Finite element methods in fluid mechanics

CHAPTER 3:

THE STOKES PROBLEM

Ramon Codina

CONTENTS

3.7 STABILIZATION METHODS

3.7.1. The Galerkin/least-squares (GLS) method

(Hughes & Franca, 1986)

Let us write the Stokes problem as follows:

$$\mathcal{L}(\boldsymbol{u},p) = \begin{pmatrix} \mathcal{L}_1(\boldsymbol{u},p) \\ \mathcal{L}_2(\boldsymbol{u},p) \end{pmatrix} := \begin{pmatrix} -\nu\Delta\boldsymbol{u} + \nabla p \\ \nabla\cdot\boldsymbol{u} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{pmatrix} =: \mathcal{F}$$

where $\nu = \mu/\rho$ is the kinematic viscosity.

The GLS method consists of adding to the terms of the Galerkin approach a least squares form of the residual $\mathcal{L}(\boldsymbol{u},p) - \mathcal{F}$ evaluated on each element. This leads to:

$$\boldsymbol{0} = \text{Galerkin terms} + \sum_e \int_{\Omega^e} (\mathcal{L}(\boldsymbol{v}_h,q_h))^{\mathrm{t}} \cdot [\boldsymbol{\tau}(\mathcal{L}(\boldsymbol{u}_h,p_h) - \mathcal{F})]$$

where $\boldsymbol{\tau}$ is a matrix. If we take it as $\boldsymbol{\tau} = \mathrm{diag}(\boldsymbol{\tau}_1, \tau_2)$, with $\boldsymbol{\tau}_1 = \tau_1\boldsymbol{I}$, we have that

$$\begin{aligned}
&(\mathcal{L}(\boldsymbol{v}_h,q_h))^{\mathrm{t}} \cdot [\boldsymbol{\tau}(\mathcal{L}(\boldsymbol{u}_h,p_h) - \mathcal{F})] \\
&\quad = \mathcal{L}_1(\boldsymbol{v}_h,q_h)\boldsymbol{\tau}_1(\mathcal{L}_1(\boldsymbol{u}_h,p_h) - \boldsymbol{f}) + \mathcal{L}_2(\boldsymbol{v}_h,q_h)\tau_2\mathcal{L}_2(\boldsymbol{u}_h,p_h) \\
&\quad = \tau_1(-\nu\Delta\boldsymbol{v}_h + \nabla q_h)\cdot(-\nu\Delta\boldsymbol{u}_h + \nabla p_h - \boldsymbol{f}) \\
&\quad + \tau_2(\nabla\cdot\boldsymbol{v}_h)(\nabla\cdot\boldsymbol{u}_h)
\end{aligned}$$

The contribution to the first equation of the original Galerkin approach is obtained with $q_h = 0$ and $\boldsymbol{v}_h \neq \boldsymbol{0}$, whereas the contribution to the second is found for $q_h \neq 0$ and $\boldsymbol{v}_h = \boldsymbol{0}$.

The final scheme is:

$$
\begin{aligned}
&a(\boldsymbol{u}_h, \boldsymbol{v}_h) - b(p_h, \boldsymbol{v}_h) - l(\boldsymbol{v}_h) \\
&\quad + \sum_e \int_{\Omega^e} \tau_1(-\nu\Delta\boldsymbol{v}_h) \cdot (-\nu\Delta\boldsymbol{u}_h + \nabla p_h - \boldsymbol{f}) \\
&\quad + \sum_e \int_{\Omega^e} \tau_2(\nabla\cdot\boldsymbol{v}_h)(\nabla\cdot\boldsymbol{u}_h) = 0 \\
&b(q_h, \boldsymbol{u}_h) + \sum_e \int_{\Omega^e} \tau_1(\nabla q_h) \cdot (-\nu\Delta\boldsymbol{u}_h + \nabla p_h - \boldsymbol{f}) = 0
\end{aligned}
$$

The parameter τ_1 is taken as $\alpha h^2/4\nu$, where α is a constant ($\alpha \approx 0.3$ for linear elements, ≈ 0.1 for quadratics). For the Stokes problem $\tau_2 = 0$.

3.7.2. Virtual bubbles and the GLS method for the P_1 element

(Bank & Welfert, 1990; Baiocchi, Brezzi & Franca, 1993)

If $\tau_2 = 0$ and $\Delta\boldsymbol{u}_h = \Delta\boldsymbol{v}_h = \mathbf{0}$ within each element, the GLS method reduces to

$$
\begin{aligned}
&a(\boldsymbol{u}_h, \boldsymbol{v}_h) - b(p_h, \boldsymbol{v}_h) - l(\boldsymbol{v}_h) = 0 \\
&b(q_h, \boldsymbol{u}_h) + \sum_e \int_{\Omega^e} \tau_1 \nabla q_h \cdot (\nabla p_h - \boldsymbol{f}) = 0
\end{aligned}
$$

