On the design of algebraic fractional step methods for viscoelastic incompressible flows

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Abstract Classical fractional step methods for viscous incompressible flows aim to uncouple the calculation of the velocity and the pressure. In the case of viscoelastic flows, a new variable appears, namely, a stress, which has an elastic and a viscous contribution. The purpose of this article is to present two families of fractional step methods for the time integration of this type of flows whose objective is to permit the uncoupled calculation of velocities, stresses and pressure, both families designed at the algebraic level. This means that the splitting of the equations is introduced once the spatial and the temporal discretizations have been performed. The first family is based on the extrapolation of the pressure and the stress in order to predict a velocity, then the calculation of a new stress, the pressure and then a correction to render the scheme stable. The second family has a discrete pressure Poisson equation as starting point; in this equation, velocities and stresses are extrapolated to compute a pressure, and from this pressure stresses and velocities can then be computed. This work presents an overview of methods previously proposed in our group, as well as some new schemes in the case of the second family.

1 Introduction

From the computational point of view, the key aspect in the complexity of the approximation of the incompressible Navier-Stokes equations is the coupling between the velocity and the pressure degrees of freedom. Apart from the difficulties in choosing a spatial interpolation for both variables that renders the final scheme stable, once the discrete problem needs to be solved one has to face with unknowns with different behavior from the standpoint of algebraic solvers. In incompressible flows, it is usually the pressure the variable that drives the whole iterative behavior

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of linear solvers, and it is certainly a waste of effort that the velocity be dragged in this process as a coupled variable. Moreover, special solvers with special preconditioners could be used for the pressure if it could solved in an uncoupled manner.

The interest in fractional step methods in incompressible flows, also known as splitting methods, started with the works of Chorin [6] and Temam [16], who attempted the uncoupling of velocity and pressure at the continuous level, segregating the calculation of the pressure from the momentum equation and then understanding the final velocity correction as a projection onto the space of solenoidal fields. Since then, many works have been devoted to a proper understanding of the original schemes, their numerical analysis, their extension to order higher than one in time and to the design of adequate boundary conditions. The reader is referred to the survey [11] for the description of all these works.

There is also the possibility to look at the problem from the purely algebraic point of view, when the equations have already been discretized in space and in time. This way to approach the problem emerged after the identification in [13] of the classical pressure segregation method as an inexact factorization of the system arising after discretization. Several authors followed this path; for a review, see [1]. This point of view has clear advantages, as for example its generality or the fact that it avoids any issue related to boundary conditions, but also some inconveniences from the convergence point of view, since estimates depend on derivatives of discrete functions whose boundedness is not easy to prove.

In the case of viscoelastic flows, the main difficulty is the appearance of a new variable, a stress, that evolves in time. Thus, there are three variables (velocity, stress and pressure) that are in principle coupled and for which uncoupling algorithms need to be devised. Obviously, the uncoupling needs to satisfy two main conditions: it has to maintain the stability of the underlying time discretization (otherwise, a simple explicit treatment of adequate terms would suffice) and it has to maintain also its temporal order of accuracy. Surprisingly, even though several fractional schemes have been proposed for this problem (see for example [15], perhaps one of the first attempts), they either do not uncouple all the variables or are not natural extensions of the most popular schemes used for viscous Newtonian flows; see for example the bibliography cited in [5].

In [5] we proposed fractional step methods for viscoelastic flows based on the segregation of the pressure and the stress in the momentum equation. The approach proposed there is completely algebraical, working with the problem arising from spatial and temporal discretization of the original initial and boundary value problem. We designed schemes of first, second and third order in time, and all motivated from two perspectives: either the extrapolation in time of variables to allow their segregation or the inexact factorization of the linear system to be solved at each time step. All the schemes were tested in convergence tests, to check the predicted order of accuracy, and in more realistic examples to experiment their robustness.

The purpose of this article is to present some fractional step methods for viscoelastic flows designed from the pure algebraic point of view. Two families of approaches will be described. The first is the same as in [5], considering pressure (and stress) extrapolations to allow for the calculation of an intermediate velocity,

whereas the second is based on the extrapolation of the velocity to allow for the calculation of the pressure. This second new approach is based in the design of fractional step schemes based on a discrete pressure Poisson equation that was proposed for viscous Newtonian flows in [2, 12].

The spatial approximation will not be discussed in detail. To fix ideas, we will describe how the approximation can be done using the finite element method using inf-sup stable approximations, although we favor the stabilized finite element approximation presented in [4]; minor modifications to the schemes to be described need to be introduced in case this stabilized formulation is used. Likewise, we will assume that the temporal discretization is performed using backward difference (BDF) schemes, although any other time integration could be employed.

The outline of the paper is as follows. In Section 2 we state the continuous problem, its finite element approximation in space and its numerical integration in time. In Section 3 we describe the schemes based on pressure extrapolation proposed in [5], whereas in Section 4 we present new schemes based on velocity extrapolation. Even though our objective is not the numerical analysis of the resulting methods, but only their design, some comments on their stability are also included in Section 5. In Section 6 we also explain how to view the schemes as inexact factorizations of the fully discrete system. Finally, some conclusions are drawn in Section 7.

2 Problem statement and numerical approximation

Let Ω be a bounded domain of $\mathbb{R}^d$ ($d = 2,3$) where the flow takes place, and let $[0,t_{\rm f}[$ be the time interval of analysis. The viscoelastic (Olroyd-B) flow problem we wish to consider consists of finding a velocity $u : \Omega\times]0,t_{\rm f}[\to \mathbb{R}^d$, a pressure $p : \Omega\times]0,t_{\rm f}[\to \mathbb{R}$ and a stress $\sigma : \Omega\times]0,t_{\rm f}[\to \mathbb{R}^d\otimes\mathbb{R}^d$ such that

$$\rho\frac{\partial u}{\partial t}+\rho u\cdot\nabla u-\nabla\cdot T+\nabla p=f \tag{1}$$

$$\nabla\cdot u=0 \tag{2}$$

with $T=2\beta\eta_0\nabla^s u+\sigma$ and

$$\frac{\lambda}{2\eta_0}\frac{\partial\sigma}{\partial t}+\frac{1}{2\eta_0}\sigma-(1-\beta)\nabla^s u+\frac{\lambda}{2\eta_0}\left(u\cdot\nabla\sigma-\sigma\cdot\nabla u-(\nabla u)^T\cdot\sigma\right)=0 \tag{3}$$

In these equations, which hold in $\Omega\times]0,t_{\rm f}[$, f is the body force, ρ the fluid density, β, η_0 and λ are positive physical parameters ($0\le\beta\le 1$) and ∇^s denotes the symmetric part of the gradient of a vector field. Appropriate initial and boundary conditions need to be added to close the problem (see [10], for example).

