Finite element methods in fluid mechanics

CHAPTER 4:

THE NAVIER-STOKES EQUATIONS

FOR INCOMPRESSIBLE FLUIDS

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CONTENTS

4.1 STATEMENT OF THE PROBLEM

4.1.1 Incompressible Navier-Stokes equations

The (transient) Navier-Stokes equations for an incompressible fluid are:

$$\begin{aligned}
\rho\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right] - 2\nabla\cdot[\mu\boldsymbol{\varepsilon}(\boldsymbol{u})] + \nabla p &= \rho\boldsymbol{f} \\
\nabla\cdot\boldsymbol{u} &= 0 \\
\boldsymbol{\varepsilon}(\boldsymbol{u}) &= \frac{1}{2}\left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{t}}\right]
\end{aligned}$$

to be solved in $\Omega\times(0,T)$. The boundary and initial conditions to be considered are

$$\begin{aligned}
\boldsymbol{u}(\boldsymbol{x},0) &= \boldsymbol{u}_0(\boldsymbol{x}) && \text{in } \Omega \\
\boldsymbol{u}(\boldsymbol{x},t) &= \boldsymbol{g}(\boldsymbol{x},t) && \text{on } \Gamma_D\times(0,T) \\
\boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{x},t) &= \boldsymbol{h}(\boldsymbol{x},t) && \text{on } \Gamma_N\times(0,T)
\end{aligned}$$

where

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}$$

is the Chauchy stress tensor.

4.1.2 Weak form

Let $\boldsymbol{v}(\boldsymbol{x})$ be a vector function of the spatial argument only. Multiplying the momentum equation by it and integrating:

$$\rho \int_\Omega \left[\boldsymbol{v} \cdot \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{v} \cdot (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\right] - 2\int_\Omega \boldsymbol{v} \cdot [\nabla \cdot (\mu \boldsymbol{\varepsilon}(\boldsymbol{u}))] \\ + \int_\Omega \boldsymbol{v} \cdot \nabla p = \int_\Omega \boldsymbol{v} \cdot \rho \boldsymbol{f}$$

Since

$$\int_\Omega \boldsymbol{v} \cdot [\nabla \cdot (\mu \boldsymbol{\varepsilon}(\boldsymbol{u}))] \\ = -\int_\Omega \nabla \boldsymbol{v} : \mu \boldsymbol{\varepsilon}(\boldsymbol{u}) + \int_{\partial\Omega} \boldsymbol{v} \cdot (\boldsymbol{n} \cdot \mu \boldsymbol{\varepsilon}(\boldsymbol{u})) \\ = -\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \mu \boldsymbol{\varepsilon}(\boldsymbol{u}) + \int_{\partial\Omega} \boldsymbol{v} \cdot (\boldsymbol{n} \cdot \mu \boldsymbol{\varepsilon}(\boldsymbol{u}))$$

and

$$\int_\Omega \boldsymbol{v} \cdot \nabla p = -\int_\Omega p \nabla \cdot \boldsymbol{v} + \int_{\partial\Omega} p \boldsymbol{n} \cdot \boldsymbol{v}$$

we can write

$$-2\int_\Omega \boldsymbol{v} \cdot [\nabla \cdot (\mu \boldsymbol{\varepsilon}(\boldsymbol{u}))] + \int_\Omega \boldsymbol{v} \cdot \nabla p \\ = 2\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \mu \boldsymbol{\varepsilon}(\boldsymbol{u}) - \int_\Omega p \nabla \cdot \boldsymbol{v} - \int_{\partial\Omega} \boldsymbol{v} \cdot (\boldsymbol{n} \cdot \boldsymbol{\sigma})$$

If $\boldsymbol{v}$ is such that $\boldsymbol{v} = \boldsymbol{0}$ on Γ_D and we make use of the Neumann boundary condition:

$$\rho \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} + \rho \int_\Omega \boldsymbol{v} \cdot (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + 2\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \mu \boldsymbol{\varepsilon}(\boldsymbol{u}) \\ - \int_\Omega p \nabla \cdot \boldsymbol{v} = \int_\Omega \rho \boldsymbol{f} \cdot \boldsymbol{v} + \int_{\Gamma_N} \boldsymbol{v} \cdot \boldsymbol{h}$$

This equation makes sense if $\boldsymbol{u}$ and $\boldsymbol{v}$ are square integrable in space and have square integrable first derivatives. The function space where they have to belong is $\boldsymbol{H}^1(\Omega)$.

