Variational Multiscale Methods in Computational Fluid Dynamics

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ABSTRACT

This article describes the Variational Multiscale Method (VMS) applied to flow problems. The main idea of the formulation in the case of stationary linear problems is explained with some detail and, at the same time, generality. However, when moving to particular problems the focus is directed to the description of different approaches that have appeared in the literature, mainly in the last decade.

KEY WORDS: Variational Multiscale; stabilized finite element methods; compatibility conditions; singularly perturbed problems; flow problems; Navier-Stokes equations; turbulence

1. INTRODUCTION

This article presents a survey of Variational Multiscale (VMS) finite element (FE) methods applied to some problems appearing in computational fluid dynamics (CFD). Since its introduction in the 90s in (Hughes, 1995; Hughes, Feijóo, Mazzei, and Quincy, 1998), the VMS framework has been applied to design *stabilized* FE methods in problems in which stability of the standard Galerkin method is not ensured. This may be due essentially to two reasons. First, singularly perturbed problems require extremely fine meshes, unaffordable in most cases from the computational point of view, to obtain stability bounds numerically meaningful and leading to convergence; in flow problems, the paradigmatic example of this situation is the convection-diffusion equation in the convection-dominated limit. Second, very often in mixed problems, involving variables of different nature, the FE spaces chosen to approximate them need to satisfy compatibility conditions to render the problem stable, i.e., satisfying the necessary and sufficient inf-sup condition for stability; the most well known example of this situation in fluid mechanics is the Stokes problem.

Both instabilities related to singular perturbation and to compatibility conditions of the interpolation can be overcome by resorting to *stabilized* FE methods. *Stabilization* is a broad term that encompasses many numerical techniques. In this article we review some of them, namely, those that can be framed within the general VMS concept and using finite elements. It is not our purpose to trace the historical development of the methods to be presented, for which the reader is referred to (Hughes, Scovazzi, and Franca, 2004). We also refer to this article for background and for additional concepts and information; in fact, the present article can be considered a follow up of the cited one, with a different style and focus, and mainly concentrating on the developments that have been produced during the last decade.

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In order to encompass a spectrum of problems as broad as possible, we start in Section 2 presenting the VMS concept for a general second order partial differential equation. This concept is presented in full generality, but without the motivation that can be found in (Hughes, Scovazzi, and Franca, 2004). There it is introduced after analyzing an acoustics problem, which involves two scales, the near field and the far field ones, say u' and $\bar{u}$. The far field component of the solution can be eliminated (using for example a Dirichlet-to-Neumann operator), thus obtaining and a problem posed for the near field solution only. This is analogous to the essential VMS concept, which consists in splitting the unknown, say u, onto a component $\bar{u}$ resolvable by the FE space and another one, u', that cannot be captured by the approximation. As for the acoustics example, the effect of u' on $\bar{u}$ is always nonlocal. In the FE context, $\bar{u}$ will be identified with the FE solution, denoted by u_h as usual, and u' will be referred to as *subgrid scale* or, simply, *subscale*.

VMS is not a method, but a framework. Different numerical methods can be obtained when the subscales are approximated. This is the objective of Section 3. In general, the exact problem for u_h can be written making use of the problem's Green's function. VMS-type methods can be considered as some sort of approximation to this function. For example, if p is the interpolation order, hierarchical p-refinement can be considered as a way to generate an approximation to the fine-scale Green's function. Different ways to tackle the subscale approximation are described. Let us clarify that we will not consider local enrichment methods as members of the VMS family, as they are not motivated as approximations to the subscale *problem*, but to enrich the FE solution.

The main objective of VMS approximations is to design stabilized FE methods, in the sense explained above. After Sections 2 and 3, we adopt a more descriptive style to acknowledge some stabilization methods that are difficult to be termed as of VMS-type but are closely related and, in the case of nonlinear stabilization, complementary.

Section 5 deals with the approximation of the incompressible Navier-Stokes equations, possibly the most important problem in CFD. There we present in some detail *a particular* VMS-type formulation and discuss the important topic of the relationship between modeling the subscales from the VMS point of view and modeling turbulence. Even if the material presented is the one closest to our research, we also refer to some of the related methods that have appeared in this context in the last years. However, we do not pretend to be exhaustive, and important works are possibly not referenced.

The descriptive point of view is again adopted in Section 6, where we refer to some VMS formulations (or closely related) that have been designed to approximate compressible flows, low Mach number flows, flow in porous media, wave problems and magneto-hydrodynamics (MHD). These are flow problems with difficulties related but not equal to those encountered in the incompressible Navier-Stokes equations. No reference is made to VMS strategies for problems in areas other than fluid mechanics. The article concludes with a description of the essential ideas in the numerical analysis of VMS methods in Section 7. An accurate description of the analytical tools required to prove stability and convergence of the methods presented would by far exceed the scope of this article; rather than rigor, our aim is to point out which is the key of the success of VMS-type formulations from the analysis point of view.

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2. THE VARIATIONAL MULTI-SCALE APPROACH: A GENERAL FRAMEWORK

2.1. Continuous problem

Let Ω be a bounded domain in $\mathbb{R}^d$, with boundary $\Gamma = \partial\Omega$. Given a function $f : \Omega \longrightarrow \mathbb{R}^n$, the problem we consider consists in finding a function $u : \Omega \longrightarrow \mathbb{R}^n$ such that

$$\mathcal{L}u = f \quad \text{in } \Omega \tag{1}$$

$$\mathcal{D}u = 0 \quad \text{on } \Gamma \tag{2}$$

where $\mathcal{L}$ is a linear differential operator (in space) and $\mathcal{D}$ is an adequate trace operator that makes the problem well posed. To simplify the exposition, we have considered the simplest homogeneous Dirichlet boundary condition (2).

In the flow problems in which we are interested, operator $\mathcal{L}$ may be either of first or of second order. We write it as

$$\mathcal{L}u = -\partial_i(K_{ij}\partial_j u) + A_{\mathrm{c},i}\partial_i u + A_{\mathrm{f},i}\partial_i u + Su \tag{3}$$

where K_{ij}, $A_{\mathrm{c},i}$, $A_{\mathrm{f},i}$ and S are $n \times n$ matrices, ∂_i denotes differentiation with respect to the i-th Cartesian coordinate x_i, indexes i and j run from 1 to d and repeated indexes imply summation. Matrices K_{ij} are assumed to define a symmetric semi-positive quadratic form on $(\mathbb{R}^n)^d \times (\mathbb{R}^n)^d$ (in particular, $K_{ij}^T = K_{ji}$), whereas matrices $A_{\mathrm{f},i}$ are those that will contribute to the fluxes of the problem (see below). Note that, for the moment, both u and f are assumed to be arrays in $\mathbb{R}^n$, without any geometric structure (they do not need to be *vectors*); in fact, they may consist of heterogeneous components. We just assume that $u^T f$ scales correctly or, in physical terms, is dimensionally consistent. Likewise, matrices in (3) may have heterogeneous components.

Particular examples of (3) appearing in flow problems will be considered below. Even if most of these problems will be nonlinear, the general framework described in this section will be restricted to the linear case.

Let $\omega \subset \Omega$. The integral of the product of two functions f_1 and f_2 defined in ω will be denoted by $\langle f_1, f_2\rangle_\omega$. If f_1 and f_2 are arrays, this is understood as the component-wise integration of the terms in $f_1^T f_2$. In case $f_1,\ f_2 \in L^2(\omega)$, the $L^2(\omega)$-inner product will be written as $(f_1, f_2)_\omega$. The domain subscript will be omitted when $\omega = \Omega$.

Let us define the operators

$$\mathcal{L}^* v := -\partial_i(K_{ij}\partial_j v) - \partial_i(A_{\mathrm{c},i}^T v) - \partial_i(A_{\mathrm{f},i}^T v) + S^T v \tag{4}$$

$$\mathcal{F}_n u := n_i K_{ij}\partial_j u - n_i A_{\mathrm{f},i} u \tag{5}$$

$$\mathcal{F}_n^* v := n_i K_{ij}\partial_j v + n_i A_{\mathrm{c},i}^T v \tag{6}$$

where n_i is the i-th component of the unit normal exterior to Γ. Operator $\mathcal{L}^*$ is defined over functions in Ω, whereas the other two over functions on Γ. Suppose that matrices K_{ij}, $A_{\mathrm{c},i}$ and $A_{\mathrm{f},i}$ are such that the image of $\mathcal{F}_n u$ is a subspace of rank $m \leq n$ of $\mathbb{R}^n$, and define $\mathcal{D}u$ to be a trace of u on this subspace; $\mathcal{D}^* v$ could defined analogously for $\mathcal{F}_n^* v$, although we will interested in cases in which $\mathcal{D}^* v = \mathcal{D}v$. If now we introduce the bilinear form B_ω on functions defined on $\omega \subset \Omega$ as

$$B_\omega(u, v) := \langle \partial_i v, K_{ij}\partial_j v\rangle_\omega + \langle v, A_{\mathrm{c},i}\partial_i u\rangle_\omega - \langle \partial_i(A_{\mathrm{f},i}^T v), u\rangle_\omega + \langle v, Su\rangle_\omega \tag{7}$$

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we may write the identities

$$B_\omega(u,v) = \langle \mathcal{L}u, v\rangle_\omega + \langle \mathcal{F}_n u, \mathcal{D}v\rangle_{\partial\omega} \tag{8}$$

$$= \langle u, \mathcal{L}^* v\rangle_\omega + \langle \mathcal{D}u, \mathcal{F}_n^* v\rangle_{\partial\omega} \tag{9}$$

Let also $L_\omega(v) = \langle f, v\rangle_\omega$. Both in B_ω and in L_ω subscripts will be omitted when $\omega = \Omega$. Problem (1)-(2) is equivalent to find *the unknown* u such that $\mathcal{D}u = 0$ on Γ and

$$B(u,v) = L(v) \tag{10}$$

for all *test functions* v such that $\mathcal{D}v = 0$ on Γ. Regularity requirements on u and v depend obviously on the problem, and will be specified when necessary. Using the additivity of the integral and (8) it can be shown in particular that $\mathcal{D}u$ needs to be continuous across interfaces in the interior Ω and the jump of the normal fluxes $\mathcal{F}_n u$ must also be zero.

2.2. Finite element discretization

For the sake of simplicity, let us assume the Ω is a polyhedral domain for which we can consider a family of FE partitions $\mathcal{P}_h = \{K\}$, where K denotes a generic FE domain, h_K will be its diameter and $h = \max_K\{h_K\}$ the mesh size. All FE functions will be identified with the subscript h. We will consider the same interpolation order p for all the elements, either simplicial or quadrilateral/hexahedral (in 2D/3D). We may however use different interpolation orders for the different components of u. The number of elements of $\mathcal{P}_h$ will be denoted by $n_{\rm el}$ and the number of nodes per element by $n_{\rm nod}$.

All approximations we shall consider are conforming, i.e., the FE spaces will be chosen as finite dimensional subspaces of the functional spaces where the problem is posed, identified for each of the problems to be considered in the following. The important case of discontinuous Galerkin approximations is thus excluded.

Let V be the functional space where the continuous problem is posed and $V_h \subset V$ the FE approximation. We will be interested in cases in which the test functions belong to the same space as the unknown.

Once the the FE approximation is set, the Galerkin approximation to problem (10) consists of finding $u_h \in V_h$ such that

$$B(u_h, v_h) = L(v_h) \quad \text{for all } v_h \in V_h \tag{11}$$

Since $B(u_h, v_h) = \sum_K B_K(u_h, v_h)$, from (8) it follows that we have to choose V_h such that $\mathcal{D}u_h$ is continuous across interlement boundaries and the solution obtained from (11) will be such that the discrete fluxes $\mathcal{F}_n u_h$ will be weakly continuous.

2.3. Scale splitting

It is well known that the solution u_h to the discrete problem (11) may suffer from numerical instabilities, which may be classified into two groups, namely, those corresponding to singularly perturbed problems and those that come from compatibility restrictions between the components of u. They all depend on the expression of the matrices that define operator $\mathcal{L}$ in (3). It is not our purpose to review them here, but only to present the VMS framework and show how it can allow one to solve those that appear in the context of flow problems. In particular, instabilities of the first group appear when diffusion (second order) terms are

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small compared to convective (first order) terms and the prototype compatibility conditions are those encountered in the Stokes and the Darcy problems described later.

The *key idea* of the VMS approach is to split space V as (Hughes, 1995; Hughes, Feijóo, Mazzei, and Quincy, 1998):

$$V = V_h \oplus V' \tag{12}$$

where V_h is the FE space and V' is any complement to it in V. Each VMS-type method will depend precisely on the way V' is approximated. We will still denote the approximation as V', since no confusion will be possible.

Splitting (12) will have the associated splitting of the unknowns and tests functions $u = u_h + u'$ and $v = v_h + v'$, with $u_h, v_h \in V_h$ and $u', v' \in V'$. Because of the linearity of the problem considered so far, we may write the continuous problem (10) as: find $u_h \in V_h$ and $u' \in V'$ such that

$$B(u_h, v_h) + B(u', v_h) = L(v_h) \quad \text{for all } v_h \in V_h \tag{13}$$

$$B(u_h, v') + B(u', v') = L(v') \quad \text{for all } v' \in V' \tag{14}$$

No approximation has been done, yet. Choosing $V' = \{0\}$ would yield the Galerkin method, but better options are expected to be found when this is unstable. If fact, the approximation to V' will be a consequence of the approximation to u' (still denoted u'). In order to avoid approximating *derivatives* of u', and require only the unknown u' itself, making use of the additivity of the integral and (9) we may write (13) as

$$B(u_h, v_h) + \sum_K \big[\langle u', \mathcal{L}^* v_h\rangle_K + \langle \mathcal{D}u', \mathcal{F}_n^* v_h\rangle_{\partial K}\big] = L(v_h) \quad \text{for all } v_h \in V_h \tag{15}$$

Since in the FE approximation v_h is piecewise polynomial, $\mathcal{L}^* v_h$ and $\mathcal{F}_n^* v_h$ are well defined element-wise.

Equation (15) will be our FE problem *once u' is approximated in terms of u_h*. The description of some possible ways to attempt this is the topic of the following section. In any case, this approximation must be obtained from (14), which using (8) may be written as

$$\begin{aligned} B(u', v') &= \sum_K \big[\langle \mathcal{L}u', v'\rangle_K + \langle \mathcal{F}_n u', \mathcal{D}v'\rangle_{\partial K}\big] \\ &= L(v') - B(u_h, v') \\ &= L(v') - \sum_K \big[\langle \mathcal{L}u_h, v'\rangle_K + \langle \mathcal{F}_n u_h, \mathcal{D}v'\rangle_{\partial K}\big] \quad \text{for all } v' \in V' \end{aligned} \tag{16}$$

All VMS-type methods consists in approximating u' from (16).