To write the weak form of the problem, let $\mathscr{V}$, $\mathscr{Q}$ and Υ be the spaces where velocities, pressures and stresses, respectively, have to belong for each $t\in]0,t_{\rm f}[$. Considering for example homogeneous velocity boundary conditions, $\mathscr{V}=H_0^1(\Omega)^d$,

$\mathscr{Q} = L^2(\Omega)/\mathbb{R}$ and Υ is the space of tensor fields with components in $L^2(\Omega)$, such that the last term in parenthesis in (3) has components in $L^2(\Omega)$ and satisfying the appropriate boundary conditions. Let $(\cdot,\cdot)$ denote the inner product in $L^2(\Omega)$ (for scalars, vectors or tensors) and $\langle\cdot,\cdot\rangle$ the integral of the product of two functions. The weak form of the problem consists then of finding $[u,p,\sigma]:]0,t_{\rm f}[\rightarrow \mathscr{X} := \mathscr{V}\times\mathscr{Q}\times\Upsilon$ such that the initial conditions are satisfied and

$$\left(\rho\frac{\partial u}{\partial t},v\right)+2\left(\beta\eta_0\nabla^s u,\nabla^s v\right)+\langle\rho u\cdot\nabla u,v\rangle+(\sigma,\nabla^s v)-(p,\nabla\cdot v)=\langle f,v\rangle \quad (4)$$

$$(q,\nabla\cdot u)=0 \quad (5)$$

$$\left(\frac{\lambda}{2\eta_0}\frac{\partial\sigma}{\partial t},\tau\right)+\left(\frac{1}{2\eta_0}\sigma,\tau\right)-\left(\left(1-\beta\right)\nabla^s u,\tau\right) + \frac{\lambda}{2\eta_0}\left(u\cdot\nabla\sigma-\sigma\cdot\nabla u-\left(\nabla u\right)^T\cdot\sigma,\tau\right)=0 \quad (6)$$

for all $[v,q,\tau]\in\mathscr{X}$, where it is assumed that f is such that $\langle f,v\rangle$ is well defined.

The fractional step schemes to be presented can be used in conjunction with any space discretization. For the sake of conciseness, suppose that the finite element method is used. From a finite element partition of the computational domain Ω we may construct conforming finite element subspaces of $\mathscr{V}$, $\mathscr{Q}$ and Υ, that we respectively denote by $\mathscr{V}_h$, $\mathscr{Q}_h$ and Υ_h, the subscript h referring to the size of the partition. We assume that these spaces render a stable approximation in space, a point that turns out to be crucial and poses stringent requirements on the choice of the finite element spaces (in the form of two inf-sup conditions). This can be circumvented by using a stabilized finite element method, in which the discrete variational form of the problem is modified with respect to the continuous form, and therefore also the final algebraic system presented below is modified. Nevertheless, since the spatial approximation is not our focus, we assume hereafter that the so-called standard Galerkin method is used and refer to [8, 4] for further discussion.

Once $\mathscr{X}$ has been approximated by $\mathscr{X}_h := \mathscr{V}_h\times\mathscr{Q}_h\times\Upsilon_h$, the unknowns and test functions can be expressed as a combination of the basis functions of each space and the arrays of nodal values. We shall respectively denote the nodal values of u_h, p_h and σ_h as U, P and Σ; these arrays are time-dependent functions before the time discretization.

Considering the time discretization prior to the splitting, any alternative could be used. To fix ideas, and to simplify the notation, we will assume that backward difference schemes (BDF) of order $k\geq 1$ are used. Let us consider a uniform partition of the interval $[0,t_{\rm f}]$ of size δt, and let us denote with a superscript the time step level at which functions are approximated. A BDF scheme of order k is based on the k-th difference of a function, which when evaluated at $t^{n+1}=(n+1)\delta t$ reads

$$\delta_k g^{n+1}=\frac{1}{\gamma_k}\left(g^{n+1}-\sum_{i=0}^{k-1}\varphi_k^i g^{n-i}\right)=:\frac{1}{\gamma_k}g^{n+1}-g^{*,n}$$

for a generic function g, and where γ_k and φ_k^i are parameters that depend on k. In particular, we will be interested in the cases $k = 1,2,3$.

We will also use the extrapolation operators of order k, defined as $\widehat{g}_k^{n+1} = g^{n+1} + \mathscr{O}(\delta t^k)$, which for $k = 1$, 2 and 3 are given by

$$\begin{aligned}
\widehat{g}_1^{n+1} &= g^n \\
\widehat{g}_2^{n+1} &= 2g^n - g^{n-1} \\
\widehat{g}_3^{n+1} &= 3g^n - 3g^{n-1} + g^{n-2}
\end{aligned}$$

As for the k-th difference of a function, proper initializations are required in the first time steps.

Assuming space is discretized using the standard Galerkin method and time using a BDF scheme of order k, the resulting algebraic structure of the approximation to problem (4)-(6) is

$$M_u \frac{\delta_k}{\delta t} U^{n+1} + K_u\left(U^{n+1}\right) U^{n+1} + G_u P^{n+1} - D_\sigma \Sigma^{n+1} = F^{n+1} \tag{7}$$

$$D_u U^{n+1} = 0 \tag{8}$$

$$M_\sigma \frac{\delta_k}{\delta t} \Sigma^{n+1} + K_\sigma\left(U^{n+1}\right) \Sigma^{n+1} - G_\sigma U^{n+1} = 0 \tag{9}$$

The identification of the matrices and arrays appearing in these algebraic equations with the terms arising from the discretization of (4)-(6) is straightforward. Let us remark that matrices G_u and D_u, coming from the gradient of the pressure and the divergence of the velocity, respectively, are related by $G_u = -D_u^T$. Similarly, matrices G_σ and D_σ coming from the symmetric gradient of the velocity and the divergence of the stress, respectively, are related by $(1-\beta)G_\sigma = -D_\sigma^T$. We have explicitly displayed the dependence of matrices K_u and K_σ on U, in the first case due to the convective term in (1) and in the second to the convective and rotational terms in (3).

Equations (7)-(9) can be written in compact form as

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & 0 \\ A_{31} & 0 & 0 \end{bmatrix} \begin{bmatrix} U^{n+1} \\ \Sigma^{n+1} \\ P^{n+1} \end{bmatrix} = \begin{bmatrix} F_1^{n+1} \\ F_2^{n+1} \\ 0 \end{bmatrix} \tag{10}$$

where only the unknowns at time step $n+1$ have been left in the left-hand-side and the identification of the different matrices and arrays is obvious.