Let us introduce the notation

$$
\begin{aligned}
(\boldsymbol{u}, \boldsymbol{v}) &= \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} && (L^2 \text{ inner product}) \\
\rho \int_\Omega \boldsymbol{v} \cdot (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} &= c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) && (\text{convective term}) \\
2 \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \mu \boldsymbol{\varepsilon}(\boldsymbol{u}) &= a(\boldsymbol{u}, \boldsymbol{v}) && (\text{viscous term}) \\
\int_\Omega p \nabla \cdot \boldsymbol{v} &= b(p, \boldsymbol{v}) \\
\int_\Omega \rho \boldsymbol{f} \cdot \boldsymbol{v} + \int_{\Gamma_N} \boldsymbol{v} \cdot \boldsymbol{h} &= l(\boldsymbol{v})
\end{aligned}
$$

The weak statement can be written as

$$
\rho \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}, \boldsymbol{v}) + c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + a(\boldsymbol{u}, \boldsymbol{v}) - b(p, \boldsymbol{v}) = l(\boldsymbol{v})
$$

with

$$
\begin{aligned}
&\boldsymbol{u}(\cdot, t), \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \\
&\boldsymbol{u}(\cdot, t) = \boldsymbol{g}, \quad \boldsymbol{v} = \boldsymbol{0} \qquad \text{on } \Gamma_D \\
&\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x})
\end{aligned}
$$

On the other hand, if q is a scalar function and we multiply the continuity eqn. by it and integrate,

$$
\int_\Omega q \nabla \cdot \boldsymbol{u} = b(q, \boldsymbol{u}) = 0
$$

This equation makes sense if q is square integrable, i.e., $q \in L^2(\Omega)$.

4.1.3 Other forms of the viscous and convective terms

If $\nabla \cdot \boldsymbol{u} = 0$ and μ is constant, it is easy to see that

$$2\nabla \cdot [\mu \boldsymbol{\varepsilon}(\boldsymbol{u})] = \mu \Delta \boldsymbol{u}$$

In this case, $a(\boldsymbol{u}, \boldsymbol{v})$ can be replaced by

$$a(\boldsymbol{u}, \boldsymbol{v}) = \mu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v}$$

but the Neumann boundary condition must be changed.

Also, the convective term can be written as:

$$\begin{aligned}(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} &= (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \boldsymbol{u}(\nabla \cdot \boldsymbol{u}) = \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) \\ &= (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \frac{1}{2}\boldsymbol{u}(\nabla \cdot \boldsymbol{u})\end{aligned}$$

The first expression is the so called conservative form. The second verifies (skew-symmetry):

$$c(\boldsymbol{u}, \boldsymbol{u}_1, \boldsymbol{u}_2) = -c(\boldsymbol{u}, \boldsymbol{u}_2, \boldsymbol{u}_1)$$

4.1.4 Finite element discretization

The transient problem is: find $\boldsymbol{u}(\cdot, t) \in \boldsymbol{H}^1(\Omega)$, with $\boldsymbol{u} = \boldsymbol{g}$ on Γ_D, $\boldsymbol{u} = \boldsymbol{u}^0$ for $t = 0$ and $p(\cdot, t) \in L^2(\Omega)$ such that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}, \boldsymbol{v}) + c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + a(\boldsymbol{u}, \boldsymbol{v}) - b(p, \boldsymbol{v}) &= l(\boldsymbol{v}) \\ b(q, \boldsymbol{u}) &= 0 \end{aligned}$$

for all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{v} = \boldsymbol{0}$ on Γ_D and for all $q \in L^2(\Omega)$.