2.4. Simple model problems

Even if the approach described is quite general, its development came through the analysis of simple model problems. In fact, for some of the issues to be described in the following it is also convenient to refer to these simple problems. These are the convection-diffusion-reaction (CDR) equation, the Stokes problem and the Darcy problem. The instabilities that appear in the Galerkin approximation of the former serve as an example of what may happen in singularly perturbed problems when diffusion is small, whereas the Stokes and the Darcy problems are

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the prototypes of problems requiring compatibility conditions between the interpolation of the variables in play.

The CDR equation we consider reads as follows: find $u : \Omega \longrightarrow \mathbb{R}$ such that

$$-\kappa \Delta u + \boldsymbol{a} \cdot \nabla u + s\, u = f \quad \text{in } \Omega \tag{17}$$

$$u = 0 \quad \text{on } \Gamma \tag{18}$$

where $\kappa > 0$ is the diffusion coefficient, $\boldsymbol{a}$ the advection velocity and s the reaction (in fact, absorption) coefficient, which have to satisfy $s - \frac{1}{2}\nabla \cdot \boldsymbol{a} \geq 0$ in order to have a well posed problem.

The Stokes problem consists of finding a velocity field $\boldsymbol{u} : \Omega \longrightarrow \mathbb{R}^d$ and a pressure field $p : \Omega \longrightarrow \mathbb{R}$ such that

$$-\nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega \tag{19}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega \tag{20}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma \tag{21}$$

where $\nu > 0$ is the kinematic viscosity and $\boldsymbol{f}$ the vector of body forces. Here and in what follows, a boldface character is used for vector fields.

Finally, the Darcy problem consists of inding a velocity field $\boldsymbol{u} : \Omega \longrightarrow \mathbb{R}^d$ and a pressure field $p : \Omega \longrightarrow \mathbb{R}$ such that

$$\sigma \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega \tag{22}$$

$$\nabla \cdot \boldsymbol{u} = g \quad \text{in } \Omega \tag{23}$$

together with appropriate boundary conditions that depend on the functional setting of the problem, which can be either $p = 0$ on Γ for the primal form of the problem or $\boldsymbol{n} \cdot \boldsymbol{u} = 0$ (zero normal component of $\boldsymbol{u}$) on Γ for the dual form. The coefficient $\sigma > 0$ is the inverse of the permeability and $\boldsymbol{f}$ and g are given data.

Table I summarizes the variational form of the three problems considered, as well as their correspondence with the general setting presented in the previous sections. The notation used in this table is standard: $H^1(\omega)$ stands for the space of functions which are in $L^2(\omega)$ with derivatives in $L^2(\omega)$, $H^1_0(\omega)$ for the space of functions in $H^1(\omega)$ that vanish on $\partial\omega$, $L^2_0(\omega)$ for the space of functions in $L^2(\omega)$ with zero mean, $H(\mathrm{div};\omega)$ for vector fields in $L^2(\omega)^d$ with divergence in $L^2(\omega)$ and $H_0(\mathrm{div};\omega)$ for the subspace made of vector fields with zero normal trace. The norm of a function f in a functional space X will be denoted by $\|f\|_X$. In the rest of the article, the same symbols in the first column of Table I will be used for all problems, being clear by the context which is the problem considered.

Different VMS-type formulations have been developed for the model problems listed, and in fact we shall refer to these model problems in the following sections. Other models could also be discussed, but the three presented are those more relevant to flow problems. Perhaps Maxwell's problem could be considered in the frontier of CFD and other areas of computational physics. While we will need to refer to it when describing VMS methods for MHD, we will not attempt to present Maxwell's problem in depth and the challenges of its numerical approximation.

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	CDR	Stokes	Darcy Primal	Darcy Dual
u	u $(n=1)$	$[\boldsymbol{u}, p]$ $(n=d+1)$	$[\boldsymbol{u}, p]$ $(n=d+1)$	$[\boldsymbol{u}, p]$ $(n=d+1)$
V	$H_0^1(\Omega)$	$H_0^1(\Omega)^d \times L_0^2(\Omega)$	$L^2(\Omega)^d \times H_0^1(\Omega)$	$H_0(\mathrm{div};\Omega) \times L^2(\Omega)$
$\mathcal{L}u$	$-\kappa\Delta u$ $+\boldsymbol{a}\cdot\nabla u + su$	$[-\nu\Delta\boldsymbol{u} + \nabla p,$ $\nabla\cdot\boldsymbol{u}]$	$[\sigma\boldsymbol{u} + \nabla p,$ $\nabla\cdot\boldsymbol{u}]$	$[\sigma\boldsymbol{u} + \nabla p,$ $\nabla\cdot\boldsymbol{u}]$
$\mathcal{F}_n u$	$\kappa\frac{\partial u}{\partial n}$	$-\nu\frac{\partial \boldsymbol{u}}{\partial n} + p\,\boldsymbol{n}$	$-\boldsymbol{u}\cdot\boldsymbol{n}$	$p\,\boldsymbol{n}$
$\mathcal{D}u$	u	$\boldsymbol{u}$	p	$\boldsymbol{n}\cdot\boldsymbol{u}$
$B(u,v)$	$\kappa(\nabla u, \nabla v)$ $+(\boldsymbol{a}\cdot\nabla u, v)$ $+s(u,v)$	$\nu(\nabla\boldsymbol{u}, \nabla\boldsymbol{v})$ $-(p, \nabla\cdot\boldsymbol{v})$ $+(q, \nabla\cdot\boldsymbol{u})$	$\sigma(\boldsymbol{u}, \boldsymbol{v})$ $+(\nabla p, \boldsymbol{v})$ $-(\nabla q, \boldsymbol{u})$	$\sigma(\boldsymbol{u}, \boldsymbol{v})$ $-(p, \nabla\cdot\boldsymbol{v})$ $+(q, \nabla\cdot\boldsymbol{u})$

Table I. Simple model problems in fluid mechanics

3. APPROXIMATION OF THE SUBGRID SCALES

In this section we discuss different strategies for the solution of the subscale problem. Because the subscale problem is infinite dimensional, some approximation needs to be introduced to make the method computationally feasible.

The fine scale problem (16) can be written as

$$\sum_K [\langle v', \mathcal{L}u'\rangle_K + \langle \mathcal{D}v', \mathcal{F}_n u'\rangle_{\partial K}] = \sum_K [\langle v', \mathcal{R}u_h\rangle_K - \langle \mathcal{D}v', \mathcal{F}_n u_h\rangle_{\partial K}]$$

where $\mathcal{R}u_h = f - \mathcal{L}u_h$ is the (strong) FE residual.

Let us denote by $\mathcal{E}_h$ the set of element edges (faces when $d=3$) including those on the boundary of the domain $\partial\Omega$, denoted by $\mathcal{E}_h^\Gamma$, and the set of internal edges, denoted by $\mathcal{E}_h^0$. Using this notation the boundary terms can be grouped as

$$\begin{aligned}\sum_K [\langle \mathcal{D}v', \mathcal{F}_n u_h\rangle_{\partial K} + \langle \mathcal{D}v', \mathcal{F}_n u'\rangle_{\partial K}] &= \sum_K \langle \mathcal{D}v', \mathcal{F}_n u\rangle_{\partial K} \\ &= \sum_{E\in\mathcal{E}_h^0} \langle \mathcal{D}v', [\![\mathcal{F}_n u]\!]\rangle_E + \sum_{E\in\mathcal{E}_h^\Gamma} \langle \mathcal{D}v', \mathcal{F}_n u\rangle_E \end{aligned} \tag{24}$$

where $[\![\mathcal{F}_n u]\!]$ denotes the jump of $\mathcal{F}_n u$, i.e. $\mathcal{F}_{n^+}u + \mathcal{F}_{n^-}u$, where $\boldsymbol{n}^\pm$ denotes the external normal to each of the elements $K^\pm$ that share edge E. Because the normal fluxes of the total unknown are continuous across the interelement boundaries the first term in the right-hand side (RHS) vanishes and we arrive to

$$\sum_K \langle v', \mathcal{L}u'\rangle_K + \sum_{E\in\mathcal{E}_h^\Gamma} \langle \mathcal{D}v', \mathcal{F}_n u'\rangle_E = \sum_K \langle v', \mathcal{R}u_h\rangle_K - \sum_{E\in\mathcal{E}_h^\Gamma} \langle \mathcal{D}v', \mathcal{F}_n u_h\rangle_E \tag{25}$$

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The boundary terms are also zero if we impose that $\mathcal{D}u' = 0$ on Γ in an essential way, so that the subscale test function also satisfies $\mathcal{D}v' = 0$ on Γ. Thus, (25) is equivalent to find $u' \in V'$ such that $\mathcal{D}u' = 0$ on Γ and

$$\mathcal{L}u' = \mathcal{R}u_h + v'^{\perp} \quad \text{in } K, \text{ for all } K \in \mathcal{P}_h \tag{26}$$

$$\mathcal{D}u' = \mathcal{D}u'_{\text{ske}} \quad \text{on } E, \text{ for all } E \in \mathcal{E}_h^0 \tag{27}$$

where u'_{ske} is the trace of the subscales, defined implicitly by the continuity of normal fluxes across interelement boundaries. In turn, $v'^{\perp}$ is any function satisfying

$$\sum_K \langle v', v'^{\perp} \rangle_K = 0, \quad \forall v' \in V' \tag{28}$$

that is, any function in $V'^{\perp}$, the orthogonal complement of V', and it is implicitly defined by a choice of the space of subscales (it is determined by the condition $u' \in V'$).

Therefore, in order to obtain a numerical method (i.e. defined in a discrete space) the following approximating steps need to be followed:

- **Approximate u'_{ske}.** Assuming that the subscale problem on each element could be exactly solved, i.e. that the inverse operator $\mathcal{L}^{-1}$ could be exactly applied, determining u'_{ske} would require the solution of a global problem, that of satisfying continuity of fluxes across edges. The same problem is faced in domain decompostion methods to obtain the traces of the unknown across subdomain boundaries and it is usually approximated by some relaxation of the continuity of fluxes to reduce the size of this coarse problem. Some two level methods (where local problems are also approximated) (Harder, Paredes, and Valentin, 2015) can be understood from this point of view. The most common approach, however, is to set $u'_{\text{ske}} = 0$, that is, to take V' as the (infinite dimensional) space of bubble functions. After this approximation, the subscale problem (26)-(27) is decoupled into n_{el} independent problems.
- **Approximately solve for u'.** The solution of the fine scale problem can be formally written as
$$u' = \mathcal{L}^{-1}\left(\mathcal{R}u_h + v'^{\perp}\right) \tag{29}$$
As already mentined, because $\mathcal{L}^{-1}$ cannot be computed some approximation is needed. Once u'_{ske} has been defined, the approximation of $\mathcal{L}^{-1}$ can be performed element by element, as it will be discussed in the following subsections, arriving to an algebraic equation of the form
$$u'|_K \approx \tau_K \left(\mathcal{R}u_h + v'^{\perp}\right)|_K \tag{30}$$
where τ_K is a matrix that approximates the inverse of the differential operator on each element K that we will call the *matrix of stabilization parameters*. Different approximations for τ_K yield different VMS methods.
- **Choose $v'^{\perp}$.** Function $v'^{\perp}$ in (26)-(27) is determined by the selection of the space of subscales. This choice was considered first in (Codina, 2000), where it is shown that the pressure stabilization provided by some classical fractional step methods for incompressible flows is similar to that provided by the VMS formulations provided the subscales are taken orthogonal to the FE space. Condition (28) can be imposed to (30) using the weighted inner product
$$(\cdot, \cdot)_\tau = \sum_K \langle \tau_K \cdot, \cdot \rangle_K$$

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and the associated projection ${P'_\tau}^\perp$ onto ${V'}^\perp$. The projection ${P'_\tau}^\perp$ differs from the $L^2(\Omega)$ projection ${P'}^\perp$ in the element-by-element weights τ_K. If τ_K is the same for all elements K we have that ${P'_\tau}^\perp = {P'}^\perp$. Testing (30) with ${w'}^\perp$ and using (28) we get ${v'}^\perp = -{P'_\tau}^\perp \mathcal{R}u_h$ and therefore

$$u'|_K = \tau_K P'_\tau \mathcal{R}u_h|_K$$

where $P'_\tau = I - {P'_\tau}^\perp$ is the projection onto the subscale space V' (I is the identity in V). A typical choice of the subscales space is given by $P'_\tau = I$ which is called in (Codina, 2002) the Algebraic Subgrid-Scale formulation (ASGS) and consists simply in taking ${v'}^\perp = 0$. In that reference the choice $P'_\tau = I - P_h := P_h^\perp$ is advocated, P_h being the $L^2(\Omega)$ projection onto the FE space. The resulting formulation is called Orthogonal Subscales Stabilization (OSS) because when τ_K is the same for all elements this choice corresponds to take V' as the orthogonal complement of V_h. If the element-by-element variation of the stabilization parameter is to be considered, in order to have $V' = V_h^\perp$ we need to take $P'_\tau = I - P_{h\tau}$ where $P_{h\tau}$ is the projection onto the FE space in the sense of $(\cdot,\cdot)_\tau$. However, as the $L^2(\Omega)$ projection is very convenient from a computational point of view, the first choice is generally considered and in this case we have $u'|_K = \tau_K P_h^\perp \mathcal{R}u_h$. The application of condition (28) to (29) was considered in (Hughes and Sangalli, 2007) where a closed expression for the subscales was obtained, namely

$$u' = \mathcal{L}^{-1}\left(I - P_h^T\left(P_h\mathcal{L}^{-1}P_h^T\right)^{-1}P_h\right)\mathcal{R}u_h$$

where P_h^T is the transposed projector.

In the following subsections we present some different strategies to build the approximation to $\mathcal{L}^{-1}$, all of them assuming $u_{\text{ske}} = 0$ and that ${v'}^\perp = 0$. This condition however could be easily relaxed by considering the appropriate projection of the FE residual instead of this residual itself. Except when explicitly stated (as in Section 5) the subscale approximation within each element K will be

$$u'|_K = \tau_K \mathcal{R}u_h|_K \tag{31}$$

3.1. Approximate Green functions

The solution of problem (26)-(27) can be obtained making use of the Green function within each element, defined as the solution of the problem

$$\mathcal{L}_x g(\boldsymbol{x},\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y}) \quad \text{in } K \tag{32}$$

$$\mathcal{D}g(\boldsymbol{x},\boldsymbol{y}) = 0 \quad \text{on } \partial K \tag{33}$$

for all $K \in \mathcal{P}_h$, where $\delta(\boldsymbol{x}-\boldsymbol{y})$ is the Dirac delta distribution and $\mathcal{L}_x$ specifies that derivatives in $\mathcal{L}$ are taken with respect to $\boldsymbol{x}$. We then have

$$u'(\boldsymbol{x})|_K = \int_K g(\boldsymbol{x},\boldsymbol{y})\mathcal{R}u_h(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \tag{34}$$

Now the problem is how to solve (32)-(33), which is infinite dimensional. One possibility is to consider the approximation

$$g(\boldsymbol{x},\boldsymbol{y})|_K \approx \tau_K(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{35}$$

from where (30) is recovered. A possible way to determine $\tau_K(\boldsymbol{x})$ is to impose that moments of $g(\boldsymbol{x},\boldsymbol{y})|_K$ and $\tau_K(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y})$ coincide. For example, when τ_K is constant:

$$\tau_K = |K|^{-1}\int_K\int_K g(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \tag{36}$$

where $|K|$ is the measure of K. See (Hughes, Scovazzi, and Franca, 2004) for additional details.

