\int_\Omega (v\ \beta_1 \frac{\partial v}{\partial x_1}+v\ \beta_2 \frac{\partial v}{\partial x_2})dx=-\int_\Omega (\beta_1 \frac{\partial v}{\partial x_1} v+\beta_2 \frac{\partial v}{\partial x_2} v)dx,$$

i e,

$$\int_\Omega (\beta_1 \frac{\partial v}{\partial x_1}+\beta_2 \frac{\partial v}{\partial x_2})v\ dx=0,$$

so that

$$a(v, v)=\int_\Omega [|\nabla v|^2+v^2]dx=\|v\|^2_{H^1(\Omega)}.$$

Existence of a unique weak solution of (2.23) now follows from Remark 2.1. Starting from (2.24) we may formulate the following finite element method for (2.23): Find $u_h \in V_h$ such that

$$\text{(2.25)} \qquad a(u_h, v)=L(v) \qquad \forall v \in V_h,$$

where V_h is a finite-dimensional subspace of V. If $\{\varphi_1, \ldots, \varphi_M\}$ is a basis for V_h we have as above that (2.25) is equivalent to the linear system $A\xi=b$ where $A=(a_{ij})$, $a_{ij}=a(\varphi_i, \varphi_j)$, and $b=(b_i)$, $b_i=(f, \varphi_i)$. Note that in this case the matrix A is *not* symmetric.

By the V-ellipticity it follows that solutions of (2.25) are unique and thus A is non-singular so that $A\xi=b$ admits a unique solution, i e, there exists a unique solution u_h of (2.25). By the same argument as in the proof of Theorem 2.4, we also have the error estimate (here $\alpha=1$):

$$\|u-u_h\|_{H^1(\Omega)} \leq \gamma \|u-v\|_{H^1(\Omega)} \quad \forall v \in V_h. \quad \square$$

Example 2.7 Let u be the temperature in a heat conducting body occupying the domain $\Omega \subset R^3$. We have in the stationary case the following relations:

$$\text{(2.26a)} \qquad -q_i=k_i(x)\frac{\partial u}{\partial x_i} \qquad \text{in } \Omega,\ i=1, 2, 3, \text{ (Fourier's law)},$$

$$\text{(2.26b)} \qquad \text{div } q=f \qquad \text{in } \Omega \text{ (conservation of energy)},$$

where the q_i denotes the heat flow in the x_i-direction, $k_i(x)$ is the heat conductivity at x in the x_i-direction and f(x) is the heat production at x. If $k_i(x)\equiv 1$, $x \in \Omega$, $i=1, 2, 3$, i e, if the heat conductivity is constant and equal in all directions, then eliminating q in (2.26), we obtain Poisson's equation $-\Delta u=f$ in Ω. With the k_i non-constant, (2.26) is an example of a partial differential equation with *variable coefficients*. However, the coefficients k_i are not assumed to depend on the solution u. If this was the case and the heat conductivities k_i depended on the temperature u, then (2.26) would be an example of a *non-linear* partial differential equation, see Chapter 13 below.

Let us now give a variational formulation of (2.26) which in the usual way can be used to formulate a finite element method for (2.26). This shows that the presence of the variable coefficients k_i do not introduce any difficulties. We complement (2.26a, b) with the following boundary conditions:

$$\text{(2.26c)} \qquad u=0 \qquad \text{on } \Gamma_1,$$

$$\text{(2.26d)} \qquad -q \cdot n=g \qquad \text{on } \Gamma_2,$$

where $\Gamma=\Gamma_1\cup\Gamma_2$ is a partition of the boundary Γ and n denotes the outward unit normal to Γ. The condition (2.26d) corresponds to a situation where the heat flow is given on Γ_2.