3 Schemes based on pressure extrapolation

The first family of schemes to be presented can be introduced using pressure and stress extrapolation in the momentum equation. This implies that these terms are

solved explicitly, and therefore this would lead to an at most conditionally stable time integration scheme. To keep the stability properties of the original BDF scheme employed, a velocity correction is required once pressure and stress have been obtained from their corresponding equations. We elaborate this idea in the next subsection, where we present the schemes already proposed in [5]. Then we write the problem posed in terms of the end-of-step unknowns, what we call equivalent monolithic formulation, which allows us to foresee the order in time of the splitting error.

3.1 Formulation of the algorithms

To motivate the schemes based on pressure extrapolation, let us write the algebraic system (7)-(9) in the equivalent form

$$M_u \frac{\delta_k}{\delta t}\tilde{U}^{n+1} + K_u(\tilde{U}^{n+1})\tilde{U}^{n+1} + G_u\hat{P}^{n+1}_{k'-1} - D_\sigma\hat{\Sigma}^{n+1}_{k'-1} = F^{n+1} \tag{11}$$

$$M_u \frac{1}{\gamma_k \delta t}(U^{n+1} - \tilde{U}^{n+1}) + N_u^{n+1} + G_u(P^{n+1} - \hat{P}^{n+1}_{k'-1}) - D_\sigma(\Sigma^{n+1} - \hat{\Sigma}^{n+1}_{k'-1}) = 0 \tag{12}$$

$$M_\sigma \frac{\delta_k}{\delta t}\tilde{\Sigma}^{n+1} + K_\sigma(\tilde{U}^{n+1})\tilde{\Sigma}^{n+1} - G_\sigma\tilde{U}^{n+1} = 0 \tag{13}$$

$$M_\sigma \frac{1}{\gamma_k \delta t}(\Sigma^{n+1} - \tilde{\Sigma}^{n+1}) + N_\sigma^{n+1} - G_\sigma(U^{n+1} - \tilde{U}^{n+1}) = 0 \tag{14}$$

$$\begin{aligned} &-D_u\tilde{U}^{n+1} + \gamma_k\delta t D_u M_u^{-1} N_u^{n+1} + \gamma_k \delta t D_u M_u^{-1} G_u(P^{n+1} - \hat{P}^{n+1}_{k'-1}) \\ &\quad - \gamma_k \delta t D_u M_u^{-1} D_\sigma(\Sigma^{n+1} - \hat{\Sigma}^{n+1}_{k'-1}) = 0 \end{aligned} \tag{15}$$

where

$$\begin{aligned} N_u^{n+1} &:= K_u(U^{n+1})U^{n+1} - K_u(\tilde{U}^{n+1})\tilde{U}^{n+1} \\ N_\sigma^{n+1} &:= K_\sigma(U^{n+1})\Sigma^{n+1} - K_\sigma(\tilde{U}^{n+1})\tilde{\Sigma}^{n+1} \end{aligned}$$

and $\tilde{U}^{n+1}$ and $\tilde{\Sigma}^{n+1}$ are intermediate unknowns. That this system is equivalent to (7)-(9) can be checked as follows: adding (11) and (12) we exactly recover (7), adding (13) and (14) we exactly recover (9), and (15) is obtained multiplying (12) by $\gamma_k \delta t D_u M_u^{-1}$ and making use of (8). The order k' used in the extrapolated variables can in principle be different from k.

Equations (11)-(15) motivate the following algorithm, which is only an approximation to (7)-(9) but allows one to compute the different variables sequentially:

1. Compute $\tilde{U}^{n+1}$ from (11).
2. Compute $\tilde{\Sigma}^{n+1}$ from (13).
3. Compute *an approximation* to P^{n+1} by solving (15) *neglecting* N_u^{n+1} and *replacing* Σ^{n+1} by $\tilde{\Sigma}^{n+1}$.

4. Compute an approximation to U^{n+1} from (12) neglecting N_u^{n+1}.
5. Compute an approximation to Σ^{n+1} from (14) neglecting N_σ^{n+1}.

Several remarks are in order:

- Steps 1 to 5 above allow one to uncouple the calculation of the different variables.
- Matrix $D_u M_u^{-1} G_u$ appearing in the pressure Poisson equation can be approximated by the classical Laplacian matrix, L, with a reduced stencil. This introduces a further approximation, except if an iterative scheme is employed where L is simply used as a preconditioner (see [7, 2]).
- For $k' = k$, the resulting scheme is of order $\mathscr{O}(\delta t^k)$ for a given spatial discretization. We will come back to this point in the following subsection.
- The resulting scheme is only stable for $k' = 1, 2$. For $k' = 3$, the extrapolation $\hat{P}_2^{n+1} = 2P^n - P^{n-1}$ is known to yield an unstable scheme (see the discussion in [12, 1]).
- For $k = 1$ we have an extension to viscoelastic flows of the classical first order fractional step method, whereas for $k = 2$ we have an extension of the second order method.

In view of these comments, we consider $k' = k = 1, 2$, obtaining the following system of equations:

First and second order pressure extrapolation schemes:

$$M_u \frac{\delta_k}{\delta t} \tilde{U}^{n+1} + K_u(\tilde{U}^{n+1})\tilde{U}^{n+1} + G_u \hat{P}_{k-1}^{n+1} - D_\sigma \hat{\Sigma}_{k-1}^{n+1} = F^{n+1} \tag{16}$$

$$M_\sigma \frac{\delta_k}{\delta t} \tilde{\Sigma}^{n+1} + K_\sigma(\tilde{U}^{n+1})\tilde{\Sigma}^{n+1} - G_\sigma \tilde{U}^{n+1} = 0 \tag{17}$$

$$\begin{aligned} &- D_u \tilde{U}^{n+1} + \gamma_k \delta t D_u M_u^{-1} G_u (P^{n+1} - \hat{P}_{k-1}^{n+1}) \\ &\qquad - \gamma_k \delta t D_u M_u^{-1} D_\sigma (\tilde{\Sigma}^{n+1} - \hat{\Sigma}_{k-1}^{n+1}) = 0 \end{aligned} \tag{18}$$

$$\frac{1}{\gamma_k \delta t} M_u (U^{n+1} - \tilde{U}^{n+1}) + G_u (P^{n+1} - \hat{P}_{k-1}^{n+1}) - D_\sigma (\tilde{\Sigma}^{n+1} - \hat{\Sigma}_{k-1}^{n+1}) = 0 \tag{19}$$

$$\frac{1}{\gamma_k \delta t} M_\sigma (\Sigma^{n+1} - \tilde{\Sigma}^{n+1}) - G_\sigma (U^{n+1} - \tilde{U}^{n+1}) = 0 \tag{20}$$

These are the first and second order pressure extrapolation algorithms proposed in [5]. In fact, it is not only the pressure, but also the stress, the variable extrapolated in the first equation.