3.2. (Residual free) bubbles

Let us introduce the space of bubble functions

$$V_K^b = \{w \in V_K : w = 0 \text{ on } \partial K\}$$

where V_K is the space of functions in V restricted to element K. Let n_{bub} be the dimension of V_K^b. Because of the assumption $u_{\text{ske}} = 0$ problem (26)-(27) can be written on each element as: find $u' \in V_K^b$ such that

$$B(u', v') = L(v') - B(u_h, v')$$

We can consider the approximation of V_K^b by a finite dimensional subspace $\text{Span}\{b_1, ..., b_{n_{\text{bub}}}\}$, which leads to the discrete problem

$$\sum_{j=1}^{n_{\text{bub}}} B(b_j, b_i)u'_j = L(b_i) - B(u_h, b_i) = \int_K b_i\,(f - \mathcal{L}u_h)$$

where u'_j are the nodal values of u'. Introducing the matrix of components $B_{ij} = B(b_j, b_i)$ we can write

$$u' = \sum_{i,j=1}^{n_{\text{bub}}} b_i B^{-1}_{ij}\int_K b_j \mathcal{R}u_h$$

The choice of the bubble functions determine the approximation of the subscale and the approximation properties of the final scheme. We may wish for example that (26)-(27) hold point-wise, i.e.

$$\mathcal{L}u' = f - \mathcal{L}u_h \iff \sum_{i=1}^{n_{\text{bub}}} \mathcal{L}b_i u'_i = f - \sum_{j=1}^{n_{\text{nod}}} \mathcal{L}N_j u_{h,j}$$

where N_j is the shape function associated to the nodal value $u_{h,j}$ of u_h. This condition is satisfied if $n_{\text{bub}} = n_{\text{nod}} + 1$ and bubbles are constructed satisfying

$$\begin{aligned} \mathcal{L}b_i &= -\mathcal{L}N_i, \quad i = 1, \ldots, n_{\text{nod}} \\ \mathcal{L}b_{n_{\text{bub}}} &= f \end{aligned}$$

case in which $u'_i = u_{h,i}$, $i = 1, \ldots, n_{\text{nod}}$ and $u'_{n_{\text{bub}}} = 1$. Because the subscale equations will then be exactly satisfied, these bubbles are called *residual free*. Their construction, as the one of the exact Green function, also requires the solution of infinite dimensional local problems. In fact both approaches have shown to be equivalent in (Brezzi, Franca, Hughes, and Russo, 1997).

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3.3. An approximate Fourier analysis

An approximate Fourier analysis of the subscale problem was proposed first in (Codina, 2002) and later extended in, e.g. (Principe and Codina, 2010) with the objective of determining the functional form the stabilization parameters, i.e. their dependence on the equation coefficients and the mesh size up to algorithmic constants. The main heuristic assumption is that u' is highly fluctuating, and therefore dominated by high wave numbers. As a consequence, we may assume that

- Values of u' on ∂K can be neglected to approximate u' in the interior of K, $K \in \mathcal{P}_h$.
- The Fourier transform can be evaluated as for functions vanishing on ∂K (and extended to $\mathbb{R}^d$ by zero).

The following developments are all considered to be made in a generic element K. If h is the size of this element, the analysis can be performed in terms of the wave number $\boldsymbol{k}/h$, with $\boldsymbol{k}$ dimensionless, or, as in (Principe and Codina, 2010), scaling the problem to a reference domain to obtain the mesh size dependence. Let us denote the Fourier transform by $\hat{}$. From (26), the Fourier transformed equation for the subscales reads

$$\hat{\mathcal{L}}(\boldsymbol{k})\hat{u}'(\boldsymbol{k}) = \widehat{\mathcal{R}u_h}(\boldsymbol{k}) \tag{37}$$

where

$$\hat{\mathcal{L}}(\boldsymbol{k}) = \left(h^{-2}k_i k_j K_{ij} + S\right) + \mathrm{i}h^{-1}k_i\left(A_{\mathrm{c},i} + A_{\mathrm{f},i}\right)$$

with $\mathrm{i} = \sqrt{-1}$. For simplicity, we have assumed that the coefficient matrices are constant. Solving (37) and using the inverse Fourier transform, we have

$$\begin{aligned} u'(\boldsymbol{x}) &= \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\hat{\mathcal{L}}^{-1}(\boldsymbol{k})\widehat{\mathcal{R}u_h}(\boldsymbol{k})\mathrm{d}\boldsymbol{k} \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}}\hat{\mathcal{L}}^{-1}(\boldsymbol{k})\mathcal{R}u_h(\boldsymbol{y})\mathrm{d}\boldsymbol{k}\mathrm{d}\boldsymbol{y} \end{aligned}$$

and from (34) we can identify the Green function of the element as

$$g(\boldsymbol{x},\boldsymbol{y}) = \int_{\mathbb{R}^d} \hat{\mathcal{L}}^{-1}(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}\mathrm{d}\boldsymbol{k} \tag{38}$$

Comparing (38) with (35) it is seen that τ_K (a matrix) can be understood as a certain constant approximation to $\hat{\mathcal{L}}^{-1}(\boldsymbol{k})$ (another matrix). Devising such approximations that yield stable formulations is in general difficult, but we may adopt a simpler approach to approximate τ_K.

First of all, let us start noting that, in general, if $f, g \in \operatorname{range}\mathcal{L}$, and $u, v \in \operatorname{dom}\mathcal{L}$, $f^T g$ and $u^T v$ may not be dimensionally meaningful. Let M be a *scaling matrix*, symmetric, positive-definite and possibly diagonal, that makes the products $f^T M g$ and $u^T M^{-1} v$ *dimensionally consistent*. Let also $|f|_M^2 := f^T M f$, $|u|_{M^{-1}}^2 := u^T M^{-1} u$ and $\|f\|_{L^2_M(K)}^2 = \int_K |f|_M^2$.

We may obtain τ_K *by imposing that* $\|\mathcal{L}\|_{L^2_M(K)} \leq \|\tau_K^{-1}\|_{L^2_M(K)}$. Using Plancherel's Theorem it can be shown that

$$\|\mathcal{L}u\|_{L^2_M(K)}^2 \leq |\widehat{\mathcal{L}}(\boldsymbol{k}^0)|_M^2 \|u\|_{L^2_M(K)}^2$$

	CDR	Stokes	Darcy Primal	Darcy Dual
τ_K	$\left(\frac{c_1\kappa^2}{h^4}+\frac{c_2\lvert\boldsymbol{a}\rvert^2}{h^2}+s^2\right)^{-1/2}$	$\mathrm{diag}\left(\frac{c_1h^2}{\nu}I_d,\nu\right)$	$\mathrm{diag}\left(\frac{c_1}{\sigma}I_d,c_2\sigma h^2\right)$	$\mathrm{diag}\left(\frac{c_1h^2}{\sigma L_0^2}I_d,c_2\sigma L_0^2\right)$

Table II. Stabilization parameters for the model problems

where $\boldsymbol{k}^0$ is a wave number whose existence is guaranteed by the Mean Value Theorem (see (Codina, González-Ondina, Díaz-Hernández, and Principe, 2008)). Therefore, we may choose τ_K such that $|\widehat{\mathcal{L}}(\boldsymbol{k}^0)|_M = |\tau_K^{-1}|_M$. In particular, if

$$\lambda_{\max}(\boldsymbol{k}^0) = \max \mathrm{spec}_{M^{-1}}(\widehat{\mathcal{L}}(\boldsymbol{k}^0)^* M \widehat{\mathcal{L}}(\boldsymbol{k}^0)) \tag{39}$$

with $\lambda \in \mathrm{spec}_{M^{-1}}A$ if there exists x such that $Ax = \lambda M^{-1}x$, we will require that $\tau_K^{-1}M\tau_K^{-1} = \lambda_{\max}M^{-1}$, that is to say, the design condition we may use is

$$M\tau_K^{-1} = \lambda_{\max}^{1/2}(\boldsymbol{k}^0)I \iff \tau_K = \lambda_{\max}^{-1/2}(\boldsymbol{k}^0)M \tag{40}$$

The components of $\boldsymbol{k}^0$ have to be understood as *algorithmic constants*.

Using matrix (40) requires to solve the eigenvalue problem (39) for a certain choice of the scaling matrix M. For the model problems presented in Table I, the expressions that can be found in the literature are slight modifications of those given Table II, where I_d stands for the $d \times d$ identity matrix and, in the case of Darcy's dual problem, L_0 is a fixed characteristic length of the computational domain.

Some remarks are in order. First, c_1 and c_2 denote algorithmic constants (different at different appearances), possibly dependent on the polynomial order of the FE interpolation, but not on the element size h or on the equation coefficients; if p is the polynomial order, it can be shown that c_1 has to be proportional to p^2 and c_2 to p (see (Principe and Codina, 2010)). Second, any expression with the same *asymptotic* behavior in terms of h and these coefficients yields similar numerical results and the same stability and error estimates. Third, all these expressions are computed locally, with $h \equiv h_K$ the diameter of element K and the physical parameters evaluated point-wise if they are variable. Finally, no anisotropy of the FE mesh is considered, so that the definition of h_K is unambiguous (see (Principe and Codina, 2010) for a discussion about this point).

The justification of the expressions in Table II can be found in (Codina, 2002; Badia and Codina, 2010). For other problems, such as the three-field Stokes problem or the mixed wave equation with convection, see (Codina, 2009) and (Codina, González-Ondina, Díaz-Hernández, and Principe, 2008), respectively.

3.4. Subscales on the element boundaries

While in many VMS formulations the contribution of the subscales in the interelement boundaries is neglected, considering this contribution can lead to enhanced and stable approximations for combinations of physical problems and interpolation spaces which would otherwise be unusable. In this section we present some of the possibilities for approximating the contribution of the subscales in the interelement boundaries, following the formulation for the subscales on the element boundaries presented and analyzed in (Codina, Principe, and Baiges, 2009). The resulting additional stabilizing terms are similar to those of other stabilized

formulations (Hughes, Franca, and Hulbert, 1987; Kechkar and Silvester, 1992; Silvester and Kechkar, 1990) but they are variationally founded. Other approaches for accounting for the subscales in the element boundaries can be found in (Araya, Barrenechea, and Valentin, 2006; Franca, Madureira, and Valentin, 2005), where the subscales are defined through the definition of local problems along the interelement boundaries.

3.4.1. The main idea Let us consider as unknowns of the problem the value of the subscales and its fluxes in the interelement boundaries. The value for the subscales can then be computed by imposing that the correct transmission conditions of the problem hold, while the fluxes in the interelement boundaries are approximated using a finite difference scheme. Some additional key ingredients of the formulation are the following: firstly, the subscales in the interelement boundaries are computed *after* the subscales in the element interiors are known. This implies that the subscales in the element interiors are computed without accounting for their boundary values. Secondly, the subscales on the element boundaries are single valued, even if this requires that they are discontinuous in the element interiors.

Once a model for the subscales on the boundaries is introduced, their contribition to the FE equation, namely (see (15)),

$$\langle \mathcal{D}u', \mathcal{F}_n^* v_h \rangle_{\partial K}$$

can be accounted for. The model for the subscales in the element boundaries is obtained by enforcing the continuity of the fluxes in the interlement boundaries (thanks to which, the first term of the RHS of (24) vanishes), that is

$$[\![\mathcal{F}_n u]\!] = 0, \quad \forall E \in \mathcal{E}_h^0$$

The previous equation can be applied to any problem of interest; however the model for the fluxes of the subscales is problem dependent. In the following we illustrate the development of the formulation for the Stokes problem.

3.4.2. Application to the Stokes problem For simplicity only velocity subscales are taken into account here. The discrete variational form accounting for the subscales on the element boundaries consists of finding $[\boldsymbol{u}_h, p_h] \in V_h \times Q_h$ and $\boldsymbol{u}' \in V'$ such that (see (15)-(16)):

$$B([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) + \sum_K \langle \boldsymbol{u}', -\nu\Delta \boldsymbol{v}_h - \nabla q_h \rangle_K + \sum_K \langle \boldsymbol{u}', \nu \partial_n \boldsymbol{v}_h + q_h \boldsymbol{n} \rangle_{\partial K} = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle \quad (41)$$

$$\sum_K \langle -\nu\Delta \boldsymbol{u}_h + \nabla p_h, \boldsymbol{v}' \rangle_K + \sum_K \langle -\nu\Delta \boldsymbol{u}', \boldsymbol{v}' \rangle_K + \sum_K \langle \nu(\partial_h \boldsymbol{u}_h + \partial_n \boldsymbol{u}') - p_h \boldsymbol{n}, \boldsymbol{v}' \rangle_{\partial K} = \langle \boldsymbol{f}, \boldsymbol{v}' \rangle \quad (42)$$

which must hold for all $[\boldsymbol{v}_h, q_h] \in V_h \times Q_h$ and all $\boldsymbol{v}' \in V'$. The last term in the left-hand-side (LHS) of equation (41) is the additional term which appears due to the contribution of the subscales in the interelement boundaries. The last term in the LHS of equation (42) is also due to the contribution of the subscales in the element boundaries, but it vanishes due to the weak continuity of stresses in the direction normal to the element boundaries.

The diffusive operator in the element interiors for the subscale equation (42) is approximated using the expressions presented in Sections 3.1, 3.2 and 3.3, which allow one to obtain the

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explicit expression for the subscales in the element interiors. However, an expression for the value of the subscales on the element boundaries is still required, which will be denoted by $\boldsymbol{u}'_E$. The problem to be solved is find $[\boldsymbol{u}_h, p_h] \in V_h \times Q_h$ and $\boldsymbol{u}' \in V'$ such that:

$$B([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) + \sum_K \langle \boldsymbol{u}', -\nu\Delta\boldsymbol{v}_h - \nabla q_h \rangle_K + \sum_K \langle \boldsymbol{u}'_E, \nu\partial_n \boldsymbol{v}_h + q_h \boldsymbol{n} \rangle_{\partial K} = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle$$
$$\sum_K \langle -\nu\Delta\boldsymbol{u}_h + \nabla p_h, \boldsymbol{v}' \rangle_K + \sum_K \tau_K^{-1} \langle \boldsymbol{u}', \boldsymbol{v}' \rangle_K = \langle \boldsymbol{f}, \boldsymbol{v}' \rangle \quad (43)$$

which must hold for all $[\boldsymbol{v}_h, q_h] \in V_h \times Q_h$ and all $\boldsymbol{v}' \in V'$.