The (Galerkin) finite element (FE) approximation consists of replacing the spaces $\boldsymbol{H}^1(\Omega)$ and $L^2(\Omega)$ by finite dimensional subspaces $\boldsymbol{V}_h$ and Q_h. The discrete version of the problem is: find $\boldsymbol{u}_h(\cdot, t) \in \boldsymbol{V}_h$, $\boldsymbol{u}_h = \boldsymbol{g}$ on Γ_D, $\boldsymbol{u}_h = \boldsymbol{u}_h^0$ for $t = 0$ and $p_h(\cdot, t) \in Q_h$ such that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}_h, \boldsymbol{v}_h) + c(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) + a(\boldsymbol{u}_h, \boldsymbol{v}_h) - b(p_h, \boldsymbol{v}_h) &= l(\boldsymbol{v}_h) \\ b(q_h, \boldsymbol{u}_h) &= 0 \end{aligned}$$

for all $\boldsymbol{v}_h \in V_h$ such that $\boldsymbol{v}_h = \boldsymbol{0}$ on Γ_D and for all $q_h \in Q_h$.

If we express $\boldsymbol{u}_h$ and p_h in terms of their nodal values $\boldsymbol{U}$ and $\boldsymbol{P}$, the discrete problem can be recast in matrix form as

$$\begin{aligned} \boldsymbol{M}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{U} + \boldsymbol{K}_c(\boldsymbol{U})\boldsymbol{U} + \boldsymbol{K}_v\boldsymbol{U} + \boldsymbol{G}\boldsymbol{P} &= \boldsymbol{F} \\ \boldsymbol{G}^{\mathrm{t}}\boldsymbol{U} &= \boldsymbol{0} \end{aligned}$$

4.1.5. Numerical problems

The finite element approximation of the incompressible Navier-Stokes equations has several numerical difficulties:

1. **Nonlinearity**. This comes from $c(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h)$ and leads to complicated physical behaviour and convergence problems.

2. **Velocity-pressure interpolation**. Not any choice of FE spaces $\boldsymbol{V}_h$ and Q_h leads to a well posed problem (solvable and numerically stable).

3. **Convection-dominated flows**. When the convective term dominates the viscous one numerical oscillations may appear. This also happens for scalar equations of convection-diffusion type.

4. **Time integration**. The continuous Navier-Stokes equations have a very complicated dynamical behaviour that is difficult (even impossible) to reproduce numerically.

4.5 SINGLE STEP AND MULTI-STEP SCHEMES FOR THE TIME DISCRETIZATION

- **Introduction**

For the methods to be described here:

space discret. + time discret. = time discret. + space discret.

Let us consider the first option for the problem:

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad & \text{in } \Omega, t > 0 \\ \nabla \cdot \boldsymbol{u} = 0 \qquad & \text{in } \Omega, t > 0 \\ \boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{0} \qquad & \text{on } \partial\Omega, t > 0 \\ \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}) \qquad & \text{in } \Omega \end{aligned}$$

The semi-discrete problem is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}_h, \boldsymbol{v}_h) + c(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) + a(\boldsymbol{u}_h, \boldsymbol{v}_h) - b(p_h, \boldsymbol{v}_h) &= l(\boldsymbol{v}_h) \\ b(q_h, \boldsymbol{u}_h) &= 0 \end{aligned}$$

- **Generalized trapezoidal rule**

$$\begin{aligned} &\frac{1}{\Delta t}(\boldsymbol{u}^n - \boldsymbol{u}^{n-1}, \boldsymbol{v}) \\ &\qquad + c(\boldsymbol{u}^n_\theta, \boldsymbol{u}^n_\theta, \boldsymbol{v}) + a(\boldsymbol{u}^n_\theta, \boldsymbol{v}) - b(p^n_\theta, \boldsymbol{v}) = l(\boldsymbol{v}) \\ &b(q, \boldsymbol{u}^n) = 0 \end{aligned}$$

where

$$\boldsymbol{u}_\theta^n := \theta \boldsymbol{u}^n + (1-\theta)\boldsymbol{u}^{n-1}, \quad \theta \in [0,1]$$

(subscript h has been dropped).

1. It can be shown that if

$$c_\theta^n(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) := \theta c(\boldsymbol{u}^n,\boldsymbol{u}^n,\boldsymbol{v}) + (1-\theta)c(\boldsymbol{u}^{n-1},\boldsymbol{u}^{n-1},\boldsymbol{v})$$

then

$$|c_\theta^n(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) - c(\boldsymbol{u}_\theta^n,\boldsymbol{u}_\theta^n,\boldsymbol{v})| = O(\Delta t^2)$$

2. $\boldsymbol{u}^n - \boldsymbol{u}^{n-1} = (\boldsymbol{u}_\theta^n - \boldsymbol{u}^{n-1})/\theta$, and therefore $\boldsymbol{u}_\theta^n$ can be taken as unknown. That makes the calculations cheaper.

