Finite elements methods in fluid mechanics

HOMEWORKS

Homework 1: Introduction

1. Let $\Omega = (0,1) \times (0,1)$ and consider the problem

$$\begin{aligned} \Delta\phi &= 0 && \text{in } \Omega \\ \phi &= 0 && \text{for } x = 0, 1,\ y = 0 \\ \phi &= x(1-x) && \text{for } y = 1 \end{aligned}$$

   Let us define the basis of functions $\{\sin(m\pi x)\sin(n\pi y)\},\ m, n = 1, 2, ...$ Obtain the Galerkin approximation to the previous problem in this basis, pointing out which are their particular features in comparison to what would be obtained using a second order finite difference scheme.

2. Let us consider a biquadratic Lagrangian element in 2D (element Q_2) with element domain $\Omega^e = (x_1, x_2) \times (y_1, y_2)$. The side $x = x_2$ lies on the boundary of the computational domain Ω. Obtain the weak form of the problem

$$\begin{aligned} -\Delta\phi &= f && \text{in } \Omega \\ \phi &= 0 && \text{on } \partial\Omega - \{x = x_2\} \\ \frac{\partial\phi}{\partial n} + \alpha\phi &= h && \text{for } x = x_2 \end{aligned}$$

   and work out the expression of the stiffness matrix of the element indicated.

3. Let us consider the one-dimensional problem

$$\begin{aligned} &-\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \lambda u = 1, \qquad 0 < x < 1 \\ &u(0) = u(1) = 0 \end{aligned}$$

   and assume that the domain $\Omega = (0,1)$ is discretized using

   (a) 6 linear elements
   (b) 3 quadratic elements
   (c) 2 cubic elements

   The size of the elements is constant in all the cases.

   (i) Obtain the analytical solution u of the continuous problem in terms of λ.
   (ii) Compute the finite element solution u_h in cases (a), (b) and (c) for $\lambda = 1$ and $\lambda = 10^3$.
   (iii) Compute the L^2 norm of the error $u - u_h$ in cases (a), (b) and (c), again for $\lambda = 1$ and $\lambda = 10^3$ (note that the number of degrees of freedom is the same in all these cases).

Homework 2: Convection-diffusion equation

1. Let us consider the one-dimensional convection-diffusion equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} - k\frac{\partial^2 u}{\partial x^2} = f, \;\; 0 < x < 1$$

with $k > 0, \; a > 0$. Let us assume that the interval $[0, 1]$ is discretized using elements of equal size h and that the SUPG method is used for the spatial discretization and the forward Euler scheme to discretize in time, approximating the mass matrix by a diagonal one obtained from a nodal quadrature rule. Prove that the algorithm satisfies the von Neumann stability condition if, and only if,

$$c \leq \frac{\gamma}{1 + \alpha\gamma}$$

where α is the upwind function of the SUPG method, γ is the Péclet number of the partition and $c := a\delta t/h$ is the Courant number, δt being the time step size.

   (a) Plot the stability region in the γ-c plane dictated by the previous condition as well as the stability region that would be obtained using the Galerkin method ($\alpha = 0$).

   (b) Prove that when $a = 0$ the algorithm is stable if, and only if,

$$\delta t \leq \frac{h^2}{2k}$$

2. Prove that the discontinuous Galerkin method in time applied to the convection-diffusion equation is stable (Hint: select an adequate v_h in Eq. (37) of the paper: R. Codina, Comparison of some finite element methods for solving the diffusion-convection-reaction equation, *Computer Methods in Applied Mechanics and Engineering*, Vol. 156 (1998), 185-210).

3. Explain why the Characteristic Galerkin method introduces diffusion along the streamlines when it is applied to the convection-diffusion equation (Hint: Use Eq. (81) of the paper mentioned in the previous exercise).

4. Let us consider again the one-dimensional equation of exercise 1, and assume now that the standard Galerkin method is used for the spatial discretization. Obtain the matrix of the final linear system when the following time discretizations are used:

   (a) The Crank-Nicolson method.

   (b) The second order Gear scheme.

   (c) The second order Adams-Brashforth scheme.

Homework 3: Stokes problem

1. Let us consider the linear system resulting from the finite element discretization of the penalized Stokes problem

$$\begin{bmatrix} \mathbf{K} & \mathbf{G} \\ \mathbf{G}^{\mathrm{t}} & -\varepsilon \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{U}^{\varepsilon} \\ \mathbf{P}^{\varepsilon} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix}.$$

   Let $\mathbf{U}$ and $\mathbf{P}$ the solution of the problem with $\varepsilon = 0$. Prove that if for all vectors $\mathbf{Q}$ of pressure variables there exists a vector $\mathbf{V}$ of velocity variables such that $\|\mathbf{V}\| = \|\mathbf{Q}\|$ and $\|\mathbf{Q}\|^2 = \mathbf{V}^{\mathrm{t}}\mathbf{G}\mathbf{Q}$, then

$$\|\mathbf{U} - \mathbf{U}^{\varepsilon}\| = O(\varepsilon) \quad \text{and} \quad \|\mathbf{P} - \mathbf{P}^{\varepsilon}\| = O(\varepsilon)$$

   for any positive definite matrix $\mathbf{M}$.