In order to determine the expression for the subscales on the element boundaries $\boldsymbol{u}'_E$, the continuity of the normal component of the total tractions in the interelement boundaries (accounting for the contribution of the subscales to these tractions) will be enforced. This is obviously a problem dependent condition. For the Stokes problem considered it reads:

$$\begin{aligned} \boldsymbol{0} &= [\![-p\boldsymbol{n} + \nu\partial_n \boldsymbol{u}]\!]_E \\ &= [\![-p_h\boldsymbol{n} + \nu\partial_n \boldsymbol{u}_h]\!]_E + [\![\nu\partial_n \boldsymbol{u}']\!]_E \end{aligned} \quad (44)$$

It remains to define an approximation for the fluxes of the subscales $[\![\nu\partial_n \boldsymbol{u}']\!]_E$ in terms of the subscales in the element interiors and the subscales on the element boundaries. In order to approximate this term, we recall that the subscales on the element boundaries $\boldsymbol{u}'_E$ are single valued and we consider a case where $[\![\nu\partial_n \boldsymbol{u}']\!]_E$ needs to be computed at the interface between two elements. We denote by $\boldsymbol{u}'_i$ the subscale in the interior of element K_i. The key point is to consider that the value for the subscale in the element interior is valid up to a certain distance δ from the element boundaries. This distance will be computed as $\delta = \delta_0 h$, where δ_0 plays the role of a stabilization parameter. With this assumption, the fluxes of the velocity subscales can be computed as:

$$[\![\nu\partial_n \boldsymbol{u}']\!]_E = \nu \frac{\boldsymbol{u}'_E - \boldsymbol{u}'_1}{\delta} + \nu \frac{\boldsymbol{u}'_E - \boldsymbol{u}'_2}{\delta} \quad (45)$$

Introducing (45) in (44) we obtain the following expression:

$$\begin{aligned} \boldsymbol{0} &= [\![-p\boldsymbol{n} + \nu\partial_n \boldsymbol{u}]\!]_E \\ &= [\![-p_h\boldsymbol{n} + \nu\partial_n \boldsymbol{u}_h]\!]_E + \nu \frac{\boldsymbol{u}'_E - \boldsymbol{u}'_1}{\delta} + \nu \frac{\boldsymbol{u}'_E - \boldsymbol{u}'_2}{\delta} \end{aligned}$$

Grouping terms we obtain the following expression for the subscales on the element boundaries:

$$\boldsymbol{u}'_E = \frac{\delta}{2\nu} [\![p_h\boldsymbol{n} - \nu\partial_n \boldsymbol{u}_h]\!]_E + \{\boldsymbol{u}'\}_E \quad (46)$$

where $\{\boldsymbol{u}'\}_E = \frac{1}{2}(\boldsymbol{u}'_1 + \boldsymbol{u}'_2)$ denotes the mean value of the subscales in the element interiors for elements 1 and 2. In view of the expression of the stabilization parameters, this term is of order $\mathcal{O}(h^2)$, whereas the first term is of order $\mathcal{O}(h)$. Therefore, neglecting $\{\boldsymbol{u}'\}_E$ in (46) and incorporating the expression of the velocity subscale (see (31)) yields:

$$B([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) + \sum_K \tau_K \langle -\nu\Delta\boldsymbol{u}_h + \nabla p_h, \nu\Delta\boldsymbol{v}_h + \nabla q_h \rangle_K$$
$$- \sum_E \frac{\delta}{2\nu} \langle [\![\nu\partial_n \boldsymbol{u}_h - p_h\boldsymbol{n}]\!] [\![\nu\partial_n \boldsymbol{v}_h + q_h\boldsymbol{n}]\!] \rangle_E = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle + \sum_K \tau_K \langle \boldsymbol{f}, \nu\Delta\boldsymbol{v}_h + \nabla q_h \rangle_K \quad (47)$$

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The stability analysis presented in (Codina, Principe, and Baiges, 2009) of formulation (47) shows that the main feature of the method is that, contrary to VMS methods which neglect the contribution of the subscales in the element boundaries, taking them into account permits to have control on the pressure jumps between elements, thus allowing one to obtain stable methods for discontinuous pressure interpolations.

The subscales on the element boundaries strategy can be extended to other problems such as the Darcy problem (Badia and Codina, 2009b; Barrenechea, Franca, and Valentin, 2007) and the incompressible Navier-Stokes equations, permitting in all these cases to use discontinuous pressure interpolations. When applied to the Helmholtz equation (Baiges and Codina, 2013), it allows one to deal with pollution errors. The subscales on the element boundaries strategy has also been used to deal with the added-mass effect in fluid-structure interaction problems (Codina and Baiges, 2011) and to enhance pressure stabilization in blast and impact simulations (Soto, Baum, and Löhner, 2010).

3.5. Open questions

It is clear that there is room for improvement in the design of the VMS-type stabilized FE methods, trying to make them computationally more efficient, more robust or more accurate. Regarding this point, apart from the options presented here, future developments could include other ways to design the space of subscales (and therefore the associated projector), other approximations on the element boundaries or other ways to treat the time dependency of the subscales in time dependent problems (see Section 5). Nevertheless, the most important point is obviously the design of τ_K for systems. Some possibilities have been described here, but general well accepted ways to select τ_K for any problem are not yet available.

4. SOME RELATED METHODS

In this section we describe some stabilization techniques in FE methods that cannot be cast within the VMS framework, but that are closely related. Even if there is no subscale to clearly identify, the standard Galerkin method is also modified with additional terms to cure its instability shortcomings. In the second subsection, we also describe methods that can be considered as complementary to linear stabilization methods in the presence of local discontinuities of the solution.

The point of view adopted here is descriptive, directing the reader to references for the details of the methods.

4.1. Symmetric projection stabilization

Among linear stabilization methods, we can roughly distinguish between methods based on FE residuals and methods based on projections. VMS formulations are in principle all based on residuals, although the OSS method described in Section 3 can be modified to a term-by-term stabilization (see (Codina, 2008b)), and be considered as a method based on projections. Residual-based methods are consistent by definition and exhibit in general optimal convergence and stability properties (see Section 7).

However, residual-based methods are some times criticized for giving unphysical pressure boundary layers (Becker and Braack, 2001), for the additional cost involved in the evaluation

of higher order derivatives and the weak inconsistency for first order approximations (Jansen, Collis, Whiting, and Shakib, 1999), the fact that the forcing term is also affected by the stabilization and the hard extension to transient problems (Codina, Principe, Guasch, and Badia, 2007; Burman, 2010), usually carried out via expensive space-time FEs. Probably, the main shortcoming of residual-based formulations is manifested when dealing with multi-physics applications, where the number of stabilization terms to be integrated notably increases; only some terms do introduce stability whereas the rest are only required to keep consistency. These additional coupling terms fill blocks of the matrix of the discrete problem that are zero for the Galerkin method.

The introduction of symmetric projection stabilization techniques represented one step further in the improvement of FE stabilization methods, since they solve all the problems commented above. Instead of considering residual-based terms, these methods introduce penalty terms over the difference between some quantities, e.g. the pressure gradient for the Stokes problem, and their projections. This family of methods does not perturb the RHS of the problem. As an example, when applied to the Stokes problem to stabilize the pressure gradient, the stabilized bilinear form reads as follows:

$$B([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) + \sum_K \tau_K (\nabla p_h - \pi_h(\nabla p_h), \nabla q_h - \pi_h(\nabla q_h))_K$$

where $\pi_h(\cdot)$ is a FE projector; different definitions for $\pi_h(\cdot)$ lead to different techniques. The resulting method is only weakly consistent, i.e. the stabilization term does not cancel for the exact solution but vanishes as the mesh size goes to zero. This way, we keep convergence and attain the desired stability.

Motivated by the inherited stability of fractional step methods, Codina and Blasco provided in (Codina and Blasco, 1997; Codina, 2000) the first algorithm of this kind, based on the (global) $L^2(\Omega)$-orthogonal projector, coined orthogonal subscales (OSS), and already discussed in the previous sections. Since the projector is global, the stabilization term leads to a dense matrix. Certainly, the method is never computed this way, and the projection is usually sent to the RHS of the linear system. In case of solving transient problems, it can simply be treated explicitly. In those situations, for reasonably small time step sizes, the OSS method has perfect sense and it is an effective and simple algorithm, since the CPU cost per time step used for the computation of the global projections in negligible. On the contrary, to send the projection term to the RHS, and make it implicit via Richardson iterations (usually merged with nonlinear iterations (Codina, 2002)) increases the number of nonlinear iterations or simply diverges; this approach is even harder to justify for linear problems as the Stokes system.

Becker and Braack envisaged in (Becker and Braack, 2001) an original way to avoid the global projections in (Codina and Blasco, 1997). Their method was later called *local projection stabilization* (LPS). The price to pay is a tighten requirement over the mesh partitions: specific hierarchical meshes were needed, since the method is based on the definition of fine and coarse FE spaces. On the other hand, the projection is not over the original FE space, as in (Codina and Blasco, 1997), but on a discontinuous space of functions. The original LPS formulation has been lately denoted as two-level LPS (Braack and Burman, 2006; Matthies, Skrzypacz, and Tobiska, 2007), due to the requirement of two nested meshes for the definition of the stabilization terms. A one-level LPS formulation has also been designed (Matthies, Skrzypacz, and Tobiska, 2007; Ganesan and Tobiska, 2008), in which the fine space is attained with an enrichment of the coarse one via additional functions of bubble type. A numerical comparison

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of both approaches can be found in (Knobloch and Lube, 2009). A recent improvement of these formulations has been recently proposed in (Badia, 2012) for pressure stability of the Stokes problem, which makes use of a particular Scott-Zhang projector which is local and does not require any particular type of mesh or FE space enrichment; this method has been named *nodal projection stabilization* (NPS), due to the nodal-wise nature of the projection. A closely related term-by-term pressure/convection stabilization has been presented in (Chacón Rebollo, Gómez Mármol, Girault, and Sánchez Muñoz, 2013).

All projection-based methods increase the connectivities of the degrees of freedom, and therefore the profile of the matrix of the final discrete system. The alternative of moving the projections to the RHS using iterative strategies has the shortcomings described for the OSS method.

4.2. Nonlinear stabilization

For convection-dominant and hyperbolic problems, linear stabilization techniques certainly reduce the oscillations of the crude Galerkin approximations, but still exhibit overshoots and undershoots around discontinuities or shocks. As a result, nonlinear stabilization techniques (traditionally called shock-capturing) have been designed, usually in the form of an artificial viscosity that depends on the solution (see, e.g., (Johnson and Szepessy, 1987; Johnson, Szepessy, and Hansbo, 1990; Szepessy, 1989)). This nonlinear viscosity must be active around shocks, where the order of accuracy needs to be sacrificed to improve stability, but it must not act in smooth regions. The way this artificial viscosity is computed leads to different families of methods. Most methods are based on residual-based viscosity (Codina, 2000; Lube and Rapin, 2006; John and Knobloch, 2007, 2008; Tezduyar and Senga, 2006) combined with residual-based linear stabilization. Recently, Guermond and co-workers have proposed entropy-viscosity methods, in which the nonlinear viscosity is defined in terms of some entropy inequality (Guermond, Pasquetti, and Popov, 2011).

The aim of the nonlinear stabilization is to eliminate oscillations around shocks or discontinuities, which implies to satisfy a discrete maximum principle (DMP). Only a few FE methods are known to satisfy some monotonicity property (see, e.g., (Mizukami and Hughes, 1985; Burman and Ern, 2002, 2005)). In the work by Burman and Ern in (Burman and Ern, 2002), they state the properties to be fulfilled by a method in order to satisfy a DMP property in a useful variational setting for nonlinear problems. The methods in (Burman and Ern, 2005; Burman and Hansbo, 2004) satisfy a DMP property for the steady-state CDR problem, but it has been observed that they are too dissipative for practical use in (John and Knobloch, 2008) and the extension to time-dependent problems is unclear. Later, Burman proposed in (Burman, 2007) a method which only includes nonlinear stabilization and satisfies monotonicity properties for the (transient) Burgers' problem in one dimension.

The combination of linear and nonlinear stabilization is a reasonable choice, since the former is an accurate method with optimal convergence properties that is effective on smooth regions whereas the latter reduces (or eliminates) oscillations around shocks or discontinuities. In particular, when using nonlinear artificial viscosity without any linear stabilization, it is common to observe the *terracing effect*, which consists in "a distortion of smooth profiles and represents an integrated, nonlinear effect of residual phase error" (Kuzmin, Löhner, and Turek, 2005; Oran and Boris, 2005). However, the combination of linear and nonlinear stabilization must be carried out with care. A naive combination of an optimally convergent linear

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stabilization and a monotonicity-preserving nonlinear stabilization can produce a method with none of these properties. It has recently been observed in (Ern and Guermond, 2013) that traditional linear stabilization terms usually harm interesting properties of the nonlinear stabilization, since they act as a hyperviscosity term. In (Ern and Guermond, 2013), the authors propose a way to blend edge stabilization (Burman and Hansbo, 2004) with entropy-viscosity and numerically observe that the resulting method converges to entropy solutions; neither a DMP nor entropy stability is proved. In (Badia and Hierro, 2014), it is designed a weighting of the projection-based stabilization in (Badia, 2012) such that, when combined with a FE discretization with a DMP (usually attained via a shock-capturing technique), it does not spoil the monotonicity properties. It is attained by switching off the linear stabilization around shocks. New nonlinear stabilization (shock-capturing) schemes based on artificial viscosity were proposed in (Badia and Hierro, 2014), based on artificial viscosity proportional to nodal jumps, which were proved to satisfy a salient strong DMP property for multidimensional time-dependent transport problems.

5. INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

In the previous sections, we have applied the VMS method to linear problems. In this section, we consider its extension to the nonlinear Navier-Stokes equations for incompressible flows. This system of equations is a classical application of these techniques, since includes the two types of instabilities that can be cured using stabilized FE and VMS techniques, namely, the one due to the fact that pressure stability is attained at the continuous level via an inf-sup condition and the presence of the convective term. Another key aspect to be considered when simulating the incompressible Navier-Stokes equations is its multiscale nature. Full-resolution direct numerical simulations that capture the smallest scales in the flow are unaffordable for large Reynolds numbers. In practice, under-resolved simulations must be performed due to the limited computational resources. On the other hand, the simulation of the smallest scales are required to have meaningful results, since at these scales it is where the dissipation mechanism is generated. In order to generate meaningful under-resolved simulations, turbulence models must be added, that somehow *model* the molecular dissipation at the unresolved smallest scales of the flow. As we will comment below, VMS (and related stabilized techniques) can be considered as turbulence models by themselves.