We introduce the space

$$V=\{v\in H^1(\Omega):\ v=0 \text{ on } \Gamma_1\},$$

multiply (2.26b) by $v\in V$ and integrate over Ω. By Green's formula we then get

$$\int_\Omega fv\,dx=\int_\Omega v \operatorname{div} q\,dx=\int_\Gamma vq\cdot n\,ds-\int_\Omega q\cdot\nabla v\,dx=$$

$$=\int_\Omega \Sigma_{i=1}^3 k_i(x)\,\frac{\partial u}{\partial x_i}\,\frac{\partial v}{\partial x_i}\,dx-\int_{\Gamma_2} gv\,ds,$$

where the last equality follows from (2.26a), (2.26d) and the fact that $v=0$ on Γ_1. Thus we are led to the following variational formulation of (2.26): Find $u\in V$ such that

(2.27) $\quad a(u,v)=L(v) \qquad \forall v\in V,$

where

$$a(v,w)=\int_\Omega \sum_{i=1}^{3} k_i(x)\,\frac{\partial v}{\partial x_i}\,\frac{\partial w}{\partial x_i}\,dx,$$

$$L(v)=\int_\Omega fv\,dx+\int_{\Gamma_2} gv\,ds.$$

We easily verify that the conditions (i)−(iv) are satisfied under the following hypothesis: There are positive constants c and C such that

$$c\le k_i(x)\le C,\ x\in\Omega,\ i=1,2,3,$$

$f\in L_2(\Omega)$, $g\in L_2(\Gamma_2)$, and the area of Γ_1 is positive.

Starting from (2.27) we may now formulate a finite element method for (2.26) by replacing V by a finite element space $V_h\subset V$. This leads to a linear system $A\xi=b$ with stiffness matrix $A=(a_{ij})$ with elements $a_{ij}=a(\varphi_i,\varphi_j)$ where $\{\varphi_1,\ldots,\varphi_M\}$ is a basis for V_h. To find the a_{ij} we have to compute integrals involving the variable coefficients $k_i(x)$. In practice we may for this purpose want to use numerical quadrature, cf Chapter 12. □

Problems

2.1 Let Ω be a square with side 1. Show that

$$\left(\int_\Omega v^2dx\right)^{1/2}\le\left(\int_\Omega|\nabla v|^2dx\right)^{1/2} \qquad \forall v\in H_0^1(\Omega).$$

2.2 Let Ω be a square with boundary Γ. Show that there is a constant C such that

$$\|v\|^2_{H^2(\Omega)}\le C\int_\Omega(\Delta v)^2dx \qquad \forall v\in H_0^2(\Omega),$$

by using the boundary conditions $v=\frac{\partial v}{\partial n}=0$ on Γ and the fact that by Green's formula,

$$\int_\Omega\left(\frac{\partial^2 v}{\partial x_1\partial x_2}\right)^2dx=\int_\Omega \frac{\partial^2 v}{\partial x_1^2}\,\frac{\partial^2 v}{\partial x_2^2}\,dx \qquad \forall v\in H_0^2(\Omega).$$

Note that if $v=0$ on Γ, then also $\frac{\partial v}{\partial s}=0$ on Γ, where $\frac{\partial}{\partial s}$ differentiation in a tangental direction to Γ.

2.3 Give a variational formulation of the problem

$$\frac{d^4u}{dx^4}=f \qquad \text{for } 0<x<1,$$
$$u(0)=u''(0)=u'(1)=u'''(1)=0,$$

and show that the conditions (i)−(iv) are satisfied. Which boundary conditions are essential and which are natural? What is the interpretation of the boundary conditions if u represents the deflection of an elastic beam?

2.4 Let Ω be a square with boundary Γ. Show that there is a constant C such that

$$\left(\int_\Gamma v^2ds\right)^{1/2}\le C\|v\|_{H^1(\Omega)} \quad \forall v\in H^1(\Omega).$$

Using this result show that the linear functional $L:H^1(\Omega)\to R$ defined by

$$L(v)=\int_\Gamma gv\,ds$$

is continuous if $g\in L_2(\Gamma)$, i e, if $\int_\Gamma g^2ds<\infty$.

2.5 Give a variational formulation of the inhomogeneous Neumann problem

$$-\Delta u+u=f \qquad \text{in } \Omega,$$
$$\frac{\partial u}{\partial n}=g \qquad \text{on } \Gamma,$$

and check if the conditions (i)–(iv) of Section 2.1 are satisfied. Give an example of a problem in mechanics that takes this form.