2. Assume that the domain $\Omega = (0,1) \times (0,1)$ is discretized using a uniform mesh of 4×4 Q_1/P_0 elements, numbered by rows from the left to the right and from the bottom to the top. This mesh is used to solve the Stokes problem with the boundary condition $\boldsymbol{u} = \mathbf{0}$ on $\partial\Omega$. If $\mathbf{G}$ is the matrix coming from the pressure gradient, prove that $\mathbf{G}\mathbf{P}_{\mathrm{esp}} = \mathbf{0}$, where $\mathbf{P}_{\mathrm{esp}}^{\mathrm{t}} = (1, -1, 1, -1, -1, 1, -1, 1, ...)$. Which conclusion can be drawn from this fact ?

3. The flow in a porous medium is modeled by the so called Darcy's problem, which consists in finding a velocity $\boldsymbol{u}$ and a pressure p such that

$$\begin{aligned} \sigma \boldsymbol{u} + \nabla p &= \mathbf{0} \quad \text{in } \Omega \\ \nabla \cdot \boldsymbol{u} &= g \quad \text{in } \Omega \\ \boldsymbol{n} \cdot \boldsymbol{u} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

   where σ is the permeability coefficient and g is a given function in $L^2(\Omega)$.

   (a) Write down a variational formulation of the problem taking $H(\mathrm{div}, \Omega)$ as velocity space and $L^2(\Omega)$ as pressure space.

   (b) Show that $\|\boldsymbol{u}\|_{H(\mathrm{div},\Omega)} \leq C\|g\|_{L^2(\Omega)}$ for a certain constant C. Obtain the inf-sup condition that allows to bound the $L^2(\Omega)$ norm of the pressure.

   (c) Obtain the matrix form of the problem after a mixed finite element approximation.

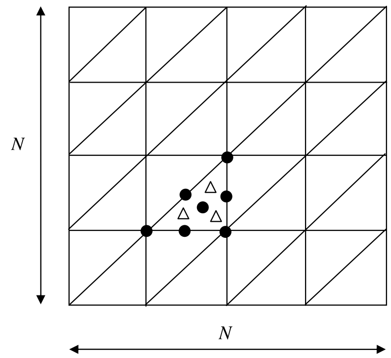
   (d) From the previous equation, obtain an equation for the nodal values of the pressure alone and compare the result with the matrix form of the approximation to the equation

$$-\frac{1}{\sigma}\Delta p = g$$

4. Consider the Stokes problem approximated using the P_2^+/P_1 element in 2D (that is to say, quadratic velocities enriched with a cubic bubble function and piecewise linear, discontinuous pressures).

   (a) Obtain the expression of the element matrix in terms of the velocity and pressure shape functions.

   (b) Suppose that the unit square is discretized using $2N^2$ elements as indicated in the following figure. If n_u is the number of velocity degrees of freedom and n_p the number of pressure degrees of freedom, compute $\lim_{N\to\infty} \frac{n_u}{n_p}$.



5. Consider the Stokes problem with porosity, which consists in finding a velocity $\boldsymbol{u}$ and a pressure p such that

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + \sigma\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \partial\Omega \end{aligned}$$

The bilinear form of the problem discretized using the algebraic subgrid scale stabilization and P_1/P_1 elements is given by

$$\begin{aligned} B([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) = {} & \nu(\nabla\boldsymbol{u}_h, \nabla\boldsymbol{v}_h) + \sigma(\boldsymbol{u}_h, \boldsymbol{u}_h) - (p_h, \nabla\cdot\boldsymbol{v}_h) + (q_h, \nabla\cdot\boldsymbol{u}_h) \\ & + \tau(\sigma\boldsymbol{u}_h + \nabla p_h, -\sigma\boldsymbol{v}_h + \nabla q_h) \end{aligned}$$

where $[\boldsymbol{u}_h, p_h]$ are the discrete velocity and pressure, $[\boldsymbol{v}_h, q_h]$ the corresponding test functions and $\tau = (\nu + 2\sigma h^2)^{-1}h^2$ the stabilization parameter. The mesh size h is considered constant.

(a) Obtain the expression of the element matrix in terms of the shape functions of the linear triangle in 2D (equal interpolation is used for the velocity and the pressure).

(b) Prove that B is coercive in the norm

$$|||[\boldsymbol{u}_h, p_h]|||^2 = \nu\|\nabla\boldsymbol{u}_h\|^2 + \sigma\|\boldsymbol{u}_h\|^2 + \tau\|\nabla p_h\|^2$$

6. Read the following paper: F. Brezzi and K.J. Bathe, A discourse on the stability conditions for mixed finite element formulations, *Computer Methods in Applied Mechanics and Engineering*, vol. 82, 27–57 (1990). Particularize the statement of Proposition 3.2 and Theorem 3.2 to the three field formulation of the Stokes problem, that is to say, using as variables $\boldsymbol{\sigma}'$ (deviatoric stress), $\boldsymbol{u}$ (velocity) and p (pressure).