Large eddy simulation (LES) turbulence models (Sagaut, 2006) are based on a scale separation that permits to reduce the computational cost with respect to direct numerical simulation (DNS). Such scale separation is traditionally achieved by filtering the original Navier-Stokes equations, which leads to an extra forcing term defined by a physical (functional or structural) model. This widely used approach is usually referred to as explicit LES (Sagaut, 2006). By contrast, implicit LES techniques (ILES) rely on purely numerical artifacts without any modification of the continuous problem. This approach was seldom followed, the MILES (Monotone Integrated LES) approach (Boris, Grinstein, Oran, and Kolbe, 1992; Fureby and Grinstein, 2002; Grinstein, Margolin, and Rider, 2007) being the main exception, until the VMS method was introduced (Hughes, 1995; Hughes, Feijóo, Mazzei, and Quincy, 1998) and subsequently proposed as an ILES method (see below). ILES techniques are usually considered to be based on the addition of purely dissipative numerical terms, see (Sagaut, 2006, Section 5.3.4). It is worth to emphasize that this is not the case of some particular VMS models, as it

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is shown in (Principe, Codina, and Henke, 2010) and discussed below.

As it has been explained, VMS was introduced as a framework for the motivation and development of stabilization techniques, which aim to overcome numerical difficulties encountered when using the standard Galerkin method. On the one hand, the velocity and pressure FE spaces need to satisfy the inf-sup compatibility condition that guarantees pressure stability and precludes the use of equal order interpolation. Mixed methods satisfying this condition can be used and their finite volume counterpart, based on staggered grids, are common in the LES community. On the other hand, global nonphysical oscillations appear in the convection dominated regime, when the mesh is not fine enough, that is, for high mesh Reynolds number flows. The only way to overcome this problem is through the addition of some form of dissipation which was recognized in the early development of stabilized methods. Let us note that the common practice in the LES community is to rely on the explicit extra term introduced by the physical model using high order approximations of the convective term. *

The first attempts to perform LES using VMS concepts, presented in (Hughes, Mazzei, and Jansen, 2000; Hughes, Oberai, and Mazzei, 2001b; Hughes, Mazzei, and Oberai, 2001a; Koobus and Farhat, 2004; John and Kindl, 2008), were performed introducing explicit subgrid modeling. The VMS models used in these works split resolved scales into large and small, introducing an explicit LES model to account for the small scales stress tensor, e.g., a Smagorinsky-type dissipative term acting on the small scales only. As a result, an important fraction of the degrees of freedom are used for the small resolved scales, whereas consistency is retained in the large resolved scales only.

ILES using a VMS approach with resolved and unresolved subgrid scales (the setting that permits to recover stabilized formulations) was suggested in (Codina, 2002) and performed in (Calo, 2004; Bazilevs, Calo, Cottrell, Hughes, Reali, and Scovazzi, 2007; Nogueira, Cueto-Felgueroso, Colominas, and Gómez, 2010). Excellent results were first presented in (Bazilevs, Calo, Cottrell, Hughes, Reali, and Scovazzi, 2007), but using isogeometric analysis for the space approximation. Compared to classical LES based on filtering, the VMS approach does not face difficulties associated to inhomogeneous non-commutative filters in wall-bounded flows. Further, it retains numerical consistency in the FE equations and optimal convergence up to the interpolation order, whereas, e.g., Smagorinsky models introduce a consistency error of order $h^{4/3}$ (see (Hughes, Mazzei, and Jansen, 2000; Hughes, Oberai, and Mazzei, 2001b; Bazilevs, Calo, Cottrell, Hughes, Reali, and Scovazzi, 2007)).

Scale separation is achieved in the VMS formalism by a variational projection. The continuous unknown is split into a resolvable FE component and an unresolvable subgrid or subscale component. The action of the subscales onto the FE scales can be approximated in different ways, leading to different VMS models but in all cases these models are *residual-based* (no eddy viscosity is introduced), which permits to retain consistency. Among the modeling possibilities is the choice of the subscale space, first discussed in (Codina, 2000), where it was enforced to be L^2-orthogonal to the FE space. Another modeling ingredient is the possibility of

*It is worth to point out that both problems (convection instability and compatibility conditions) are also present in the *linear* Oseen problem. One of the inconsistencies of an explicit LES approach without a numerical dissipation term is that convection is stabilized by a term that comes from the physical model of the nonlinear Navier Stokes equations and such a term is not present when the linear Oseen problem is considered.

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considering time-dependent subscales and to keep the VMS decomposition in all the nonlinear terms, which was studied in (Codina, 2002; Codina, Principe, Guasch, and Badia, 2007). Clear improvements have been observed when using dynamic and fully nonlinear models for the simulation of laminar flows (Codina, Principe, Guasch, and Badia, 2007; Avila, Principe, and Codina, 2011).

The original VMS formulation (Hughes, 1995; Hughes, Feijóo, Mazzei, and Quincy, 1998) was developed having linear problems in mind and its extension to the Navier-Stokes equations was implicitly based on a "linearization", fixing the advection velocity and applying the multiscale splitting to the rest of the terms. A nonlinear scale splitting was used in (Hughes, Mazzei, and Jansen, 2000; Hughes, Oberai, and Mazzei, 2001b) together with an explicit resolution of the small scales in which a Smagorinsky damping was introduced. A nonlinear scale splitting with modeled subscales was used in (Codina, 2002; Bazilevs, Calo, Cottrell, Hughes, Reali, and Scovazzi, 2007) and in (Codina, Principe, Guasch, and Badia, 2007), where it was shown that it leads to global conservation of momentum.

In this section, we consider the application of VMS formulations to the incompressible Navier-Stokes equations. In particular, issues related to stability, the dissipative structure of the formulation, the energy transfer between unresolved and resolved scales and the interpretation of these techniques as turbulence models (including a discussion about the possibility to produce backscatter) will be addressed.

5.1. Continuous problem

The Navier-Stokes problem for an incompressible fluid consist of finding a velocity $\boldsymbol{u}$ and a pressure p solution of the initial and boundary value problem

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega,\ t>0 \tag{48}$$

$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega,\ t>0 \tag{49}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma,\ t>0 \tag{50}$$

$$\boldsymbol{u} = \boldsymbol{u}^0 \quad \text{in } \Omega,\ t=0 \tag{51}$$

where $\boldsymbol{u}^0$ is the initial condition.

Let $V = H^1_0(\Omega)^d$ and $Q = L^2_0(\Omega)$, and let also $L^2(0,T;V)$ be the set of functions whose V-norm in space is L^2 in time, and $\mathcal{D}'(0,T;Q)$ the set of "functions" whose Q-norm in space is a distribution in time. The weak form of the Navier-Stokes equations can be written as follows: find $[\boldsymbol{u},p] \in L^2(0,T;V)\times\mathcal{D}'(0,T;Q)$ such that

$$(\partial_t\boldsymbol{u},\boldsymbol{v}) + \langle\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v}\rangle + \nu(\nabla\boldsymbol{u},\nabla\boldsymbol{v}) - (p,\nabla\cdot\boldsymbol{v}) = \langle\boldsymbol{f},\boldsymbol{v}\rangle, \quad \forall\boldsymbol{v}\in V \tag{52}$$

$$(q,\nabla\cdot\boldsymbol{u}) = 0, \qquad \forall q\in Q \tag{53}$$

5.2. Nonlinear scale splitting

As commented above, the starting point of the VMS framework is a *two scale* decomposition of the unknowns, the velocity $\boldsymbol{u}$ and the pressure p. In the case of the incompressible Navier-Stokes equations, stabilization is required to avoid the need for using velocity-pressure interpolations satisfying the inf-sup condition and to deal with convection-dominated flows. In all what follows, the time variable will be left continuous. Let $V_h \subset V$ and $Q_h \subset Q$ be FE spaces to approximate the velocity and the pressure, respectively, *in space*. In order to apply the VMS

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framework to the Navier-Stokes equations, we split the velocity and the pressure as

$$\boldsymbol{u} = \boldsymbol{u}_h + \boldsymbol{u}', \quad p = p_h + p' \tag{54}$$

where $\boldsymbol{u}_h$, p_h belong to the FE spaces and $\boldsymbol{u}'$ and p' are what we have been calling the subgrid scales or *the subscales*. The way these are modeled defines the particular numerical approximation. We can identify the FE components of the solution as the *resolved* scales, whereas the subscales are the *unresolved* scales. A first important topic to be considered when applying the VMS formulation is the appearance of nonlinear terms involving $\boldsymbol{u}'$ in the Navier-Stokes equations. Let us analyze the implications of keeping the nonlinear terms. Using (54), the convective term will lead to

$$\begin{aligned} \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) &= \nabla \cdot (\boldsymbol{u}_h \otimes \boldsymbol{u}_h) + \nabla \cdot (\boldsymbol{u}_h \otimes \boldsymbol{u}') + \nabla \cdot (\boldsymbol{u}' \otimes \boldsymbol{u}_h) + \nabla \cdot (\boldsymbol{u}' \otimes \boldsymbol{u}') \\ &\equiv \text{(I)} + \text{(II)} + \text{(III)} + \text{(IV)} \end{aligned} \tag{55}$$

Obviously, (I) would be the only term appearing in a Galerkin approximation, whereas the rest are the contributions from the velocity subscale. It can be shown that when this subscale is modeled, the term that provides numerical stability is (II), in the sense that it provides control of the convective derivative and the pressure gradient. One can also show that (III) leads to global momentum conservation (Codina, Principe, Guasch, and Badia, 2007).

Because of its resemblance with similar terms in turbulence models, (IV) in (55) raises the question of whether keeping the contribution from the subscales in the convective term could be viewed as a turbulence model or not. This possibility was mentioned in (Codina, 2002), which contrasts with the option in (Hughes, Mazzei, and Jansen, 2000), where a LES model is used to represent the subgrid scales (see Remark 6 in (Codina, 2002) and, for background on LES models, (Pope, 2000)). In (Bazilevs, Calo, Cottrell, Hughes, Reali, and Scovazzi, 2007) the possibility to model turbulence using only numerical ingredients within the VMS context is fully and successfully exploited. The role of numerical stabilization terms as turbulence models was also envisaged in (de Sampaio, Hallak, Coutinho, and Pfeil, 2004; Hoffman and Johnson, 2006; Colomés, Badia, Codina, and Principe, 2015), for example.

By analogy with LES models, the different terms appearing in (55) could be termed as follows:

$$\begin{aligned} \text{(II)+(III)} &= \boldsymbol{u}_h \otimes \boldsymbol{u}' + \boldsymbol{u}' \otimes \boldsymbol{u}_h : && \text{Cross stress} \\ \text{(IV)} &= \boldsymbol{u}' \otimes \boldsymbol{u}' : && \text{Reynolds stress} \\ \text{(II)+(III)+(IV)} &= \boldsymbol{u}_h \otimes \boldsymbol{u}_h - \boldsymbol{u} \otimes \boldsymbol{u} : && \text{Subgrid scale tensor} \end{aligned}$$

Some comments in the line of viewing an approximation to the subscales in (55) as an effective LES model will be provided in the following.

Another crucial point is how to treat the time dependency of the velocity subscale. Using the VMS splitting, the velocity time derivative can be split as

$$\partial_t \boldsymbol{u} = \partial_t \boldsymbol{u}_h + \partial_t \boldsymbol{u}' \tag{56}$$

The first term would be the only one kept if the time derivative of the subscales is neglected. In this situation, the subscales are termed *quasi-static* in (Codina, 2002), in contrast to *dynamic subscales* if they are considered to be time-dependent. As shown in (Codina, Principe, Guasch, and Badia, 2007), the second term leads to a correct behavior of time integration schemes and better accuracy. In particular, in (Badia and Codina, 2009a) stability and convergence for the Stokes problem is proved without any restriction on the time step size and the stabilization parameters on which the formulation depends.

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5.3. Subgrid scale decomposition

As explained earlier, the starting point of the formulation to be presented is the splitting (54). For simplicity, we will not consider pressure subscales (see (Codina, 2001, 2002) for an analysis of their inclusion). Thus, if $\boldsymbol{u}'$ is a certain approximation to the exact velocity subscale, the splitting we consider is $\boldsymbol{u} \approx \boldsymbol{u}_* := \boldsymbol{u}_h + \boldsymbol{u}',\ p \approx p_h$. When inserted into (52)-(53) this yields:

$$
\begin{aligned}
&(\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h) + \langle \boldsymbol{u}_* \cdot \nabla \boldsymbol{u}_h, \boldsymbol{v}_h \rangle + \nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - (p_h, \nabla \cdot \boldsymbol{v}_h) + (q_h, \nabla \cdot \boldsymbol{u}_h) \\
&\quad + (\partial_t \boldsymbol{u}', \boldsymbol{v}_h) - \sum_K \langle \boldsymbol{u}', \boldsymbol{u}_* \cdot \nabla \boldsymbol{v}_h + \nu \Delta \boldsymbol{v}_h + \nabla q_h \rangle_K \\
&\quad + \sum_K \langle \boldsymbol{u}', \nu \boldsymbol{n} \cdot \nabla \boldsymbol{v}_h + q_h \boldsymbol{n} \rangle_{\partial K} = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle \qquad (57) \\
&(\partial_t \boldsymbol{u}', \boldsymbol{v}') + \sum_K \langle \boldsymbol{u}_* \cdot \nabla \boldsymbol{u}' - \nu \Delta \boldsymbol{u}', \boldsymbol{v}' \rangle_K + \sum_K \langle \nu \boldsymbol{n} \cdot \nabla \boldsymbol{u}', \boldsymbol{v}' \rangle_{\partial K} \\
&\quad + \sum_K \langle \partial_t \boldsymbol{u}_h + \boldsymbol{u}_* \cdot \nabla \boldsymbol{u}_h - \nu \Delta \boldsymbol{u}_h + \nabla p_h, \boldsymbol{v}' \rangle_K \\
&\quad + \sum_K \langle \nu \boldsymbol{n} \cdot \nabla \boldsymbol{u}_h - p_h \boldsymbol{n}, \boldsymbol{v}' \rangle_{\partial K} = \langle \boldsymbol{f}, \boldsymbol{v}' \rangle \qquad (58)
\end{aligned}
$$

These discrete variational equations must hold for all test functions $[\boldsymbol{v}_h, q_h] \in V_h \times Q_h$ and $\boldsymbol{v}' \in V'$, where V' is the space of subscales to be defined. It is observed that some terms have been integrated by parts within each element.