Consider now the P_1^+/P_1 element and the standard Galerkin method. Within each element:

$$
\boldsymbol{u}_h^b(\boldsymbol{x}) = \boldsymbol{u}_h(\boldsymbol{x}) + \sum_j \boldsymbol{B}_j(\boldsymbol{x}) u_j^b
$$

where $\boldsymbol{B}_j(\boldsymbol{x})$ is the vector that has the bubble function $B(\boldsymbol{x})$ in the jth component and zero the rest. Using the first equation to be solved we have that, for each element,

$$
\begin{aligned}
a(\boldsymbol{u}_h^b, \boldsymbol{B}_k) - b(p_h, \boldsymbol{B}_k) &= l(\boldsymbol{B}_k) \\
u_k^b a(\boldsymbol{B}_k, \boldsymbol{B}_k) &= l(\boldsymbol{B}_k) + b(p_h, \boldsymbol{B}_k) - a(\boldsymbol{u}_h, \boldsymbol{B}_k) \\
&= \int_{\Omega^e} \boldsymbol{B}_k \cdot (\boldsymbol{f} - \nabla p_h + \nu \Delta \boldsymbol{u}_h) \\
&+ \int_{\partial\Omega^e} \boldsymbol{B}_k \cdot (p\boldsymbol{n} - \nu \boldsymbol{n} \cdot \nabla \boldsymbol{u}_h) \\
&= \int_{\Omega^e} \boldsymbol{B}_k \cdot (\boldsymbol{f} - \nabla p_h), \\
u_k^b &= \frac{1}{a(\boldsymbol{B}_k, \boldsymbol{B}_k)} \int_{\Omega^e} \boldsymbol{B}_k \cdot (\boldsymbol{f} - \nabla p_h)
\end{aligned}
$$

This expression holds for each element. For the second equation to be solved:

$$
\begin{aligned}
& b(q_h, \boldsymbol{u}_h^b) = b(q_h, \boldsymbol{u}_h) + \sum_e \sum_j u_j^b b(q_h, \boldsymbol{B}_j)|_e = b(q_h, \boldsymbol{u}_h) \\
& + \sum_e \sum_j \frac{1}{a(\boldsymbol{B}_j, \boldsymbol{B}_j)} \left[\int_{\Omega^e} \boldsymbol{B}_j \cdot (\boldsymbol{f} - \nabla p_h) \right] \left[\int_{\Omega^e} \boldsymbol{B}_j \cdot \nabla q_h \right] \\
& = b(q_h, \boldsymbol{u}_h) + \sum_e \int_{\Omega^e} \nabla q_h \cdot \hat{\boldsymbol{\tau}} \cdot (\boldsymbol{f} - \nabla p_h) = 0
\end{aligned}
$$

where $\hat{\boldsymbol{\tau}}$ behaves like h^2. Therefore, the use of the P_1^+/P_1 element is very similar to the use of the P_1/P_1 element and the GLS formulation, the only difference being the definition of the stabilization matrix ($\hat{\boldsymbol{\tau}}$ in one case, $\tau_1 \boldsymbol{I}$ in the other).

3.7.3. Inclusion of jumps for discontinuous pressures

(Franca & Stenberg, 1991; Silvester & Kechkar, 1990)

The Q_1/P_0 element (for example) does not satisfy the BB condition:

- There are spurious pressure modes
- $\beta = \beta(h)$ in the inf-sup condition. This happens at least for regular meshes.

Let us consider a partition of Ω into patches of 2×2 Q_1/P_0 elements (in 2D) and let Γ_p be the collection of interior edges of patch number p. The Silvester-Kechkar stabilization consists of replacing the incompressibility condition $b(q_h, \boldsymbol{u}_h) = 0$ by

$$b(q_h, \boldsymbol{u}_h) + \sum_p \alpha h \int_{\Gamma_p} [\![q_h]\!][\![p_h]\!] = 0$$

where $[\![\cdot]\!]$ is the jump operator.

The GLS method combined with the second term in this equation converges if at least one of the following conditions is verified (Franca & Stenberg):

- Pressures are continuous (the jump disappears).
- $\alpha > 0$.
- The order of velocity interpolation is greater of equal to the space dimension.

3.7.4. Other stabilization methods

- Douglas & Wang, 1988

$$
\begin{aligned}
0 = {} & \text{Galerkin terms} \\
& + \sum_e \int_{\Omega^e} \tau(+\nu\Delta \boldsymbol{v}_h + \nabla q_h) \cdot (-\nu\Delta \boldsymbol{u}_h + \nabla p_h - \boldsymbol{f})
\end{aligned}
$$

This method motivated the Subgrid Scale Methods (Hughes, 1995).

- Brezzi & Douglas, 1988

$$
\begin{aligned}
0 = {} & b(q_h, \boldsymbol{u}_h) \\
& + \sum_e \tau \left[\int_{\Omega^e} \nabla q_h \cdot (\nabla p_h - \boldsymbol{f}) - \int_{\partial\Omega^e \cap \partial\Omega} q_h \nu \boldsymbol{n} \cdot \Delta \boldsymbol{u}_h \right]
\end{aligned}
$$

- Stabilization based on the projection of the pressure gradient (Codina & Blasco, 1995)

$$
\begin{aligned}
0 = {} & b(q_h, \boldsymbol{u}_h) \\
& + \sum_e \tau \int_{\Omega^e} \nabla q_h \cdot (\nabla p_h - \boldsymbol{y}_h)
\end{aligned}
$$

where $\boldsymbol{y}_h$ is the projection of the pressure gradient onto the velocity space, that is, it is the solution of

$$(\boldsymbol{y}_h, \boldsymbol{v}_h) = (\nabla p_h, \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h$$

In matrix form

$$\boldsymbol{M}\boldsymbol{Y} = \boldsymbol{G}\boldsymbol{P}$$

The matrix form of the continuity equation is now

$$\boldsymbol{G}^{\mathrm{t}}\boldsymbol{U} + \tau(\boldsymbol{L} - \boldsymbol{G}^{\mathrm{t}}\boldsymbol{M}^{-1}\boldsymbol{G})\boldsymbol{P} = \boldsymbol{0}$$

The matrix $\boldsymbol{L} - \boldsymbol{G}^{\mathrm{t}}\boldsymbol{M}^{-1}\boldsymbol{G}$ may be thought of as the difference between two discrete Laplacian operators computed in a different manner.