Finite element methods in fluid mechanics

HOMEWORK 1

1. Let $\Omega = (0,1)\times(0,1)$ and consider the problem

$$
\begin{aligned}
\Delta\phi &= 0 && \text{in } \Omega,\\
\phi &= 0 && \text{for } x = 0,1,\ y = 0,\\
\phi &= x(1-x) && \text{for } y = 1.
\end{aligned}
$$

Let us define the basis of functions $\{\sin(m\pi x)\sin(n\pi y)\}$, $m,n = 1,2,\dots$. Obtain the Galerkin approximation to the previous problem in this basis, pointing out which are their particular features in comparison to what would be obtained using a second order finite difference scheme.

2. Let us consider a biquadratic Lagrangian element in 2D (element Q_2) with element domain $\Omega^e = (x_1,x_2)\times(y_1,y_2)$. The side $x = x_2$ lies on the boundary of the computational domain Ω. Obtain the weak form of the problem

$$
\begin{aligned}
-\Delta\phi &= f && \text{in } \Omega,\\
\phi &= 0 && \text{on } \partial\Omega - \{x = x_2\},\\
\frac{\partial\phi}{\partial n} + \alpha\phi &= h && \text{for } x = x_2,
\end{aligned}
$$

and work out the expression of the stiffness matrix of the element indicated.

3. Let us consider the one-dimensional problem

$$
\begin{aligned}
&-\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \lambda u = 1, \qquad 0 < x < 1,\\
&u(0) = u(1) = 0,
\end{aligned}
$$

and assume that the domain $\Omega = (0,1)$ is discretized using

a) 6 linear elements
b) 3 quadratic elements
b) 2 cubic elements

The size of the elements is constant in all the cases.

i) Obtain the analytical solution u of the continuous problem in terms of λ.
ii) Compute the finite element solution u_h in cases a), b) and c).
iii) Compute the L^2 norm of the error $u - u_h$ in cases a), b) and c) (note that the number of degrees of freedom is the same in all these cases).

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HOMEWORK 2

1. Let us consider the one-dimensional convection-diffusion equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} - k\frac{\partial^2 u}{\partial x^2} = f, \quad 0 < x < 1,$$

with $k > 0,\ a > 0$. Let us assume that the interval $[0, 1]$ is discretized using elements of equal size h and that the SUPG method is used for the spatial discretization and the forward Euler scheme to discretize in time, approximating the mass matrix by a diagonal one obtained from a nodal quadrature rule. Prove that the algorithm satisfies the von Neumann stability condition if, and only if,

$$c \leq \frac{\gamma}{1 + \alpha\gamma}$$

where α is the upwind function of the SUPG method, γ is the Péclet number of the partition and $c := a\delta t/h$ is the Courant number, δt being the time step size.

Plot the stability region in the γ-c plane dictated by the previous condition as well as the stability region that would be obtained using the Galerkin method ($\alpha = 0$).

Prove that when $a = 0$ the algorithm is stable if, and only if,

$$\delta t \leq \frac{h^2}{2k}$$

2. Prove that the discontinuous Galerkin method in time applied to the convection-diffusion equation is stable (Hint: select an adequate v_h in Eq. (37) of the CMAME paper *Comparison of some finite element methods for solving the diffusion-convection-reaction equation*).

3. Explain why the Characteristic Galerkin method introduces diffusion along the streamlines when it is applied to the convection-diffusion equation (Hint: Use Eq. (81) of the previous CMAME paper).

4. Let us consider again the one-dimensional equation of exercise 1, and assume now that the standard Galerkin method is used for the spatial discretization. Obtain the matrix of the final linear system when the following time discretizations are used:

a) The Crank-Nicolson method.
b) The second order Gear scheme.
c) The second order Adams-Brashforth scheme.

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HOMEWORK 3

1. Assume that the domain $\Omega = (0,1) \times (0,1)$ is discretized using a uniform mesh of 4×4 Q_1/P_0 elements, numbered by rows from the left to the right and from the bottom to the top. This mesh is used to solve the Stokes problem with the boundary condition $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$. If $\boldsymbol{G}$ is the matrix coming from the pressure gradient, prove that $\boldsymbol{G}\boldsymbol{P}_{\text{esp}} = \boldsymbol{0}$, where $\boldsymbol{P}_{\text{esp}}^{\text{t}} = (1, -1, 1, -1, ...)$. Which conclusion can be drawn from this fact ?

2. One of the possible linealizations of the stationary Navier-Stokes equations is the so called fixed point or Picard's method. At each iteration, the problem to be solved is:

$$\begin{aligned} -\nu\Delta\boldsymbol{u}_i + (\boldsymbol{u}_{i-1}\cdot\nabla)\boldsymbol{u}_i + \nabla p_i = \boldsymbol{f}, &\quad \text{in } \Omega \\ \nabla\cdot\boldsymbol{u}_i = 0, &\quad \text{in } \Omega \\ \boldsymbol{u}_i = \boldsymbol{0}, &\quad \text{on } \partial\Omega \end{aligned}$$

where the subscript denotes the iteration counter.

a) Obtain the variational formulation of this problem (at the continuous level) and prove that, for each i, there exists a unique solution regardless of the value of ν.
b) Obtain an 'a priori' estimate for the norm of $\boldsymbol{u}_i$ and of p_i in terms of $\boldsymbol{f}$ in the appropriate functional space.
c) Describe the finite element approximation of this problem using the SUPG formulation and including the gradient of the pressure test function in the stabilization term.
d) Obtain an 'a priori' estimate in the appropriate norm of the discrete solution $\boldsymbol{u}_{h,i}$ and $p_{h,i}$ in terms of $\boldsymbol{f}$ without using any kind of inf-sup condition on the interpolation spaces.

3. Some mathematical models that consider the resistance to flow due to the porosity of the medium include a term of the form $\alpha\boldsymbol{u}$, with $\alpha > 0$, in the Navier-Stokes equations. The resulting boundary value problem is:

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \alpha\boldsymbol{u} + \nabla p = \boldsymbol{f}, &\quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} = 0, &\quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0}, &\quad \text{on } \partial\Omega. \end{aligned}$$

Describe the finite element approximation to this problem using Picard's linearization method, equal velocity-pressure interpolation and the algebraic subgrid scale method.

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HOMEWORK 4

Choose a bidimensional domain to solve the stationary incompressible Navier-Stokes equations for incompressible fluids with the appropriate boundary conditions. For example:

- A square with the velocity prescribed on the upper wall.
- A U-shaped domain.
- A bifurcation.
- An expansion (step problem).

Take values of the density and the viscosity so that the Reynolds number be small (if you have convergence problems, increase the viscosity).

Undertake ONE of the following studies:

1. Use the penalty method with a discontinuous pressure element (for example the Q_2/P_1). Solve the problem with 4 different values of the penalty parameter ϵ and plot the curve $\|\boldsymbol{G}^{\mathrm{t}}\boldsymbol{U}\|$ vs ϵ.
2. Using any stable method and 3 different viscosity values, plot the logarithm of the residual vs the iteration counter using both the Picard and the Newton-Raphson methods.
3. Compare the maximum and minimum values of the pressure and the stream-function using the following elements (with meshes of approximately the same number of nodes): Q_1/P_0, Q_2/P_1, P_2/P_0 and P_2^+/P_1.