5.4. Simplifying assumptions

Apart from taking the pressure subscale to be zero, no approximations have been made to arrive at (57)-(58). The different approximations discussed in Section 3 will lead to different formulations within the same framework. One usually considers $\partial_t \boldsymbol{u}' \approx \boldsymbol{0}$ and takes $\boldsymbol{u}_* \approx \boldsymbol{u}_h$ as advection velocity in (57)-(58). However, these are precisely approximations that can be relaxed. On the other hand, as commented above, the *space of subscales* V', that is, the space where $\boldsymbol{u}'$ belongs for t fixed, can be enforced to be L^2 *orthogonal to the FE space*, leading to the OSS approach.

Another classical assumption is to approximate $\boldsymbol{u}' \approx \boldsymbol{0}$ on ∂K for each element domain K of the FE partition. That could be understood as approximating the velocity subscale by a space of bubble functions. Nevertheless, this approximation can be relaxed following the ideas discussed in Section 3.4.

Finally, the essential approximation is

$$
\sum_K \langle \boldsymbol{u}_* \cdot \nabla \boldsymbol{u}' - \nu \Delta \boldsymbol{u}', \boldsymbol{v}' \rangle_K \approx \sum_K \tau_K^{-1} \langle \boldsymbol{u}', \boldsymbol{v}' \rangle_K, \qquad (59)
$$

where τ_K is a set of *scalar* algorithmic parameters computed within each element K as

$$
\tau_K^{-1} = \frac{c_1 \nu}{h_K^2} + \frac{c_2 \|\boldsymbol{u}_*\|_{L^\infty(K)}}{h_K}. \qquad (60)
$$

c_1 and c_2 are algorithmic constants that depend only on the degree of the FE approximation being used. This corresponds to the first component of the matrix of stabilization parameters

for the Stokes problem given in Table II extended to the Navier-Stokes equations. In fact, what is important is its asymptotic behavior in terms of h_K, ν and $\|\boldsymbol{u}_*\|_{L^\infty(K)}$, and thus a slightly simpler expression of τ_K has been used if it is compared with the one corresponding to the CDR equation in Table II. We also would like to stress that the introduction of τ_K comes from the approximation of a *spatial* operator, as it is clearly seen from (59). Let us comment on expression (60):

- The influence of the constants c_1 and c_2 is discussed in (Colomés, Badia, Codina, and Principe, 2015). A theoretical way to determine them would be to impose that the numerical dissipation they introduce be equal to the molecular dissipation in turbulent regimes, as explained in (Guasch and Codina, 2013).
- The definition of τ_K in (60) is not standard, in the sense that the one used often depends on the time step size of the time discretization, δt. Instead of (60), it is often used $\tau_{K,\delta t}^{-1} = \tau_K^{-1} + \delta t^{-1}$ (see, e.g., (Hsu, Bazilevs, Calo, Tezduyar, and Hughes, 2010; Gamnitzer, Gravemeier, and Wall, 2010)).
- In this article we do not consider the case of time dependent computational domains, required for example in the important application of fluid-structure interaction problems. For the application of VMS-type stabilization techniques to this situation and the way to compute the stabilization parameter τ_K in this case, see for example (Tezduyar, 2003; Tezduyar and Sathe, 2007).

With the simplifying assumptions described, the equation for the velocity subscales (58) becomes

$$(\partial_t \boldsymbol{u}', \boldsymbol{v}') + \sum_K \tau_K^{-1} \langle \boldsymbol{u}', \boldsymbol{v}' \rangle_K = \sum_K \langle \mathcal{R}_u[\boldsymbol{u}_h, p_h], \boldsymbol{v}' \rangle_K$$

where $\mathcal{R}_u$ is the FE residual of the momentum equation. If P' is the appropriate projection onto V', we may write this equation point-wise within each element K:

$$\partial_t \boldsymbol{u}'|_K + \tau_K^{-1} \boldsymbol{u}'|_K = P' \mathcal{R}_u[\boldsymbol{u}_h, p_h]|_K \tag{61}$$

5.5. Dynamic and quasi-static subscales

As the numerical instabilities of flow problems have essentially a spatial nature, the time dependency of the subscales was not considered, and the standard choice (Hughes, Mazzei, and Jansen, 2000; Hughes, Oberai, and Mazzei, 2001b; Bazilevs, Calo, Cottrell, Hughes, Reali, and Scovazzi, 2007) was to neglect the temporal derivative of the subscales. From (61) we then have

$$\boldsymbol{u}'|_K = \tau_K P' \mathcal{R}_u[\boldsymbol{u}_h, p_h]|_K \tag{62}$$

In this case, the subscales may be called *quasi-static*, since their time derivative will be neglected even if they will vary in time because of the time variation of $\mathcal{R}_u[\boldsymbol{u}_h, p_h]$.

The subscale as a time dependent variable of the problem was introduced in (Codina, 2002; Codina, Principe, Guasch, and Badia, 2007). It gives rise to important properties like commutativity of space and time discretization, stability without restrictions on the time step size (Codina, Principe, Guasch, and Badia, 2007; Badia and Codina, 2009a) and, combined with orthogonal subscales, to important stability and convergence properties in time. These include improved stability estimates with respect to the Galerkin method, both in bounded

time intervals and in the long term behavior (Badia, Codina, and Gutiérrez-Santacreu, 2010), and convergence towards weak solutions of the Navier-Stokes equations (Badia and Gutiérrez-Santacreu, 2014). From the physical point of view, dynamic subscales open the door to model backscatter (Codina, Principe, and Badia, 2011; Principe, Codina, and Henke, 2010), as explained below.

To deal with *dynamic* subscales one needs to approximate in time (61), using for example finite differences. This requires the storage at the integration points of the velocity subscales, since at a certain time step the subscales of the previous one are required. Further details can be found in the above references. There is also the possibility to integrate the subscales in time analytically, as explained in (Codina and Principe, 2007).

5.6. Final formulation

The approximations described allow us to formulate a method that can be effectively implemented. It consists of finding $\boldsymbol{u}_h \in L^2(0,T;V_h)$ and $p_h \in \mathcal{D}'(0,T;Q_h)$ such that

$$(\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h) + \langle \boldsymbol{u}_* \cdot \nabla \boldsymbol{u}_h, \boldsymbol{v}_h \rangle + \nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - (p_h, \nabla \cdot \boldsymbol{v}_h) + (q_h, \nabla \cdot \boldsymbol{u}_h) - \sum_K \langle \boldsymbol{u}', \boldsymbol{u}_* \cdot \nabla \boldsymbol{v}_h + \nu \Delta \boldsymbol{v}_h + \nabla q_h \rangle_K = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle \tag{63}$$

$$(\partial_t \boldsymbol{u}', \boldsymbol{v}') + \sum_K \tau_K^{-1} \langle \boldsymbol{u}', \boldsymbol{v}' \rangle_K + \sum_K \langle \boldsymbol{u}_* \cdot \nabla \boldsymbol{u}_h - \nu \Delta \boldsymbol{u}_h + \nabla p_h, \boldsymbol{v}' \rangle_K = \langle \boldsymbol{f}, \boldsymbol{v}' \rangle \tag{64}$$

These equations must hold for all $[\boldsymbol{v}_h, q_h] \in V_h \times Q_h$ and $\boldsymbol{v}' \in V'$.

A complete numerical analysis of (63)-(64) would include stability and convergence estimates as well as a qualitative analysis of the associated dynamical system. Moreover, in the context of stabilized FE methods this analysis should be conducted in *norms that do not explode as* $\nu \to 0$ *and allow for any velocity-pressure interpolation.* Whereas the second requirement could be considered not essential by those that favor the use of inf-sup stable velocity-pressure interpolations, the first is a must. From the numerical point of view, estimates that explode with ν are *completely useless* if the formulation is intended to be applied to large Reynolds number flows and, obviously, to model turbulence; see (Badia, Codina, and Gutiérrez-Santacreu, 2010) for results in this direction.

5.7. Dissipative structure

Let R be a region formed by a patch of elements, and let $\boldsymbol{t}_R$ be the flux on ∂R, which may include both the flux of stresses (tractions) and convective fluxes. For simplicity, suppose that τ is constant (computed with a characteristic velocity and element length). If $\mathcal{L}_u \boldsymbol{v} := \boldsymbol{u}_* \cdot \nabla \boldsymbol{v} - \nu \Delta \boldsymbol{v}$, taking $\boldsymbol{v}_h = \boldsymbol{u}_h$, $q_h = p_h$ and $\boldsymbol{v}' = \boldsymbol{u}'$ in (57)-(58), using (59) and neglecting the subscales on the interelement boundaries, we get:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_h\|_R^2 + \nu\|\nabla \boldsymbol{u}_h\|_R^2 + \langle \partial_t \boldsymbol{u}', P'(\boldsymbol{u}_h)\rangle_R + \sum_{K \subset R} \langle \boldsymbol{u}', P'(\mathcal{L}_u^* \boldsymbol{u}_h - \nabla p_h)\rangle_K = W_R(\boldsymbol{u}_h) \tag{65}$$

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$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}'\|_R^2+\tau^{-1}\|\boldsymbol{u}'\|_R^2+\langle P'(\partial_t\boldsymbol{u}_h),\boldsymbol{u}'\rangle_R \\ +\sum_{K\subset R}\langle\boldsymbol{u}',P'(\mathcal{L}_u\boldsymbol{u}_h+\nabla p_h)\rangle_K=\langle\boldsymbol{f},\boldsymbol{u}'\rangle_R \tag{66}$$

where $W_R(\boldsymbol{u}_h)=\langle\boldsymbol{f},\boldsymbol{u}_h\rangle_R+\langle\boldsymbol{t}_R,\boldsymbol{u}_h\rangle_{\partial R}$, $\mathcal{L}_u^*\boldsymbol{v}:=-\boldsymbol{u}_*\cdot\nabla\boldsymbol{v}-\nu\Delta\boldsymbol{v}$ and $\|\cdot\|_R$ is the $L^2(R)$-norm. In (65) we have neglected the term $\langle\boldsymbol{u}_*\cdot\nabla\boldsymbol{u}_h,\boldsymbol{u}_h\rangle$; $\boldsymbol{u}_*$ is not divergence free in the approximated problem, but $\langle\boldsymbol{u}_*\cdot\nabla\boldsymbol{u}_h,\boldsymbol{v}_h\rangle$ in (57) can be replaced by the skew-symmetric form

$$\frac{1}{2}\langle\boldsymbol{u}_*\cdot\nabla\boldsymbol{u}_h,\boldsymbol{v}_h\rangle-\frac{1}{2}\langle\boldsymbol{u}_*\otimes\boldsymbol{u}_h,\nabla\boldsymbol{v}_h\rangle \\ =\langle\boldsymbol{u}_*\cdot\nabla\boldsymbol{u}_h,\boldsymbol{v}_h\rangle+\frac{1}{2}\langle\nabla\cdot\boldsymbol{u}_*,\boldsymbol{u}_h\cdot\boldsymbol{v}_h\rangle-\frac{1}{2}\langle\boldsymbol{n}\cdot\boldsymbol{u}_*,\boldsymbol{u}_h\cdot\boldsymbol{v}_h\rangle_\Gamma$$

without altering the stability and consistency of the formulation. The last term in this expression has been kept to show that only the definition of the flux $\boldsymbol{t}_R$ will change if the integral is performed in a region R interior to Ω.

From (65)-(66) we may draw the first important conclusion. Suppose that $\nu\Delta\boldsymbol{u}_h$ is negligible (because $\boldsymbol{u}_h$ is linear within each element or because ν is very small or because $P'(\Delta\boldsymbol{u}_h)\approx\boldsymbol{0}$, as we will see below). Let us define

$$\begin{aligned}
\mathcal{K}_h&:=\frac{1}{2}\|\boldsymbol{u}_h\|_R^2,\quad \mathcal{K}':=\frac{1}{2}\|\boldsymbol{u}'\|_R^2 && \text{Kinetic energy of } \boldsymbol{u}_h \text{ and of } \boldsymbol{u}'\\
\mathcal{M}_h&:=\nu\|\nabla\boldsymbol{u}_h\|_R^2,\quad \mathcal{M}':=\tau^{-1}\|\boldsymbol{u}'\|_R^2 && \text{Dissipation of } \boldsymbol{u}_h \text{ and of } \boldsymbol{u}'\\
\mathcal{P}_h&:=W_R(\boldsymbol{u}_h),\quad \mathcal{P}':=\langle\boldsymbol{f},\boldsymbol{u}'\rangle_R && \text{External power on } \boldsymbol{u}_h \text{ and on } \boldsymbol{u}'\\
\mathcal{T}&:=\sum_{K\subset R}\langle\boldsymbol{u}',P'(\mathcal{L}_u^*\boldsymbol{u}_h-\nabla p_h)\rangle_K && \text{Energy transfer}
\end{aligned}$$

Note that the energy transfer term $\mathcal{T}$, when considered in the equation for the FE component, can be thought as the *numerical dissipation of the formulation.*

From (65)-(66) we see that only if V' is a subspace of $V_h^\perp$ the energy balance in region R can be written as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}_h+\mathcal{M}_h+\mathcal{T}&=\mathcal{P}_h\\
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}'+\mathcal{M}'-\mathcal{T}&=\mathcal{P}'
\end{aligned} \tag{67}$$

Therefore, *there is a scale separation in the kinetic energy balance only if the subscales are orthogonal to the FE space.*

5.8. Global kinetic energy balance equations

That the numerical approximation of the velocity subscales defines a LES model is sustained not only by numerical experiments, but also by some physical reasoning. We present one of these physical arguments next.

For the following discussion we may assume quasi-static subscales (an important implication of considering dynamics subscales is presented in the next subsection) and no body forces applied to the fluid. Neglecting $\nu\Delta\boldsymbol{u}_h$ and calling $P'=P_h^\perp$ the projection orthogonal to the FE space V_h, from (64) it follows that

$$\boldsymbol{u}'=-\tau P_h^\perp(\boldsymbol{u}_*\cdot\nabla\boldsymbol{u}_h+\nabla p_h)$$

so that the energy balance equation (65) when $R = \Omega$ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_h\|^2 + \nu\|\nabla\boldsymbol{u}_h\|^2 + \int_\Omega \varepsilon_{\mathrm{num}} = 0 \tag{68}$$

where $\|\cdot\| \equiv \|\cdot\|_{L^2(\Omega)}$ and

$$\varepsilon_{\mathrm{num}} := \tau|P_h^\perp(\boldsymbol{u}_* \cdot \nabla\boldsymbol{u}_h + \nabla p_h)|^2 \geq 0 \tag{69}$$

is the global numerical dissipation at each point. We may now compare the energy budget (68) with what would be obtained for the continuous problem, using a LES model and approximating numerically a LES model. Let $\varepsilon_{\mathrm{LES}}$ be the pointwise dissipation associated to a certain LES model (see (Pope, 2000)) and $\bar{\boldsymbol{u}}$ the filtered velocity field resulting from this model. If the LES model is approximated numerically, let $\bar{\boldsymbol{u}}_h$ be the approximation to $\bar{\boldsymbol{u}}$, $\varepsilon^h_{\mathrm{LES}}$ the approximation to the LES dissipation and $\bar{\varepsilon}_{\mathrm{num}}$ the numerical dissipation inherent to the scheme, for example (69) replacing $\boldsymbol{u}_h$ by $\bar{\boldsymbol{u}}_h$ if the formulation we are describing is used.