3. It is unnecessary to recover the pressure p^n from p_θ^n. In any case, the pressure must be taken as unknown in order to have a solvable problem.

4. Cases of interest are:
$\theta = 1/2$: Crank-Nicolson scheme
$\theta = 1$: Backward Euler method.

5. Heywood & Rannacher showed that for $\theta = 1/2$ the scheme is second order accurate in velocities and first order accurate in pressures, that is,

$$\begin{aligned} \|\boldsymbol{u}^n - \boldsymbol{u}(t^n)\|_0 &= O(\Delta t^2) \\ \|p^n - p(t^n)\|_0 &= O(\Delta t) \end{aligned}$$

However, these estimates fail as $t \to 0^+$. It is recommended to take $\theta = 1$ for the first few time steps (damping of highly oscillating harmonics).

- **Some multi-step schemes**

  - **Method 1**

$$
\begin{aligned}
&\frac{1}{\Delta t}(\boldsymbol{u}^n - \boldsymbol{u}^{n-1}, \boldsymbol{v}) \\
&\quad + \frac{1}{2}\left[3c(\boldsymbol{u}^{n-1}, \boldsymbol{u}^{n-1}, \boldsymbol{v}) - c(\boldsymbol{u}^{n-2}, \boldsymbol{u}^{n-2}, \boldsymbol{v})\right] \\
&\quad + \frac{1}{2}\left[a(\boldsymbol{u}^n, \boldsymbol{v}) + a(\boldsymbol{u}^{n-1}, \boldsymbol{v})\right] - b(p, \boldsymbol{v}) = l(\boldsymbol{v}) \\
&b(q, \boldsymbol{u}^n) = 0
\end{aligned}
$$

1. The matrix of the algebraic system to be solved at each time step is always the same. It has the structure

$$
\begin{pmatrix} \boldsymbol{A} & \boldsymbol{G} \\ \boldsymbol{G}^{\mathrm{t}} & \boldsymbol{0} \end{pmatrix}
$$

   where

$$
\boldsymbol{A} = \frac{1}{\Delta t}\boldsymbol{M} + \frac{1}{2}\boldsymbol{K}
$$

2. Since the convective term is treated explicitly, there is a stability limit for this scheme, i.e., it is conditionally stable.

3. The scheme is second order accurate in velocities, i.e., the error is $O(\Delta t^2)$.

4. It is a very popular method, especially in the finite difference literarure. It is often referred to as the ABCN (Adams-Bashforth/Crank-Nicolson) method.

- **Method 2**

$$\frac{1}{\Delta t}\left(\frac{1}{2}(3\boldsymbol{u}^n - 4\boldsymbol{u}^{n-1} + \boldsymbol{u}^{n-2}), \boldsymbol{v}\right) \\ + c(\boldsymbol{u}^n, \boldsymbol{u}^n, \boldsymbol{v}) + a(\boldsymbol{u}^n, \boldsymbol{v}) - b(p, \boldsymbol{v}) = l(\boldsymbol{v})$$
$$b(q, \boldsymbol{u}^n) = 0$$

1. Fully implicit scheme, unconditionally stable.

2. Second order accurate.

3. It is more robust than the Crank-Nicolson method. However, $\boldsymbol{u}^{n-2}$ must be stored.

4. Useful for complex flows (nonlinear fluids, coupled problems).

- **Method 3**

$$\frac{1}{\Delta t}\left(\frac{1}{2}(3\boldsymbol{u}^n - 4\boldsymbol{u}^{n-1} + \boldsymbol{u}^{n-2}), \boldsymbol{v}\right) \\ + 2c(\boldsymbol{u}^{n-1}, \boldsymbol{u}^{n-1}, \boldsymbol{v}) - c(\boldsymbol{u}^{n-2}, \boldsymbol{u}^{n-2}, \boldsymbol{v}) \\ + a(\boldsymbol{u}^n, \boldsymbol{v}) - b(p, \boldsymbol{v}) = l(\boldsymbol{v})$$
$$b(q, \boldsymbol{u}^n) = 0$$

1. Similar to method 1, although more diffusive.

2. Since convection is treated explicitly, it is conditionally stable.

3. It is a second order accurate scheme.

The finite element interpolation in time must be discontinuous. Otherwise the soltion in time would be coupled.