A third order scheme can be obtained with a different approximation to (11)-(15), which can be related to Yosida's factorization (see [5]). The steps are the following:

1. Compute $\tilde{U}^{n+1}$ from (11) with $k = 3$ and $k' = 2$.
2. Compute $\tilde{\Sigma}^{n+1}$ from (13) with $k = 3$ and $k' = 2$.

3. Compute *an approximation* to P^{n+1} by solving (15) *neglecting* N_u^{n+1}, *replacing* Σ^{n+1} by $\tilde{\Sigma}^{n+1}$ and taking $k=3$ and $k'=2$.
4. Compute an approximation to U^{n+1} from (12) *without neglecting* N_u^{n+1}.
5. Compute an approximation to Σ^{n+1} from (14) *neglecting* N_σ^{n+1}.

Even if only a first order extrapolation is used for the pressure and the elastic stresses in the momentum equation, including N_u^{n+1} in the fourth step allows one to obtain *third* order accuracy.

The system of equations to be solved is presented next.

Third order pressure extrapolation scheme:

$$M_u\frac{\delta_3}{\delta t}\tilde{U}^{n+1}+K_u(\tilde{U}^{n+1})\tilde{U}^{n+1}+G_uP^n-D_\sigma\Sigma^n=F^{n+1} \tag{21}$$

$$M_\sigma\frac{\delta_3}{\delta t}\tilde{\Sigma}^{n+1}+K_\sigma(\tilde{U}^{n+1})\tilde{\Sigma}^{n+1}-G_\sigma\tilde{U}^{n+1}=0 \tag{22}$$

$$\begin{aligned}&-D_u\tilde{U}^{n+1}+\gamma_3\delta tD_uM_u^{-1}G_u(P^{n+1}-P^n)\\&\qquad-\gamma_3\delta tD_uM_u^{-1}D_\sigma(\tilde{\Sigma}^{n+1}-\Sigma^n)=0\end{aligned} \tag{23}$$

$$\begin{aligned}&\frac{1}{\gamma_3\delta t}M_u(U^{n+1}-\tilde{U}^{n+1})+K_u(U^{n+1})U^{n+1}-K_u(\tilde{U}^{n+1})\tilde{U}^{n+1}\\&\qquad+G(P^{n+1}-P^n)-D_\sigma(\tilde{\Sigma}^{n+1}-\Sigma^n)=0\end{aligned} \tag{24}$$

$$\frac{1}{\gamma_3\delta t}M_\sigma(\Sigma^{n+1}-\tilde{\Sigma}^{n+1})-G_\sigma(U^{n+1}-\tilde{U}^{n+1})=0 \tag{25}$$

3.2 Equivalent monolithic formulations

A way to predict formally the order of approximation of the splitting schemes introduced is to write the equations for the final unknowns, after the correction steps, and see which is the perturbation with respect to the original monolithic equations. Let us start with the first and second order schemes introduced earlier. Adding up (16) and (19) on the one hand, and (17) and (20) on the other, we obtain

$$\begin{aligned}&M_u\frac{\delta_k}{\delta t}U^{n+1}+K_u(U^{n+1})U^{n+1}+G_uP^{n+1}-D_\sigma\Sigma^{n+1}\\&\qquad-N_u^{n+1}-D_\sigma(\tilde{\Sigma}^{n+1}-\Sigma^{n+1})=F^{n+1}\\&M_\sigma\frac{\delta_k}{\delta t}\Sigma^{n+1}+K_\sigma(U^{n+1})\Sigma^{n+1}-G_\sigma U^{n+1}-N_\sigma^{n+1}=0\end{aligned}$$

from where we observe that the perturbation of the momentum equation is $-N_u^{n+1}-D_\sigma(\tilde{\Sigma}^{n+1}-\Sigma^{n+1})$ and the perturbation of the stress equation is $-N_\sigma^{n+1}$, as it could

be expected from the steps followed. These are the only perturbations, since from (18) and (19) it follows that

$$D_u U^{n+1} = 0$$

i.e., the continuity equation is not perturbed (it would be perturbed if the classical Laplacian matrix L is used, as mentioned earlier).

Let us analyze which is the expected order of accuracy. Combining (19) and (20) we get

$$\left[\frac{1}{\gamma_k \delta t} M_u + \gamma_k \delta t D_\sigma M_\sigma^{-1} G_\sigma\right](U^{n+1} - \tilde{U}^{n+1}) + G_u(P^{n+1} - \hat{P}_{k-1}) - D_\sigma(\Sigma^{n+1} - \hat{\Sigma}_{k-1}) = 0$$

from where we see that $U^{n+1} - \tilde{U}^{n+1}$ is of order $\mathscr{O}(\delta t^k)$ (in an adequate norm). Knowing this, it follows from (20) that $\Sigma^{n+1} - \tilde{\Sigma}^{n+1}$ is of order $\mathscr{O}(\delta t^{k+1})$. From this we conclude that *the perturbation terms* $-N_u^{n+1} - D_\sigma(\tilde{\Sigma}^{n+1} - \Sigma^{n+1})$ *and* $-N_\sigma^{n+1}$ *are of order* $\mathscr{O}(\delta t^k)$ *and, in fact, the correction step (20) is not needed to have a splitting error of order* $\mathscr{O}(\delta t^k)$. This last remark is relevant, since using the classical factorization point of view described in [5] this last step does not appear.

Let us move our attention to the third order pressure extrapolation scheme. Adding up (21) and (24) on the one hand, and (22) and (25) on the other, we obtain

$$M_u \frac{\delta_3}{\delta t} U^{n+1} + K_u(U^{n+1})U^{n+1} + G_u P^{n+1} - D_\sigma \Sigma^{n+1} - D_\sigma(\tilde{\Sigma}^{n+1} - \Sigma^{n+1}) = F^{n+1}$$
$$M_\sigma \frac{\delta_3}{\delta t} \Sigma^{n+1} + K_\sigma(U^{n+1})\Sigma^{n+1} - G_\sigma U^{n+1} - N_\sigma^{n+1} = 0$$

from where it follows that the perturbation of the momentum equation is only $-D_\sigma(\tilde{\Sigma}^{n+1} - \Sigma^{n+1})$ and the perturbation of the stress equation is $-N_\sigma^{n+1}$. Combining (23) and (24) one gets

$$D_u U^{n+1} + \gamma_3 \delta t D_u M_u^{-1} N_u^{n+1} = 0$$

Let us verify formally which should be the order of accuracy of the scheme. Combining (24) and (25) we get

$$\left[\frac{1}{\gamma_3 \delta t} M_u + K_u(U^{n+1}) - K_u(\tilde{U}^{n+1}) + \gamma_3 \delta t D_\sigma M_\sigma^{-1} G_\sigma\right](U^{n+1} - \tilde{U}^{n+1}) + G_u(P^{n+1} - P^n) - D_\sigma(\Sigma^{n+1} - \Sigma^n) = 0$$

Noting that $K_u(U)$ is linear in U, from this expression it follows that $U^{n+1} - \tilde{U}^{n+1}$ is of order $\mathscr{O}(\delta t^2)$ (in an adequate norm). Knowing this, from (25) it follows that $\Sigma^{n+1} - \tilde{\Sigma}^{n+1}$ is of order $\mathscr{O}(\delta t^3)$. Contrary to the first and second order schemes, the correction step (25) is now crucial, since it guarantees that the perturbation of the momentum equation is $\mathscr{O}(\delta t^3)$, which is of the same order as the perturbation of the stress equation and the perturbation of the continuity equation of the mono-

lithic scheme. Therefore, we can expect (21)-(25) to be a third order fractional step scheme. This was numerically checked in [5].