The counterpart of (68) for the continuous problem, a LES model and a numerical approximation of a LES model would respectively be:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|^2 + \int_\Omega \varepsilon_{\mathrm{mol}} = 0, \qquad \varepsilon_{\mathrm{mol}} = \nu|\nabla\boldsymbol{u}|^2 \tag{70}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{\boldsymbol{u}}\|^2 + \nu\|\nabla\bar{\boldsymbol{u}}\|^2 + \int_\Omega \varepsilon_{\mathrm{LES}} = 0 \tag{71}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{\boldsymbol{u}}_h\|^2 + \nu\|\nabla\bar{\boldsymbol{u}}_h\|^2 + \int_\Omega \bar{\varepsilon}_{\mathrm{num}} + \int_\Omega \varepsilon^h_{\mathrm{LES}} = 0 \tag{72}$$

Let us discuss Lilly's argument (Lilly, 1967). Suppose that the flow is turbulent, with fully developed and isotropic turbulence, and that a LES model is used to capture the main flow features. In the inertial range of the Kolmogorov spectrum one may assume that all the kinetic energy of the flow is contained in the large scales, and that the molecular dissipation of these scales is negligible, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|^2 \sim \frac{\mathrm{d}}{\mathrm{d}t}\|\bar{\boldsymbol{u}}\|^2, \qquad \nu\|\nabla\bar{\boldsymbol{u}}\|^2 \sim 0$$

Comparing the localized versions of (70) and (71) it turns out that these assumptions imply

$$\varepsilon_{\mathrm{LES}}(\bar{\boldsymbol{u}}) \sim \varepsilon_{\mathrm{mol}}(\boldsymbol{u}) \tag{73}$$

This is the basic requirement of a LES model: the dissipation it introduces must be proportional (equal, in the best case) to the molecular dissipation.

The question now is whether the numerical dissipation (69) satisfies this requirement. It is shown in (Guasch and Codina, 2013) that *if the mesh size h belongs to the inertial range and the classical assumptions of statistical fluid mechanics apply* (as described for example in (Monin and Yaglom, 1971)), then

$$\varepsilon_{\mathrm{num}}(\boldsymbol{u}_h) \sim \varepsilon_{\mathrm{mol}}(\boldsymbol{u}) \tag{74}$$

Furthermore, like in LES models we may consider that $\boldsymbol{u}_h$ carries all (or most of) the kinetic energy of the flow. In this case, the energy balance equations (68) and (71) are formally identical. Moreover, if (73) and (74) both hold, from (72) it follows that *approximating a LES model with a numerical scheme with a dissipation satisfying (74) is clearly redundant.* Of course the numerical approximation is unavoidable. Hence, what is unnecessary is the use of a LES model.

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5.9. Backscatter

In this subsection we use again a heuristic reasoning to analyze the possibility to model backscatter by the numerical formulation described.

Using the notation introduced heretofore, backscatter can be defined by condition $\mathcal{T} < 0$, where $\mathcal{T}$ is the energy transfer term appearing in (67). This means that energy is supplied from the unresolved (small) scales to the resolved (FE) scales. Physically, it is known that this can happen only at isolated spatial points and time instants. Numerically, the model should be such that $\mathcal{T}$ can be negative only if region R is small enough (possibly a single element) and at a few time steps of the time discretization.

As before, we will consider the subscales orthogonal to the FE space and that $\boldsymbol{f} \in V_h$. From (64) we have that

$$\boldsymbol{u}' = -\tau[\partial_t \boldsymbol{u}' + P'(\mathcal{L}_u \boldsymbol{u}_h + \nabla p_h)]$$

which upon substitution in the expression of $\mathcal{T}$ yields

$$\mathcal{T} = \underbrace{\sum_{K \subset R} \tau \langle P'(\mathcal{L}_u \boldsymbol{u}_h + \nabla p_h), P'(-\mathcal{L}_u^* \boldsymbol{u}_h + \nabla p_h) \rangle_K}_{(\int_R \varepsilon_{\text{num}})} + \underbrace{\sum_{K \subset R} \tau \langle \partial_t \boldsymbol{u}', P'(-\mathcal{L}_u^* \boldsymbol{u}_h + \nabla p_h) \rangle_K}_{(\int_R \beta_{\text{num}})}$$

If viscous terms are negligible (or $\Delta \boldsymbol{v}_h$ is approximated by the discrete Laplacian $\Delta_h \boldsymbol{v}_h$) and $\boldsymbol{r} = -P_h^{\perp}(\boldsymbol{u}_* \cdot \nabla \boldsymbol{u}_h + \nabla p_h)$ we have

$$\mathcal{T} = \int_R \varepsilon_{\text{num}} + \int_R \beta_{\text{num}} = \sum_K \tau \left(\langle \boldsymbol{r}, \boldsymbol{r} \rangle_K - \langle \partial_t \boldsymbol{u}', \boldsymbol{r} \rangle_K \right) \tag{75}$$

From this expression it immediately follows that *when orthogonal subscales are used, backscatter is possible only if the subscales are dynamic*, since otherwise the second term in (75) vanishes and the first one is obviously non-negative.

6. OTHER PROBLEMS

In this section we describe some of the works that have been published applying VMS formulations to flow problems other than those considered heretofore, trying to emphasize which is the main difficulty that needs to be faced and that is not present in the incompressible Navier-Stokes equations. The objective is to give an overview of the literature (surely incomplete and based on the perspective of the authors) and not to describe the different formulations in detail.

6.1. Compressible flows and shallow water flows

The compressible Navier-Stokes equations and the shallow water equations can be written as a system of the form

$$\partial_t u + \partial_i F_i + \partial_i G_i = f \tag{76}$$

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where F_i and G_i $(i = 1, \ldots, d)$ are respectively the convective and the viscous fluxes and the array of unknowns u depends on the problem and the choice of variables. For the shallow water equations, these are a velocity averaged in depth and the water elevation, whereas for compressible flows using conservation variables these are density, momentum and total energy (see (Hauke and Hughes, 1998) for a discussion about different options).

The convective term can be written as $\partial_i F_i = A_i \partial_i u$, where $A_i = \partial_u F_i$ is the Jacobian of F_i, and the viscous fluxes can be expressed in terms of u as $G_i = -K_{ij}\partial_j u$ for appropriate matrices K_{ij}, so that (76) has the form $\partial_t u + \mathcal{L}u = f$, with $\mathcal{L}$ formally given as in (3) but with the coefficient matrices depending on u.

Both for the shallow water equations and for compressible flows, apart from the problem nonlinearity, the possible dominance of convective terms and the possible need to satisfy compatibility conditions between variables, there is another numerical difficulty compared to the incompressible Navier-Stokes equations, which is the possible existence of shocks, i.e., discontinuities in u across surfaces. These need to be treated using the techniques described in Section 4.2, which, to our knowledge, have not been directly related to VMS methods.

The literature about stabilized FE methods for compressible flows and shallow water equations is vast, and we will not discuss it here (see (Shakib, Hughes, and Johan, 1991) for one of the early works in this line). Let us only describe some papers that use the VMS concept to design stabilized FE formulations for compressible flows, even if the approximation of the subscales is not exactly the one we have presented. The problematic of shallow water equations is very similar.

Subscales were introduced to the compressible flow equations in (Koobus and Farhat, 2004). The authors proposed a Finite Volume/FE formulation with density, velocity and temperature variables, treating each equation separately and consequently stabilizing the formulation using a sixth order upwind effect in the advective term. They separated the scales a priori and used a Finite Volume cell-agglomeration projector to define the subscales. The effect of the subscales into the resolved scales was taken into account by means of a Reynolds stress calculation.

The authors in (van der Bos, van der Vegt, and Geurts, 2007) contrasted that initial work in (Koobus and Farhat, 2004), specifically in the subscale cell-agglomeration projector definition. They considered the compressible Navier-Stokes equations in terms of density, velocity and total energy and again separated the scales independently for each equation. A general discontinuous Galerkin method cared for evaluating the subscales only inside the elements. Again, they accounted for the effect of the subscales into the resolved scales with the Reynolds stress, but used a Fourier-Spectral projector to define the subscales that was self-adjoint and commutative. Within this branch, in (Levasseur, Sagaut, Chalot, and Davroux, 2006) a stabilized FE method (GLS) was used to deal with the compressible Navier-Stokes equations written with the entropy variables in system form. The authors proposed a filtering procedure with subgrid entropy variables and modeled the effects of subscales into the resolved scales with a turbulence model.

In (Dahmen, Gotzen, Müller, and Schäfer, 2010) a multilevel representation for the scales applied to the compressible Navier-Stokes equations was presented. The authors used conservative variables but again considered each equation independently, and they stabilized using suitable bases for the Finite Volume method. Velocity, pressure, shear stress and temperature variables were used in the mixed terms that resulted after averaging the compressible equations. The influence of the subscales was modeled using the Smagorinsky turbulence model in order to solve turbulent compressible flows.

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Stabilization methods proposed strictly in the frame of the VMS method were first attempted in the compressible Navier-Stokes equations in (Rispoli and Saavedra, 2006). The authors extended satisfactorily the VMS stabilized FE formulation previously developed for the incompressible Navier-Stokes equations in (Rispoli, Corsini, and Tezduyar, 2007). The compressible equations were written in system form using density, momentum and total energy conservation variables. The expression for the stabilization parameters employed was an extension of the one used for the CDR problem (see Table II) using quadratic elements. More recently, the VMS method was applied in (Marras, Moragues, Vázquez, Jorba, and Houzeaux, 2013) to stabilize the compressible flow equations. Convergence of the method in a widespread range of flow regimes, including low Mach numbers, was demonstrated.

6.2. Low speed thermally coupled flows

Even though the compressible Navier Stokes equations accurately describe thermally coupled flows, their application to low speed (compared to the sound velocity) flows posses extra challenges. The essential problem is that the flow has the same difficulties in the pressure treatment as incompressible flows, which generates stability problems, as already discussed. Another related problem with compressible flows is the presence of acoustic phenomena, which are pressure and density waves of small amplitude and fast propagation velocity (the sound speed) that satisfy the system of equations. It is easy to see that a wave equation for the pressure can be deduced from the full compressible equations. When the Mach number (ratio of the flow speed and sound speed) is small this hyperbolic wave equation for the pressure becomes the well known elliptic pressure Poisson equation.

It is therefore desirable to obtain a simplified set of equations taking the zero Mach number limit of the compressible Navier-Stokes equations. For details about this asymptotic expansion procedure, see (Principe and Codina, 2009; Majda and Sethian, 1985). As a particular result of this process, the total pressure is split into two parts, the thermodynamic part $p^{\text{th}}(t)$, which is uniform in space, and the hydrodynamic part $p(\boldsymbol{x}, t)$ which is several orders of magnitude smaller than p^{th} and is therefore omitted in the state and energy equations. This leads to a removal of the acoustic modes but large variations of density due to temperature variations are allowed. This system of equations is commonly used to describe problems of combustion in the form of deflagrations (i.e., flames at low speed).

From the point of view of their numerical approximation, the zero Mach number equations present the same difficulties as the incompressible Navier Stokes equations, namely the compatibility conditions between the velocity and pressure spaces, and the instabilities due to convection terms. A VMS formulation for the low Mach number equations was presented in (Gravemeier and Wall, 2010) and also in (Avila, Principe, and Codina, 2011; Avila, Codina, and Principe, 2014), where the full nonlinear and time dependent subscales described in Section 5 were used (it is worth noting that the low Mach number equations present very strong nonlinearities).

However, the most widely used model in the context of thermally coupled flows is the so called Boussinesq approximation. In this approximation, whose asymptotic derivation is delicate (Principe and Codina, 2009), density variations are neglected everywhere except in the forcing term, including the continuity equations, which makes the flow incompressible. A VMS approximation of the Boussinesq equations with time dependent and nonlinear subscales was proposed in (Codina and Principe, 2007) and applied to turbulent flows in (Codina,

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Principe, and Avila, 2010).

6.3. Darcy's problem and Waves

Darcy's problem (22)-(23) governs the flow of an incompressible fluid through a porous medium. It is composed by the Darcy law that relates the fluid velocity (the flux) and the pressure gradient and the mass conservation equation. In flow in porous media, a proper functional setting for this problem the dual form (see Table I), since the flux is usually the variable of interest. This yields a saddle-point problem that is well posed due to inf-sup conditions known to hold at the continuous level, and that allow one to obtain stability estimates for the pressure and the velocity divergence.

As for the Stokes problem, the Galerkin approximation of this indefinite system is a difficult task, because the continuous inf-sup conditions are not naturally inherited by most FE velocity-pressure spaces. One could avoid these problems by resorting to the primal form, getting a pressure Poisson problem; this is an elliptic problem that can be easily approximated by the Galerkin technique and Lagrangian elements. The fluxes can be obtained as a post-process by using a L^2-projection. This approach is computationally appealing because pressure and velocity computations are decoupled and the implementation is easy. Unfortunately, this approach has two drawbacks: the loss of accuracy for the velocity and the very weak enforcement of the mass conservation equation. Improved post-processing techniques that reduce these problems can be found e.g. in (Correa and Loula, 2007; Coutinho, Dias, Alves, Landau, Loula, Malta, Castro, and Garcia, 2004). This approach has been restricted to continuous (H^1-conforming) pressure FE spaces. However, the exact pressure admits discontinuities, e.g. in regions with jumps of the physical properties, and this approach leads to poor accuracy in the vicinity of these regions.

The indefinite problem can be approximated by the Galerkin technique and mixed FE formulations (see (Boffi, Brezzi, and Fortin, 2013)) that satisfy the inf-sup conditions required for the well-posedness of the discrete problem. An example of inf-sup stable pair is the Raviart-Thomas FE space, composed by velocities with continuous normal traces and discontinuous tangential traces on the element boundaries and discontinuous pressures of one order of interpolation less. The element unknowns are the normal fluxes on the faces, but all components are needed inside every element domain. This makes the implementation involved, specially for three dimensional problems.