2.6 Give a variational formulation of the problem

$$-\Delta u = f \qquad \text{in } \Omega,$$
$$\gamma u + \frac{\partial u}{\partial n} = g \qquad \text{on } \Gamma,$$

where γ is a constant. When are conditions (i)–(iv) satisfied? Give an interpretation of the boundary condition (which is sometimes referred to as a Robin (or third type) boundary condition).

2.7 Consider the variational problem (2.27) with variable coefficients. Suppose that Ω is composed of two parts Ω_1 and Ω_2 with common boundary S (see Fig 2.1) and suppose the coefficients $k_i(x)$ are defined by

$$k_i(x) = \begin{cases} \varkappa_1 & \text{for } x \in \Omega_1, \\ \varkappa_2 & \text{for } x \in \Omega_2, \end{cases}$$

where the $\varkappa_i$ are positive constants.

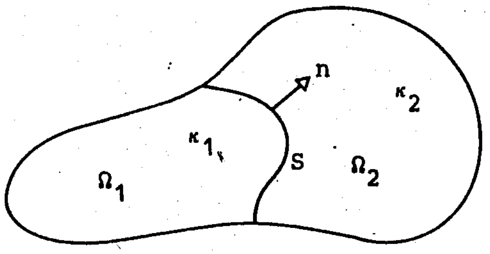

Fig 2.1

In this case (2.27) models stationary heat conduction in an isotropic body composed of two materials with heat conductivity coefficients $\varkappa_1$ and $\varkappa_2$ occupying the regions Ω_1 and Ω_2. Show (formally) that $u \in V$ satisfies (2.27) if and only if

(2.28a) $\quad -\varkappa_j \Delta u = f \qquad \text{in } \Omega_j,\ j=1, 2,$

(2.28b) $\quad u = 0 \qquad \text{on } \Gamma_1,$

(2.28c) $\quad q \cdot n = g \qquad \text{on } \Gamma_2,$

(2.28d) $\quad \varkappa_1 \frac{\partial u_1}{\partial n} = \varkappa_2 \frac{\partial u_2}{\partial n} \qquad \text{on } S,$

where $\frac{\partial u_j}{\partial n}$ denotes the derivative of $u_j = u|_{\Omega_j}$ in a direction n normal to S.

Notice that (2.28d) represents a balance of heat flowing between Ω_1 and Ω_2. Observe that this relation is "automatically built in" in the variational formulation (2.27).

2.8 Show (formally) that u is the solution of the variational problem

(2.29) $$\min_{v \in H_0^1(I)} \left[\frac{1}{2} \int_I k(x)(v')^2 dx - \int_I v\, dx\right],$$

where $I = (0, 1)$, and

$$k(x) = \begin{cases} 1 & \text{if } x \in I_1 = (0, \frac{1}{2}), \\ \frac{1}{2} & \text{if } x \in I_2 = (\frac{1}{2}, 1), \end{cases}$$

if and only if u satisfies

$$-k(x)u''(x) = 1 \text{ in } I_1 \text{ and } I_2,$$

(2.30) $$u_1 = u_2,\ 2\frac{du_1}{dx} = \frac{du_2}{dx} \qquad \text{for } x = \frac{1}{2},$$
$$u(0) = u(1) = 0,$$

where $u_i = u|_{I_i}$, $i=1, 2$. Then formulate a finite element method for (2.30) using piecewise linear functions. Determine the corresponding linear system in the case of a uniform partition and give an interpretation of this system as a difference method for (2.30).