4 Schemes based on velocity extrapolation

In the schemes presented heretofore, pressure and stress have been extrapolated in the momentum equation. This permits to compute a first guess for the velocity that needs to be corrected. The idea now is to write an equation for the pressure and extrapolate the velocity and the stress. That should allow one to compute a first guess for the pressure, that may need to be corrected (or not). But such an equation for the pressure is not explicit in (1)-(2), and so we will start reformulating the continuous problem, although we shall see that it is not an appropriate option.

4.1 The continuous problem

We may replace the continuous equation (2) by the equation that is obtained taking the divergence of (1) and using that fact that u must be divergence free. This leads to:

$$\Delta p = \nabla \cdot (f + 2\beta\eta_0 \nabla \cdot \nabla^s u - \rho u \cdot \nabla u + \nabla \cdot \sigma)$$

which has to hold in Ω and in the time interval $]0,t_{\rm f}[$. The appropriate boundary condition for this equation turn out to be that the normal derivative of the pressure on $\partial\Omega$ be equal to the normal component of the term within parenthesis. If q is a pressure test function, the weak form of this equation reads:

$$(\nabla q, \nabla p) = (\nabla q, f + 2\beta\eta_0 \nabla \cdot \nabla^s u - \rho u \cdot \nabla u + \nabla \cdot \sigma) \tag{26}$$

for all test functions q. The continuous variational problem determined by equations (4), (5) and (6) can be replaced by the problem made by equations (4), (26) and (6). However, two remarks are needed:

- The regularity of the problem has changed. This is obvious from (26). It is well posed for example for pressures in $H^1(\Omega)$ in space, not only in $L^2(\Omega)$, velocities in $H^2(\Omega)$ and stresses in $H(\mathrm{div};\Omega)$. The regularity of these variables could be relaxed at the expenses of taking q in $H^2(\Omega)$. This additional need of regularity is not only a theoretical problem, but also could complicate enormously the numerical approximation.
- For divergence free velocities, $\nabla \cdot \nabla \cdot \nabla^s u = 0$, and therefore the term $2\beta\eta_0 \nabla \cdot \nabla^s u$ could be removed from (26). However, this does not only change the natural boundary condition, but also yields an ill-posed problem (see the discussion and references in [1]).

In view of these comments, it seems clear that system (4), (26) and (6) is not a good alternative. However, we could mimic the obtention of (26) at the algebraic level, and design effective fractional step schemes from the resulting equations.

4.2 Formulation of the algorithms

Let us consider problem (7)-(9). Multiplying the first equation by $\gamma_k \delta t D_u M_u^{-1}$ and using the fact that $D_u U^{n+1} = 0$ we obtain

$$\begin{aligned}
&\gamma_k \delta t D_u M_u^{-1} G_u P^{n+1} \\
&\quad = \gamma_k \delta t D_u M_u^{-1} (F^{n+1} - K_u(U^{n+1})U^{n+1} + D_\sigma \Sigma^{n+1}) + D_u U^{*,n} \qquad (27)
\end{aligned}$$

$$M_u \frac{\delta_k}{\delta t} U^{n+1} + K_u\left(U^{n+1}\right) U^{n+1} + G_u P^{n+1} - D_\sigma \Sigma^{n+1} = F^{n+1} \qquad (28)$$

$$M_\sigma \frac{\delta_k}{\delta t} \Sigma^{n+1} + K_\sigma\left(U^{n+1}\right) \Sigma^{n+1} - G_\sigma U^{n+1} = 0 \qquad (29)$$

This system is equivalent to (7)-(9), with the difference that now we have an equation for the pressure in terms of the velocity and the stress that is invertible, of Poisson type, obtained from the original monolithic discretization of the problem. To this system we can apply the same ideas as for the algorithms based on pressure extrapolation:

- Compute an approximation to the pressure using a velocity and a stress extrapolation in (27).
- Compute an approximation to the velocity using the pressure obtained and a stress extrapolation in (28).
- Compute the stress using the velocity obtained in (29).
- Correct the velocity to cancel the effect of the extrapolated stress in (28).
- Correct the pressure to cancel the effect of the extrapolated velocity and stress in (27).

To have an overall scheme of order k, the extrapolations need to be of order $k-1$. The equations to be solved are thus the following:

First, second and third order velocity extrapolation schemes:

$$\begin{aligned}
&\gamma_k \delta t D_u M_u^{-1} G_u \tilde{P}^{n+1} \\
&\quad = \gamma_k \delta t D_u M_u^{-1} (F^{n+1} - K_u(\hat{U}_{k-1}^{n+1})\hat{U}_{k-1}^{n+1} + D_\sigma \hat{\Sigma}_{k-1}^{n+1}) + D_u U^{*,n} \qquad (30)
\end{aligned}$$

$$M_u \frac{\delta_k}{\delta t} \tilde{U}^{n+1} + K_u(\tilde{U}^{n+1})\tilde{U}^{n+1} + G_u \tilde{P}^{n+1} - D_\sigma \hat{\Sigma}_{k-1}^{n+1} = F^{n+1} \qquad (31)$$

$$M_\sigma \frac{\delta_k}{\delta t} \Sigma^{n+1} + K_\sigma(\tilde{U}^{n+1})\Sigma^{n+1} - G_\sigma \tilde{U}^{n+1} = 0 \qquad (32)$$

$$\frac{1}{\gamma_k \delta t} M_u (U^{n+1} - \tilde{U}^{n+1}) - D_\sigma (\Sigma^{n+1} - \hat{\Sigma}^{n+1}_{k-1}) = 0 \tag{33}$$

$$D_u M_u^{-1} G_u (P^{n+1} - \tilde{P}^{n+1}) = D_u M_u^{-1} (K_u(\hat{U}^{n+1}_{k-1}) \hat{U}^{n+1}_{k-1} - D_\sigma \hat{\Sigma}^{n+1}_{k-1}) + D_u M_u^{-1} (-K_u(U^{n+1}) U^{n+1} + D_\sigma \Sigma^{n+1}) \tag{34}$$