Let

$$
\begin{aligned}
&0 = t_0 < t_1 < \ldots < t_N = T \qquad \text{(partition of } [0,T]) \\
&I_n = (t_n, t_{n+1}) \\
&Q_n = \Omega \times I_n \\
&Q_n^e = \Omega^e \times I_n
\end{aligned}
$$

Integrating the Navier-Stokes equations multiplied by the test functions over Q_n rather than over Ω and considering the test function $\boldsymbol{v}_h$ time dependent:

$$
\begin{aligned}
\int_{I_n} \left(\frac{\partial \boldsymbol{u}_h}{\partial t}, \boldsymbol{v}_h \right) &+ \int_{I_n} c(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) \\
&+ \int_{I_n} a(\boldsymbol{u}_h, \boldsymbol{v}_h) - \int_{I_n} b(p_h, \boldsymbol{v}_h) = \int_{I_n} l(\boldsymbol{v}_h)
\end{aligned}
$$

But

$$
\begin{aligned}
&\int_{I_n} \left(\frac{\partial \boldsymbol{u}_h}{\partial t}, \boldsymbol{v}_h \right) = \int_{I_n} \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{u}_h, \boldsymbol{v}_h) - \int_{I_n} \left(\boldsymbol{u}_h, \frac{\partial \boldsymbol{v}_h}{\partial t} \right) \\
&= \left(\boldsymbol{u}_h(t_{n+1}^-), \boldsymbol{v}_h(t_{n+1}^-) \right) - \left(\boldsymbol{u}_h(t_n^+), \boldsymbol{v}_h(t_n^+) \right) - \int_{I_n} \left(\boldsymbol{u}_h, \frac{\partial \boldsymbol{v}_h}{\partial t} \right)
\end{aligned}
$$

Prescribing weakly the condition $\boldsymbol{u}_h(t_n^+) = \boldsymbol{u}_h(t_n^-)$:

$$
\begin{aligned}
\int_{I_n} \left(\frac{\partial \boldsymbol{u}_h}{\partial t}, \boldsymbol{v}_h \right) &= - \int_{I_n} \left(\boldsymbol{u}_h, \frac{\partial \boldsymbol{v}_h}{\partial t} \right) \\
&= \left(\boldsymbol{u}_h(t_{n+1}^-), \boldsymbol{v}_h(t_{n+1}^-) \right) - \left(\boldsymbol{u}_h(t_n^-), \boldsymbol{v}_h(t_n^+) \right)
\end{aligned}
$$

Remark: for piecewise constant interpolations

$$\int_{I_n} \left(\frac{\partial \boldsymbol{u}_h}{\partial t}, \boldsymbol{v}_h \right) = \big(\boldsymbol{u}_h(t_{n+1}^-), \boldsymbol{v}_h(t_{n+1}^-) \big) - \big(\boldsymbol{u}_h(t_n^-), \boldsymbol{v}_h(t_n^+) \big)$$
$$= (\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n, \boldsymbol{v}_h^{n+1})$$

The extension of the GLS formulation using space-time finite element interpolations leads to:

– Momentum equations:

$$\begin{aligned} 0 = &- \int_{I_n} \left(\boldsymbol{u}_h, \frac{\partial \boldsymbol{v}_h}{\partial t} \right) \\ &+ \big(\boldsymbol{u}_h(t_{n+1}^-), \boldsymbol{v}_h(t_{n+1}^-) \big) - \big(\boldsymbol{u}_h(t_n^-), \boldsymbol{v}_h(t_n^+) \big) \\ &+ \int_{I_n} [c(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) + a(\boldsymbol{u}_h, \boldsymbol{v}_h) - b(p_h, \boldsymbol{v}_h) - l(\boldsymbol{v}_h)] \\ &+ \sum_e \int_{I_n} \int_{\Omega^e} \tau_1 \left[\frac{\partial \boldsymbol{v}_h}{\partial t} + (\boldsymbol{u}_h \cdot \nabla) \boldsymbol{v}_h - \nu \Delta \boldsymbol{v}_h \right] \\ &\quad \cdot \left[\frac{\partial \boldsymbol{u}_h}{\partial t} + (\boldsymbol{u}_h \cdot \nabla) \boldsymbol{u}_h - \nu \Delta \boldsymbol{u}_h + \nabla p_h - \boldsymbol{f} \right] \\ &+ \sum_e \int_{I_n} \int_{\Omega^e} \tau_2 (\nabla \cdot \boldsymbol{v}_h)(\nabla \cdot \boldsymbol{u}_h) \end{aligned}$$