2.9 Show that if u is the solution of the Dirichlet problem

(2.31) $$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma,$$

where $f \in L_2(\Omega)$ and $\Omega \subset R^2$, then $p = \nabla u$ is the solution of the minimization problem

(2.32) $$\min_{q \in H_f} \frac{1}{2} \int_\Omega |q|^2 dx,$$

where

$$H_f = \{q \in H:\ \text{div } q + f = 0 \text{ in } \Omega\},$$
$$H = \{q = (q_1, q_2):\ q_i \in L_2(\Omega)\}.$$

The minimization problem (2.32) corresponds to the *Principle of minimum complementary energy* in mechanics. Starting from (2.32), replacing H_f by a finite-dimensional subspace, one may construct finite element methods of so-called *equilibrium* type (for such a method the equilibrium condition $\operatorname{div} q+f=0$ will be satisfied exactly in the discrete model). Methods of this type may in certain cases have advantages as compared to the conventional finite element methods, so-called *displacement methods*, that we have studied above (in a displacement method for (2.26) the compatibility relation (2.26a) is satisfied exactly). Hint: First show that $p \in H_f$ is a solution of (2.32) if and only if

$$\int_\Omega p \cdot q \, dx=0 \qquad \forall q \in H_0,$$

where $H_0=\{q \in H, \operatorname{div} q=0 \text{ in } \Omega\}$.

2.10 Solve Problem 2.3 with the following alternative boundary conditions:

$$u(0)=-u''(0)+\gamma u'(0)=0, \; u(1)=u''(1)+\gamma u'(1)=0,$$

where γ is a positive constant. Also give a mechanical interpretation of the boundary conditions.

2.11 Consider the Neumann problem

(2.33a) $\quad -\Delta u=f$ in Ω,

(2.33b) $\quad \dfrac{\partial u}{\partial n}=g$ on Γ,

(2.33c) $\quad \int_\Omega u dx=0.$

Note that if u satisfies (2.33a, b), then so does $u+c$ for any constant c, and that the condition (2.33c) is added to give uniqueness. Give a variational formulation of (2.33) using the space

$$V=\{v \in H^1(\Omega): \; \int_\Omega v dx=0\},$$

and prove that the conditions (i)–(iv) are satisfied.

3. Some finite element spaces

3.1 Introduction. Regularity requirements

We shall now present some commonly used finite element spaces V_h. These spaces will consist of piecewise polynomial functions on subdivisions or "triangulations" $T_h=\{K\}$ of a bounded domain $\Omega \subset R^d$, $d=1, 2, 3$, into elements K. For $d=1$, the elements K will be intervals, for $d=2$, triangles or quadrilaterals and for $d=3$ tetrahedrons for instance.

We will need to satisfy either $V_h \subset H^1(\Omega)$ or $V_h \subset H^2(\Omega)$, corresponding to second order or fourth order boundary value problems, respectively. Since the space V_h consists of piecewise polynomials, we have

(3.1) $\quad V_h \subset H^1(\Omega) \Leftrightarrow V_h \subset C^0(\bar{\Omega})$,
(3.2) $\quad V_h \subset H^2(\Omega) \Leftrightarrow V_h \subset C^1(\bar{\Omega})$,

where $\bar{\Omega}=\Omega \cap \Gamma$ and

$$C^0(\bar{\Omega})=\{v: v \text{ is a continuous function defined on } \bar{\Omega}\},$$

$$C^1(\bar{\Omega})=\{v \in C^0(\bar{\Omega}): D^\alpha v \in C^0(\bar{\Omega}), \quad |\alpha|=1\}.$$

Thus, $V_h \subset H^1(\Omega)$ if and only if the functions $v \in V_h$ are continuous, and $V_h \subset H^2(\Omega)$ if an only if the functions $v \in V_h$ *and* their first derivatives are continuous. The equivalence (3.1) depends on the fact that the functions v in V_h are polynomials on each element K so that if v is continuous across the common boundary of adjoining elements, then the first derivatives $D^\alpha v$, $|\alpha|=1$, exist and are piecewise continuous so that $v \in H^1(\Omega)$. On the other hand, if v is not continuous across a certain inter-element boundary, i e $v \notin C^0(\bar{\Omega})$, then the derivatives $D^\alpha v$, $|\alpha|=1$, do not exist as functions in $L_2(\Omega)$ and thus $v \notin H^1(\Omega)$ (if v is discontinuous across an element side S, then $D^\alpha v$, $|\alpha|=1$, would be a δ-function supported by S which is not a square-integrable function). In a similar way we realize that (3.2) holds.

To define a finite element space V_h we will have to specify:

(a) the triangulation $T_h=\{K\}$ of the domain Ω,