This algorithms admits several modifications and requires some remarks:

- Matrix $D_u M_u^{-1} G_u$ has a wide stencil. In principle, one could use the approximation $D_u M_u^{-1} G_u \tilde{P}^{n+1} \approx L \tilde{P}^{n+1} + (D_u M_u^{-1} G_u - L) \hat{P}_{k-1}$. However, the resulting scheme turns out to be unstable for $k = 3$ because of the second order pressure extrapolation, and thus it cannot be used to design a third order formulation. The alternative could be to use L only as a preconditioner in an iterative scheme. See [12] for further discussion.
- If instead of $\hat{\Sigma}^{n+1}_{k-1}$ one uses $\hat{\Sigma}^{n+1}_k$ in (30), the fourth step (33) would be unnecessary from the accuracy point of view. However, stability would be affected, since the intermediate velocities obtained from (31) depend on extrapolated stresses, i.e., to an explicit treatment of the stress in the momentum equation.
- For the exact problem, $D_u U^{*,n} = 0$. However, this does not hold with the approximations done, and the term $D_u U^{*,n}$ has to be kept in (30) to obtain a stable scheme (see [2]).
- A very important point from the computational point of view is that the fifth step (34) is in fact *not needed*, since pressure is not an evolution variable for incompressible flows. However, it is formally convenient to maintain (34), since it shows how the pressure should be corrected in case it is needed.

4.3 Equivalent monolithic formulation

As for the schemes based on pressure extrapolation, let us obtain the equivalent monolithic system solved by (30)-(33). The resulting momentum equation is obtained adding up (31) and (33) and the resulting stress equation is directly (32). These equations can be written as

$$M_u \frac{\delta_k}{\delta t} U^{n+1} + K_u(U^{n+1}) U^{n+1} + G_u \tilde{P}^{n+1} - D_\sigma \Sigma^{n+1} - N_u^{n+1} = F^{n+1} \tag{35}$$

$$M_\sigma \frac{\delta_k}{\delta t} \Sigma^{n+1} + K_\sigma(U^{n+1}) \Sigma^{n+1} - G_\sigma U^{n+1} + [K_\sigma(\tilde{U}^{n+1}) - K_\sigma(U^{n+1})] \Sigma^{n+1} - G_\sigma(\tilde{U}^{n+1} - U^{n+1}) = 0 \tag{36}$$

Multiplying (35) by $\gamma_k \delta t D_u M_u^{-1}$ and making use of (30) it is found that

$$D_u U^{n+1} + \gamma_k \delta t D_u M_u^{-1} [K_u(U^{n+1}) U^{n+1} - K_u(\hat{U}^{n+1}_{k-1}) \hat{U}^{n+1}_{k-1}]$$

$$+\gamma_k \delta t D_u M_u^{-1}[-D_\sigma(\Sigma^{n+1}-\hat{\Sigma}_{k-1}^{n+1})-N_u^{n+1}]=0 \tag{37}$$

From (33) it follows that $U^{n+1}-\tilde{U}^{n+1}$ is or order δt^k (in the appropriate norm). Identifying $\tilde{P}^{n+1}$ with the pressure to be computed, we observe that (35)-(37) is a perturbation of the original system (7)-(9) with all the perturbation terms of order δt^k.

5 Comments on stability

The obvious way to undertake the numerical analysis of the algorithms presented is to evaluate the stability and convergence properties of the segregated schemes with respect to their monolithic counterpart, and then rely on the estimates of stability and convergence of the monolithic formulations with respect to the continuous problem. The difficulty of this approach relies on the fact that convergence estimates of the first step will depend on norms of discrete solutions. While in some cases it is possible to prove bounds for these norms (see [3] for an application of this technique to a first order scheme), in general this boundedness has to be assumed. The order of accuracy of the formulations has to be based solely on the formal derivation presented before, comparing the fractional step schemes with their monolithic versions.

However, stability can be proved rigorously and at the pure algebraic level. This was shown first in [7] for Newtonian fluids, and then the approach was followed in [9, 2] with other schemes (see [1] for a review and additional references).

It is outside the scope of this article to present the stability proofs of the different schemes presented. We will just describe the results that can be obtained in a descriptive manner. To this end, given arrays X and Y of m components and a positive definite $m\times m$ matrix A, we define

$$(X,Y)_A := X^T A Y, \quad \|X\|_A := (X^T A X)^{1/2}, \quad \|X\|_{-A} := \sup_{Y\neq 0} \frac{X^T Y}{\|Y\|_A}$$

Given a sequence of arrays $\{X^n\}$, $n=1,2,\ldots,N$, we define

$$\{X^n\}\in \ell^\infty(A) \iff \|X^n\|_A<\infty \text{ for all } n=1,2,\ldots,N$$

$$\{X^n\}\in \ell^p(A) \iff \sum_{n=1}^{N} \delta t \|X^n\|_A^p<\infty \;\; 1\le p<\infty$$

where $\delta t = t_{\mathrm{f}}/N$. We will apply these definitions to the sequences $\{U^n\}$, $\{\tilde{U}^n\}$, $\{\Sigma^n\}$, $\{\tilde{\Sigma}^n\}$ and $\{P^n\}$ obtained using the *first and second order* schemes presented. The third order formulations proposed have been based on the third order BDF time integration scheme, which is only conditionally stable; therefore, unconditional stability for the split schemes cannot be expected.

Let us denote by $K_{u,0}$ the symmetric part of K_u. From the original term in (4) from which matrix comes, it is seen that it is zero when $\beta=0$. The results one can

prove for all the methods presented are the following:

$$\{U^n\} \in \ell^\infty(M_u), \quad \{\tilde{U}^n\} \in \ell^\infty(M_u) \cap \ell^2(K_{u,0})$$
$$\{\Sigma^n\} \in \ell^\infty(M_\sigma), \quad \{\tilde{\Sigma}^n\} \in \ell^\infty(M_\sigma)$$

provided $\sum_{n=1}^N \delta t \|F^n\|_{M_u}^2 < \infty$. If $\beta > 0$, the stability for $\{\tilde{U}^n\}$ is optimal, and in fact one only needs to have $\sum_{n=1}^N \delta t \|F^n\|_{-L_+}^2 < \infty$, where $L_+ = -L$ and L is the Laplacian matrix, as before, but now extended to vector fields (the sequence of arrays $\{F^n\}$ comes from the approximation of the forcing term f). However, if $\beta = 0$ (or β is very small), we do not have stability of $\{\tilde{U}^n\}$ in the discrete counterpart of $L^2(0,t_{\rm g};H^1(\Omega)^d)$, which is precisely $\ell^2(K_{u,0})$ if $\beta > 0$ or $\ell^2(L_+)$ if $\beta = 0$ (again, L_+ is applied to vector fields).