Homework 4: Navier-Stokes equations

1. One of the possible linealizations of the stationary Navier-Stokes equations is the so called fixed point or Picard's method. At each iteration, the problem to be solved is:

$$\begin{aligned} -\nu\Delta \boldsymbol{u}_i + (\boldsymbol{u}_{i-1}\cdot\nabla)\boldsymbol{u}_i + \nabla p_i &= \boldsymbol{f} \quad \text{in } \Omega \\ \nabla\cdot\boldsymbol{u}_i &= 0 \quad \text{in } \Omega \\ \boldsymbol{u}_i &= \boldsymbol{0} \quad \text{on } \partial\Omega \end{aligned}$$

   where the subscript denotes the iteration counter.

   (a) Obtain the variational formulation of this problem (at the continuous level) and prove that, for each i, there exists a unique solution regardless of the value of $\nu > 0$.

   (b) Obtain an 'a priori' estimate for the norm of $\boldsymbol{u}_i$ and of p_i in terms of $\boldsymbol{f}$ in the appropriate functional space.

   (c) Describe the finite element approximation of this problem using the SUPG formulation and including the gradient of the pressure test function in the stabilization term.

   (d) Obtain an 'a priori' estimate in the appropriate norm of the discrete solution $\boldsymbol{u}_{h,i}$ and $p_{h,i}$ in terms of $\boldsymbol{f}$ without using any kind of inf-sup condition on the interpolation spaces.

2. Some mathematical models that consider the resistance to flow due to the porosity of the medium include a term of the form $\sigma\boldsymbol{u}$, with $\sigma \geq 0$, in the Navier-Stokes equations. The resulting boundary value problem is:

$$\begin{aligned} -\nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \sigma\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \partial\Omega \end{aligned}$$

   Describe the finite element approximation to this problem using Picard's linearization method, equal velocity-pressure interpolation and the algebraic subgrid scale method.

3. When the Navier-Stokes equations are written in a rotating frame of reference two additional terms appear, namely, the Coriolis and the centrifugal forces. The problem to be solved is to find a velocity $\boldsymbol{u}$ and a pressure p such that

$$\begin{aligned} \frac{\partial\boldsymbol{u}}{\partial t} + 2\,\boldsymbol{\omega}\times\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{f} + \boldsymbol{\omega}\times\boldsymbol{x}\times\boldsymbol{x} \\ \nabla\cdot\boldsymbol{u} &= 0 \end{aligned}$$

   where $\boldsymbol{\omega}$ is the angular velocity of the frame of reference and $\boldsymbol{x}$ the position vector.

   (a) Propose a fully implicit second order fractional step method to integrate the problem in time. Is it possible to uncouple the calculation of the velocity components ?

   (b) Obtain the element matrices using resulting after a Galerkin finite element approximation in space using the Taylor Hood P_2/P_1 interpolation in 2D. Indicate which of these matrices are symmetric and which are skew-symmetric. Consider the three steps of the algorithm.

4. Three finite element methods to solve the incompressible Navier-Stokes equations are described next. The problem to be solved is to find a velocity $\boldsymbol{u}$ and a pressure p solution of

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}$$
$$\nabla \cdot \boldsymbol{u} = 0$$

to be solved in the domain Ω with homogeneous Dirichlet boundary conditions and a given initial condition. The spatial discretization is done using the finite element method in all cases. For each method, indicate

(a) Expected temporal accuracy and possible problems that can be encountered when advancing in time.

(b) Possible spatial discretizations, commenting advantages and disadvantages.

(c) Linearized and fully discrete equations of the algorithm, as well as a succinct flow diagram for the computer implementation. If a certain step has several possibilities, choose one.

The methods are:

Method A

- Temporal discretization: Crank-Nicolson scheme for the viscous term and a second order explicit approximation for the convective term.
- Linearization of the convective term: not applicable.
- Stabilization of the convective term: SUPG method.
- Treatment of incompressibility: mixed inf-sup stable velocity-pressure interpolations.

Method B

- Temporal discretization: characteristic-Galerkin method.
- Linearization of the convective term: Picard's fixed point method (along the characteristics).
- Stabilization of the convective term: Not applicable.
- Treatment of incompressibility: mixed inf-sup stable velocity-pressure interpolations.

Method C

- Temporal discretization: second order Gear scheme (BDF2).
- Linearization of the convective term: Newton-Raphson's method.
- Stabilization of the convective term: algebraic subgrid scale stabilization.
- Treatment of incompressibility: algebraic subgrid scale stabilization.

5. Read the paper: S. Badia and R. Codina, Algebraic pressure segregation methods for the incompressible Navier-Stokes equations, *Archives of Computational Methods in Engineering*, to appear. Propose a modification of the time integration of problem (29) so that the resulting scheme is second order accurate in time.