– Continuity equation:

$$\begin{aligned} 0 = \int_{I_n} b(q_h, \boldsymbol{u}_h) &+ \sum_e \int_{I_n} \int_{\Omega^e} \tau_1 (\nabla q_h) \\ &\cdot \left[\frac{\partial \boldsymbol{u}_h}{\partial t} + (\boldsymbol{u}_h \cdot \nabla) \boldsymbol{u}_h - \nu \Delta \boldsymbol{u}_h + \nabla p_h - \boldsymbol{f} \right] \end{aligned}$$

4.7 FRACTIONAL STEP METHODS

- **Classical method**
  (Chorin, 1968; Temam, 1969)

  Given a vector field $\boldsymbol{w}$, there exists a unique pair $\boldsymbol{v}, \phi$ such that

$$\boldsymbol{w} = \boldsymbol{v} + \nabla\phi \qquad \& \qquad \nabla \cdot \boldsymbol{v} = 0$$

  with $\boldsymbol{n} \cdot \boldsymbol{v} = 0$ on the boundary. We can write

$$\boldsymbol{w} = P_{\text{div}}(\boldsymbol{w}) + P_{\text{grad}}(\boldsymbol{w})$$

  The Navier-Stokes equations can be written as

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + \nabla p &= -(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nu\Delta\boldsymbol{u} + \boldsymbol{f} \\ \nabla \cdot \boldsymbol{u} &= 0 \end{aligned}$$

  Since $\nabla \cdot (\partial\boldsymbol{u}/\partial t) = 0$, we have that

$$\frac{\partial \boldsymbol{u}}{\partial t} = P_{\text{div}}\left[\nu\Delta\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \boldsymbol{f}\right]$$

  The idea of the fractional step method is:

  1. Solve
$$\frac{\partial \tilde{\boldsymbol{u}}}{\partial t} = \nu\Delta\tilde{\boldsymbol{u}} - (\tilde{\boldsymbol{u}} \cdot \nabla)\tilde{\boldsymbol{u}} + \boldsymbol{f}$$

  2. Project
$$\frac{\partial \boldsymbol{u}}{\partial t} = P_{\text{div}}\left(\frac{\partial \tilde{\boldsymbol{u}}}{\partial t}\right)$$

that is, find ϕ and $\frac{\partial \boldsymbol{u}}{\partial t}$ such that

$$\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} + \nabla\phi &= \frac{\partial \tilde{\boldsymbol{u}}}{\partial t} \\
\nabla \cdot \boldsymbol{u} &= 0
\end{aligned}$$

At the numerical level, there will be an error due to the uncoupling of the two steps. The numerical version of the previous equations is:

$$\begin{aligned}
&\frac{1}{\Delta t}(\tilde{\boldsymbol{u}}^n - \boldsymbol{u}^{n-1}) = \nu\Delta\tilde{\boldsymbol{u}}^n - (\tilde{\boldsymbol{u}}^n \cdot \nabla)\tilde{\boldsymbol{u}}^n + \boldsymbol{f} \\
&\frac{1}{\Delta t}(\boldsymbol{u}^n - \boldsymbol{u}^{n-1}) + \nabla\phi = \frac{1}{\Delta t}(\tilde{\boldsymbol{u}}^n - \boldsymbol{u}^{n-1}) \\
&\qquad \iff \frac{1}{\Delta t}(\boldsymbol{u}^n - \tilde{\boldsymbol{u}}^n) + \nabla\phi = \boldsymbol{0} \\
&\nabla \cdot \boldsymbol{u}^n = 0
\end{aligned}$$

- **Other methods**

Different schemes are obtained depending on the time discretization. Some popular options are the following:

1. Explicit viscous and convective terms

$$\begin{aligned}
&\frac{1}{\Delta t}(\tilde{\boldsymbol{u}}^n - \boldsymbol{u}^{n-1}) = \nu\Delta\boldsymbol{u}^{n-1} - (\boldsymbol{u}^{n-1} \cdot \nabla)\boldsymbol{u}^{n-1} + \boldsymbol{f} \\
&\frac{1}{\Delta t}(\boldsymbol{u}^n - \tilde{\boldsymbol{u}}^n) + \nabla\phi = \boldsymbol{0} \\
&\nabla \cdot \boldsymbol{u}^n = 0
\end{aligned}$$

In this case, $\phi = p^n$.

2. ABCN based scheme

$$
\begin{aligned}
&\frac{1}{\Delta t}(\tilde{\boldsymbol{u}}^n - \boldsymbol{u}^{n-1}) = \frac{1}{2}\left[\nu\Delta\tilde{\boldsymbol{u}}^n + \nu\Delta\boldsymbol{u}^{n-1}\right] \\
&\quad - \frac{1}{2}\left[3(\boldsymbol{u}^{n-1}\cdot\nabla)\boldsymbol{u}^{n-1} - (\boldsymbol{u}^{n-2}\cdot\nabla)\boldsymbol{u}^{n-2}\right] + \boldsymbol{f} \\
&\frac{1}{\Delta t}(\boldsymbol{u}^n - \tilde{\boldsymbol{u}}^n) + \nabla\phi = \boldsymbol{0} \\
&\nabla\cdot\boldsymbol{u}^n = 0
\end{aligned}
$$

- **Pressure Poisson equation**

  In the projection step:

$$
\begin{aligned}
&\frac{1}{\Delta t}(\boldsymbol{u}^n - \tilde{\boldsymbol{u}}^n) + \nabla\phi = \boldsymbol{0} \\
&\nabla\cdot\boldsymbol{u}^n = 0
\end{aligned}
$$

  Taking the divergence of the first equation and using the second yields:

$$
\Delta\phi = \frac{1}{\Delta t}\nabla\cdot\tilde{\boldsymbol{u}}^n
$$

  There are 2 advantatges of doing this:

1. The calculation of $\boldsymbol{u}^n$ and ϕ is uncoupled.

2. At the discrete level:

$$\begin{aligned}\frac{1}{\Delta t}\boldsymbol{M}\left(\boldsymbol{U}^n - \tilde{\boldsymbol{U}}^n\right) + \boldsymbol{G}\boldsymbol{\Phi} &= \boldsymbol{0}\\ -\boldsymbol{G}^{\mathrm{t}}\boldsymbol{U}^n &= \boldsymbol{0}\\ \Delta t \boldsymbol{L}\boldsymbol{\Phi} - \boldsymbol{G}^{\mathrm{t}}\tilde{\boldsymbol{U}}^n &= \boldsymbol{0}\end{aligned}$$

Eliminating $\tilde{\boldsymbol{U}}^n$ from the first equation and inserting it into the last one yields:

$$-\boldsymbol{G}^{\mathrm{t}}\boldsymbol{U}^n + \Delta t\left(\boldsymbol{L} - \boldsymbol{G}^{\mathrm{t}}\boldsymbol{M}^{-1}\boldsymbol{G}\right)\boldsymbol{\Phi} = \boldsymbol{0}$$

It is possible to use equal interpolation for $\boldsymbol{u}$ and ϕ.

- **Boundary conditions**

Suppose that $\boldsymbol{u} = \boldsymbol{g}$ on $\partial\Omega$ is prescribed for the original continuous problem. The classical strategy for fractional step methods is to prescribe

$$\begin{aligned}\tilde{\boldsymbol{u}} &= \boldsymbol{g} && \text{on } \partial\Omega\\ \boldsymbol{n}\cdot\boldsymbol{u} &= \boldsymbol{n}\cdot\boldsymbol{g} && \text{on } \partial\Omega\\ \frac{\partial\phi}{\partial n} &= 0 && \text{on } \partial\Omega\end{aligned}$$

However, experience shows that it is possible to prescribe numerically $\boldsymbol{u} = \boldsymbol{g}$ on $\partial\Omega$. The explanation to this fact and the study of methods that allow this is a current research problem.