To obtain the missing stability one has to make use of the inf-sup conditions that need to be satisfied between the approximation of pressures and velocities on the one hand and on the approximation of velocities and stresses on the other. Alternatively, one can use stabilized finite element formulations (see [8] and references therein for further discussion). Using the first option, the conditions that need to be satisfied can be written as follows. Let P be an array in the space coming from the discretization of the pressure and let M_p be the matrix coming from the $L^2(\Omega)$ inner product in the pressure space. Let also U, V be generic arrays in the space coming from the discretization of the velocity and Ψ an array in the space coming from the discretization of the stress. Then, we assume that there exist $\beta_1 > 0$ and $\beta_2 > 0$, constants, such that

$$\text{For all } P \text{ there exists } V \text{ such that } \beta_1 \|P\|_{M_p} \|V\|_{L_+} \leq P^T D_u V$$
$$\text{For all } U \text{ there exists } \Psi \text{ such that } \beta_2 \|U\|_{L_+} \|\Psi\|_{M_\sigma} \leq \Psi^T G_\sigma U$$

Under this assumption, one can prove that

$$\{\tilde{U}^n\} \in \ell^2(L_+), \quad \{P^n\} \in \ell^2(M_p)$$

With this, we have all the stability results that could be expected. In fact, for schemes based on velocity extrapolation one can prove some additional stability results that do not have a counterpart at the continuous level (see the review in [1]).

6 The inexact factorization point of view

Let us apply the inexact factorization point of view to fractional step schemes for viscoelastic flows. This idea was proposed in [13]; see also [14] for an interesting elaboration.

Let A be the matrix of system (10), which we may factorize as $A = L_A U_A$, with L_A lower diagonal per blocks and U_A upper diagonal. Writing (10) as $AX^{n+1} = R^{n+1}$, we may solve the sequence $L_A \tilde{X}^{n+1} = R^{n+1}$ and $U_A X^{n+1} = \tilde{X}^{n+1}$, the advantage be-

ing that in each system we can solve sequentially for the different unknowns. The problem is that this process involves the inversion of A_{11} and A_{22}, which is computationally expensive. Therefore, the idea of inexact factorizations is to *approximate* A_{11}^{-1} and A_{22}^{-1}, this yielding approximations to L_A and U_A respectively denoted by L^* and U^*. Thus, the matrix of the approximate factorization is $A^* = L^*U^*$, and the error matrix is $E^* = A - A^*$. We will apply this idea to the first order schemes based on pressure extrapolation and on velocity extrapolation. For the application to second and third order schemes based on pressure extrapolation, see [5].

6.1 First order pressure extrapolation scheme as inexact factorization

To simplify the notation, let us introduce the abbreviations

$$B := D_u M_u^{-1} G_u, \quad C_u := \frac{1}{\delta t} M_u + K_u, \quad C_\sigma := \frac{1}{\delta t} M_\sigma + K_\sigma$$

It is understood in all what follows that matrices K_u and K_σ are evaluated with $\tilde{U}^{n+1}$.

If in algorithm (16)-(20) we take $k = 1$ and replace (20) by $\Sigma^{n+1} = \tilde{\Sigma}^{n+1}$ (that can be done for the reasons explained in Subsection 3.2), we may understand this algorithm as the sequence of solving first $L^*\tilde{X}^{n+1} = R^{n+1}$:

$$\begin{aligned}
&C_u \tilde{U}^{n+1} = F_1^{n+1} \\
&C_\sigma \tilde{\Sigma}^{n+1} - G_\sigma \tilde{U}^{n+1} = F_2^{n+1} \\
&-D_u \tilde{U}^{n+1} + \delta t B \tilde{P}^{n+1} - \delta t D_u M_u^{-1} D_\sigma \tilde{\Sigma}^{n+1} = 0
\end{aligned}$$

and then solving $U^* X^{n+1} = \tilde{X}^{n+1}$:

$$\begin{aligned}
&P^{n+1} = \tilde{P}^{n+1} \\
&\Sigma^{n+1} = \tilde{\Sigma}^{n+1} \\
&U^{n+1} + \delta t M_u^{-1} G_u P^{n+1} - \delta t M_u^{-1} D_\sigma \Sigma^{n+1} = \tilde{U}^{n+1}
\end{aligned}$$

Matrices L^* and U^* are now given by

$$L^* = \begin{bmatrix} C_u & 0 & 0 \\ -G_\sigma & C_\sigma & 0 \\ -D_u & -\delta t D_u M_u^{-1} D_\sigma & \delta t B \end{bmatrix}, \quad U^* = \begin{bmatrix} I & -\delta t M_u^{-1} D_\sigma & \delta t M_u^{-1} G_u \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

Thus, matrix A has effectively been approximated by $A \approx A^* = L^*U^*$, where

$$A^* = \begin{bmatrix} C_u & -D_\sigma - \delta t K_u M_u^{-1} D_\sigma & G_u + \delta t K_u M_u^{-1} G_u \\ -G_\sigma & C_\sigma + \delta t G_\sigma M_u^{-1} D_\sigma & -\delta t G_\sigma M_u^{-1} G_u \\ -D_u & 0 & 0 \end{bmatrix}$$

The error matrix of the splitting scheme is

$$E^* := A - A^* = \begin{bmatrix} 0 & \delta t K_u M_u^{-1} D_\sigma & -\delta t K_u M_u^{-1} G_u \\ 0 & -\delta t G_\sigma M_u^{-1} D_\sigma & \delta t G_\sigma M_u^{-1} G_u \\ 0 & 0 & 0 \end{bmatrix}$$

This error matrix allows us to observe which are the terms approximated and that they are of first order in time.

6.2 First order velocity extrapolation scheme as inexact factorization

Schemes based on pressure extrapolation can be cast as a classical inexact *LU* factorization. However, velocity correction schemes fit better as inexact *general* factorizations of the system matrix into block triangular matrices. For Newtonian flows, it was shown in [1] that they can be written as a factorization of the system matrix A into two block triangular matrices, but not the canonical *LU* factorization. In the case of viscoelastic flows, it is convenient to organize the unknowns as $(\Sigma^{n+1}, U^{n+1}, P^{n+1})$ and split the matrix of the system to be solved as the product of *three triangular matrices*. If in algorithm (30)-(34) we take $k = 1$ and neglect (34) (for the reasons explained Subsection 4.2) this splitting is as follows:

$$\begin{aligned} A &= \begin{bmatrix} C_\sigma & -G_\sigma & 0 \\ -D_\sigma & C_u & G_u \\ 0 & D_u & 0 \end{bmatrix} \\ &\approx \begin{bmatrix} I_\sigma & 0 & 0 \\ 0 & I_u & 0 \\ 0 & \delta t D_u M_u^{-1} & -\delta t D_u M_u^{-1} G_u \end{bmatrix} \begin{bmatrix} C_\sigma & -G_\sigma & 0 \\ 0 & C_u & G_u \\ 0 & 0 & I_p \end{bmatrix} \begin{bmatrix} I_\sigma & 0 & 0 \\ -\delta t M_u^{-1} D_\sigma & I_u & 0 \\ 0 & 0 & I_p \end{bmatrix} \\ &= \begin{bmatrix} C_\sigma & -G_\sigma & 0 \\ -D_\sigma - E_{u\sigma} & C_u & G_u \\ -E_{p\sigma} & D_u - E_{pu} & 0 \end{bmatrix} =: A^* \end{aligned} \tag{38}$$

where I_σ, I_u and I_p are the identity matrices corresponding to stress, velocity and pressure, respectively, and the error terms are:

$$\begin{aligned} E_{u\sigma} &= \delta t K_u M_u^{-1} D_\sigma = \mathscr{O}(\delta t) \\ E_{p\sigma} &= \delta t^2 D_u M_u^{-1} C_u M_u^{-1} D_\sigma = \mathscr{O}(\delta t) \\ E_{pu} &= -\delta t M_u^{-1} K_u = \mathscr{O}(\delta t) \end{aligned}$$

which are all of order δt.

In order to check that this splitting corresponds to (30)-(33), let us write now the approximate factorization (38) as $A^* = T_{(1)} T_{(2)} T_{(3)}$, where matrices $T_{(i)}$, $i = 1, 2, 3$,

are all block triangular. This is what allows us to solve for the different unknowns in an uncoupled way. Problem $T_{(1)}X_{(1)} = R^{n+1}$, with $X_{(1)} = (\Sigma_{(1)}^{n+1}, U_{(1)}^{n+1}, P_{(1)}^{n+1})$ and $R^{n+1} = (\frac{1}{\delta t}M_\sigma \Sigma^n, F^{n+1} + \frac{1}{\delta t}M_u U^n, 0)$ yields:

$$\begin{aligned}
&\Sigma_{(1)}^{n+1} = \frac{1}{\delta t}M_\sigma \Sigma^n \\
&U_{(1)}^{n+1} = F^{n+1} + \frac{1}{\delta t}M_u U^n \\
&\delta t D_u M_u^{-1} G_u P_{(1)}^{n+1} = \delta t D_u M_u^{-1} U_{(1)}^{n+1} = \delta t D_u M_u^{-1} F^{n+1} + D_u U^n
\end{aligned}$$

from where it follows that $P_{(1)}^{n+1} = \tilde{P}^{n+1}$ is the solution of (30) (with $k = 1$). Solving now $T_{(2)}X_{(2)} = X_{(1)}$ yields:

$$\begin{aligned}
&P_{(2)}^{n+1} = P_{(1)}^{n+1} = \tilde{P}^{n+1} \\
&C_u U_{(2)}^{n+1} + G_u P_{(2)}^{n+1} = U_{(1)}^{n+1} \iff C_u U_{(2)}^{n+1} = F^{n+1} + \frac{1}{\delta t}M_u U^n - G_u \tilde{P}^{n+1} \\
&C_\sigma \Sigma_{(2)}^{n+1} - G_\sigma U_{(2)}^{n+1} = \Sigma_{(1)}^{n+1} \iff C_\sigma \Sigma_{(2)}^{n+1} - G_\sigma U_{(2)}^{n+1} = \frac{1}{\delta t}M_\sigma \Sigma^n
\end{aligned}$$

from where it follows that $U_{(2)}^{n+1} = \tilde{U}^{n+1}$ is the solution of (31) and $\Sigma_{(2)}^{n+1} = \Sigma^{n+1}$ the solution of (32), with $k = 1$ in both cases. Finally, solving $T_{(3)}X_{(3)} = X_{(2)}$ yields:

$$\begin{aligned}
&\Sigma_{(3)}^{n+1} = \Sigma_{(2)}^{n+1} = \Sigma^{n+1} \\
&-\delta t M_u^{-1} D_\sigma \Sigma_{(3)}^{n+1} + U_{(3)}^{n+1} = U_{(2)}^{n+1} \iff U_{(3)}^{n+1} = \tilde{U}^{n+1} + \delta t M_u^{-1} D_\sigma \Sigma^{n+1} \\
&P_{(3)}^{n+1} = P_{(2)}^{n+1} = \tilde{P}^{n+1}
\end{aligned}$$

from where $U_{(3)}^{n+1} = U^{n+1}$ is the solution of (33) with $k = 1$. Therefore, $X_{(3)}$ is the solution of the first order version of (30)-(33), thus proving that this algorithm corresponds to the inexact factorization (38).

7 Conclusions

In this article we have explained the main aspects related to the design of fractional step schemes for viscoelastic flows at the purely algebraic level. The design of the algorithms has taken as starting point the fully discrete problem, discretized both in space and in time. The driving idea in all cases is to extrapolate one variable to allow the uncoupled calculation of the others and then to make a correction to maintain the implicitness of the original time integration. Two families of schemes have been presented, one based on pressure (and stress) extrapolation and the other based on velocity (and stress) extrapolation. In the former case, the modifications required to design a third order scheme have been explained, whereas the latter has

been motivated from a discrete pressure Poisson equation that does not have the theoretical difficulties of the continuous one.

A first way to understand the properties of the schemes proposed, and in particular their order of accuracy, is to write the equivalent monolithic problem. This shows which equations of the original system are approximated a how. The interpretation of the schemes as inexact factorization serves the same target, and is also a source of inspiration to design other fractional steps schemes.

Comments about the stability of the schemes have been also provided. Summarizing, one can prove at the discrete level the same stability results as those that hold for the continuous counterpart, although using purely algebraic concepts.

Many of the points treated deserve further research. Related to the last point, for example, the stability of third order schemes has not been undertaken, and the analysis of either inf-sup stable or stabilized formulations has many gaps to be filled, although we have tried to explain the main lines. The same happens with the identification of inexact factorizations for all the schemes proposed, and even the analysis of modifications that these factorizations suggest. Needless to say that all what has been presented could be applied to time integration schemes other that BDF. The usefulness of algebraic fractional step schemes to design preconditioners for linear solvers has not even been touched. Nevertheless, our objective has been to provide a global picture of this way to approach fractional step methods in computational fluid mechanics, particularly applied to viscoelastic fluids.

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