As explained earlier, an alternative is to resort to stabilization techniques, so that arbitrary interpolations can be used. Residual-based stabilization techniques for the Darcy problem were originally designed in (Masud and Hughes, 2002). Therein, the stabilized problem mimics the mixed Laplacian functional setting (the pressure belongs to $H^1(\Omega)$ and the velocity belongs to $L^2(\Omega)^d$) and leads to the same order of convergence that is attained when using the pressure Poisson problem plus post-processing. This method was extended to discontinuous FE spaces for velocities and pressures in (Brezzi, Hughes, Marini, and Masud, 2005; Hughes, Masud, and Wan, 2006). The methods proposed there correspond to approximation (31). In (Correa and Loula, 2008), a stabilized conforming FE formulation is presented that gives very strong stability bounds; both velocity and pressure are in $H^1(\Omega)$. However, no convergence is attained for the natural norm and only L^2-norms of the errors can be bounded using elliptic regularity properties.

In (Badia and Codina, 2010), stabilized methods based on the VMS decomposition were

developed for the Darcy problem. The stabilization parameters were designed based on a heuristic Fourier analysis and are summarized in Table II. The definition of this matrix involves a characteristic length scale. The choice of this characteristic length, which can be either the element size or the diameter of the domain, leads to stabilized methods with different stability and convergence properties. In this frame, we can recover numerical methods that mimic the typical setting in Darcy's flow (dual form) as well as others that mimic the mixed Laplacian formulation (primal form). Intermediate settings with unclear continuous counterpart but interesting convergence properties are also designed. Roughly speaking, we can increase the velocity stability reducing pressure stability and vice-versa, and analogously for the convergence rate. The optimal method depends on the velocity and pressure approximation order.

The use of stabilized methods for the Darcy problem is also appealing when we consider coupled Stokes-Darcy problems. The Galerkin approximation of both the Stokes and the Darcy problems requires the use of velocity-pressure interpolations that satisfy the adequate inf-sup conditions. Different interpolation pairs are known to satisfy this condition for each problem independently, but the key issue is to find interpolations that satisfy *both* at the same time. The design of this kind of mixed velocity-pressure interpolations, even in the non-conforming case for the Stokes problem, is a difficult task (see (Arbogast and Brunson, 2007)). Furthermore, the resulting mixed interpolations are expensive and in some cases restricted to specific typologies of meshes. When using stabilized methods, it is possible to use *any* velocity-pressure pair, provided the discrete spaces are conforming. The most important feature, however, is that *the same* formulation works for the Stokes and the Darcy problems. A thorough presentation of stabilized FE methods for the Stokes-problem, and its complete numerical analysis, including Aubin-Nitsche arguments, can be found in (Badia and Codina, 2009b). Therein, classical stabilized techniques for the Stokes problem were blended with the formulations presented in (Badia and Codina, 2010) for the Darcy problem. Several other attempts based on some sort of stabilization can be found in the literature. For example, in (Burman and Hansbo, 2007) it is proposed to use a continuous linear velocity and piecewise constant pressure interpolation, since a common stabilization procedure can be designed for the Stokes and the Darcy problems using this element (even though these two problems are studied independently).

The stabilized FE techniques based on the VMS concept that have been developed for the Darcy problem can be also applied to the wave equation in mixed form. These were proposed in (Codina, 2008a). Following the ideas in (Badia and Codina, 2010), stabilized FE formulations that can mimic three functional settings proposed at the continuous level were designed and analyzed in (Badia, Codina, and Espinoza, 2014a).

6.4. Incompressible Magnetohydrodynamics

Incompressible visco-resistive MHD equations model incompressible viscous and electrically conducting fluids under the influence of electromagnetic fields. If $\boldsymbol{b}$ is the magnetic (induction) field, the Navier-Stokes problem (48)-(51) has to be solved, adding the Lorentz force $-\varrho(\nabla \times \boldsymbol{b}) \times \boldsymbol{b}$ to the LHS of the momentum equation (48) and solving for $\boldsymbol{b}$ from the equations

$$\varrho \partial_t \boldsymbol{b} + \lambda \nabla \times \nabla \times \boldsymbol{b} - \nabla \times (\boldsymbol{u} \times \varrho \boldsymbol{b}) = \boldsymbol{0} \tag{77}$$

$$\nabla \cdot \boldsymbol{b} = 0 \tag{78}$$

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where ϱ and λ are physical properties and appropriate initial and boundary conditions need to be added to (77)-(78).

Condition (78) holds exactly for all time if it holds at the initial time, but this is not so at the numerical level. To enforce it, the gradient of a scalar unknown r can be added to the LHS of (77), acting as a sort of Lagrange multiplier and giving a saddle point structure to the stationary and linear version of (77)-(78), i.e, Maxwell's problem. Another alternative is to add a penalty term to (77) enforcing (78).

The introduction of ∇r in (77) leads to a double-saddle-point formulation for the MHD system. A Galerkin FE approximation of the resulting problem was proposed and analyzed in (Schötzau, 2004), based on Nedelec's or edge elements. Saddle-point formulations require to choose particular mixed FE spaces satisfying discrete versions of the adequate inf-sup conditions, which as in the previous problems complicates implementation issues, e.g. the database structure, the computation of the coupling terms and the graphs needed for the compressed storage of the system matrix. Furthermore, inf-sup stable formulations still require stabilization terms when dealing with under-resolved convection dominant flows.

With the aim to solve the Maxwell problem with Lagrangian FEs, the differential operator of the problem can be transformed into an elliptic one by adding an exact penalty term containing the divergence of $\boldsymbol{b}$. Since it is zero at all times at the continuous level, the magnetic stabilization reads as a div-div term, which can be understood as an exact penalty formulation. Under some assumptions on the computational domain, the magnetic field is smoother than for the original problem (see e.g. (Gerbeau, Le Bris, and Lelièvre, 2006)). An inf-sup stable Galerkin FE formulation for the flow sub-problem and the regularized exact penalty formulation for the magnetic sub-problem was proposed in (Gunzburger, Meir, and Peterson, 1991), whereas a stabilized FE version of this formulation can be found in (Gerbeau, 2000). A stabilized FE formulation based on the VMS concept of the regularized saddle-point version of the system was proposed in (Codina and Hernández, 2006) (see also (Codina and Hernández, 2011) for the thermally coupled problem). A related stabilization technique can be found in (Ben Salah, Soulaimani, and Habashi, 2001). The formulations in (Ben Salah, Soulaimani, and Habashi, 2001; Gerbeau, 2000; Gerbeau, Le Bris, and Lelièvre, 2006; Codina and Hernández, 2006, 2011; Shadid, Pawlowski, Banks, Chacon, Lin, and Tuminaro, 2010) are based on the regular functional setting of the problem, and so, restricted to smooth or convex domains (see (Hazard and Lenoir, 1996; Costabel and Dauge, 2002)). They are accurate for regular magnetic solutions but tend to spurious (unphysical) solutions otherwise.

In order to solve this problem, the missing ingredient is a Lagrangian FE scheme for the Maxwell sub-problem that can converge to singular solutions. In (Costabel and Dauge, 2002) a rehabilitation of H^1-conforming $\mathcal{C}^0$ nodal (i.e. Lagrangian) FEs is proposed, based on a weighted version of the penalty term that was able to converge to the"good" solution in nonconvex domains. In order to use the resulting numerical method, singularity regions have to be identified *a priori*, and proper weighted functions constructed based on this information. In the negative side, this clearly complicates the numerical integration (of the weighted term), loses computational efficiency and complicates the automatization of the simulations. An alternative approach to solve Maxwell's problem is the decomposition of the solution into singular and smooth part (see (Assous, Jr., Labrunie, and Segré, 2003; Hazard and Lenoir, 1996)) but this method is harder to generalize, specially in three dimensions. In (Duan, Jia, Lin, and Tan, 2009) a method based on local projections that uses a FE space composed of cubic nodal elements enriched with edge and element bubbles was designed. The introduction

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of the local projection in the penalty term allows one to converge to nonsmooth solutions, but the same projection weakens convergence, which is only attained in the L^2 norm. There are other nodal-based FE methods, but they converge to spurious solutions in nonconvex domains (see e.g. (Jin, 1993)). The weighted exact penalty formulation was applied to the MHD system in (Hasler, Schneebeli, and Schötzau, 2004).

More recently, new mixed FE formulations for Lagrangian FEs, based on a stabilized VMS-type approximation of a novel augmented formulation of the Maxwell problem, were proposed in (Badia and Codina, 2012). We also refer to (Bonito and Guermond, 2011) for a similar approach, regarding the eigenvalue problem. The essential idea in these works to converge to singular solutions is to avoid the spurious control on the $L^2(\Omega)$-norm of the divergence of $\boldsymbol{b}$, typical of penalized or curl-div formulations. Instead of avoiding this by using weighted $L^2(\Omega)$-inner products, the idea is to resort to the introduction of ∇r in (77), as explained above. However, to ensure stability in the appropriate functional setting, a novel augmented formulation has to be introduced, which consists of adding a Laplacian of the multiplier in the zero divergence restriction. Since the multiplier is zero in the continuous problem, consistency remains unaltered. The final ingredient is to use a stabilized formulation at the discrete level, consisting only in adding a least-square form of the zero divergence condition. The stabilizing term is multiplied by the square of the mesh size, so that it mimics stability of the divergence of the unknown in $H^{-1}(\Omega)$ (dual of $H^1(\Omega)$), not in $L^2(\Omega)$, as curl-div formulations wrongly do. This new term is also responsible for obtaining stability in $L^2(\Omega)$ of $\boldsymbol{b}$. Finally, in order to have approximability for linear Lagrangian elements, particular mesh typologies must be used for singular solutions that can easily be generated by a cheap post-processing of any original triangular or quadrilateral mesh, both in two and three dimensions (see (Badia and Codina, 2012) for more details).

In (Badia, Codina, and Planas, 2013, 2014b), a nodal-based stabilized FE formulation of the MHD problem based on the VMS concept was proposed, in which the magnetic sub-problem is approximated following the ideas in (Badia and Codina, 2012). The resulting scheme always converges to the correct solution, solving the problems related to previous approaches. Furthermore, the introduction of stabilization terms changes the nature of the problem, and subsequently the system matrix. Whereas the system matrix obtained by using mixed FEs is indefinite, with null diagonal-blocks, the one from the stabilized FE formulation is positive definite. It simplifies the numerical linear algebra strategy to solve the final linear system in an efficient (optimal) way (see (Shadid, Pawlowski, Banks, Chacon, Lin, and Tuminaro, 2010) for a detailed discussion).

7. ON THE NUMERICAL ANALYSIS OF VMS-BASED STABILIZED FINITE ELEMENT METHODS

To give an overview of the numerical analysis of the formulations presented in this article is clearly outside our scope. However, we describe in the following some of the *key* ideas to obtain a priori stability and convergence estimates and a posteriori error estimates . These basic ideas can be presented for an abstract *linear* problem.

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7.1. A priori results

Let us consider the simplest of the situations analyzed in Sections 2 and 3, namely, subscales proportional to FE residuals and vanishing on the inter-element boundaries. From (15) and (31) it follows that the problem to be solved can be written as follows: find $u_h \in V_h$ such that

$$B_h(u_h, v_h) = L_h(v_h) \tag{79}$$

for all $v_h \in V_h$, where

$$B_h(u_h, v_h) = B(u_h, v_h) + \sum_K \langle \tau_K \mathcal{L} u_h, -\mathcal{L}^* v_k \rangle_K$$

$$L_h(v_h) = L(v_h) + \sum_K \langle \tau_K f, -\mathcal{L}^* v_k \rangle_K$$

Let us split $\mathcal{L}$ into a symmetric part $\mathcal{L}_1$ and a skew-symmetric one $\mathcal{L}_2$, so that

$$\mathcal{L} u_h = \mathcal{L}_1 u_h + \mathcal{L}_2 u_h, \quad \mathcal{L}^* v_h = \mathcal{L}_1 v_h - \mathcal{L}_2 v_h$$

since u' is assumed to vanish on the element boundaries.

Let $\gtrsim$ stand for $\geq$ up to constants (dimensionless and independent of the numerical discretization), and similarly for $\lesssim$, and let us denote by $\|\cdot\|_{\mathrm{G}}$ the norm in which the Galerkin method is stable, which may result from an inf-sup condition (without restrictions on the interpolation spaces) or from a coercivity property of the form $B(v_h, v_h) \gtrsim \|v_h\|_{\mathrm{G}}^2$. Suppose the latter situation for simplicity. To prove stability of problem (79) the simplest is to start taking $v_h = u_h$, which yields

$$B_h(u_h, u_h) \gtrsim \|u_h\|_{\mathrm{G}}^2 - \sum_K \langle \tau_K \mathcal{L}_1 u_h, \mathcal{L}_1 u_h \rangle_K + \sum_K \langle \tau_K \mathcal{L}_2 u_h, \mathcal{L}_2 u_h \rangle_K$$

Assuming τ_K positive definite, we may consider *the stabilized norm* $|||\cdot|||$, defined by

$$|||v_h|||^2 = \|u_h\|_{\mathrm{G}}^2 + \sum_K \langle \tau_K \mathcal{L}_2 u_h, \mathcal{L}_2 u_h \rangle_K \tag{80}$$

In essence, the analysis of VMS-type stabilized FE methods consists of the following steps:

- Show that $\|u_h\|_{\mathrm{G}}^2 - \sum_K \langle \tau_K \mathcal{L}_1 u_h, \mathcal{L}_1 u_h \rangle_K \gtrsim \|u_h\|_{\mathrm{G}}^2$. This usually poses conditions on the parameters on which τ_K depends. If this is proved, we immediately have the coercivity property $B_h(u_h, u_h) \gtrsim |||u_h|||^2$. In the case in which the projection onto the space of subscales P' is not the identity on FE residuals, this coercivity has to be replaced by a more general inf-sup condition which requires some more elaboration.
- Show that L_h is continuous with the norm $|||\cdot|||$. From this step, which is usually trivial, and the previous one, stability of problem (79) is guaranteed in the norm $|||\cdot|||$.
- By construction, residual-based FE methods are consistent, i.e., the exact solution u satisfies $B_h(u, v_h) = L_h(v_h)$ for all $v_h \in V_h$. However, bounds for the consistency error have to be obtained for some of the methods presented in Section 4.
- Obtain the interpolation error in the norm $|||\cdot|||$. If $\tilde{u}_h$ is the best interpolant of u in V_h, this error $E(h)$ will be such that $|||u - \tilde{u}_h||| \lesssim E(h)$.
- Show that $B(u - \tilde{u}_h, v_h) \lesssim |||v_h||| E(h)$. This condition is weaker than the continuity of B in the norm $|||\cdot|||$, but it is what is really needed.

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- From the last steps and the triangle inequality, it easily follows the a priori estimate $\|\!|u - u_h|\!\| \lesssim E(h)$, i.e., the method is convergent with error function $E(h)$.

As a result of this analysis, the method that leads to (79) is stable and convergent in the norm $\|\!|\cdot|\!\|$. Then one may proceed trying to obtain analytical results in other finer norms, but the gain with respect to the Galerkin method is clear: control (stability and convergence) is now attained for the skew-symmetric part of operator $\mathcal{L}_2$, the one "unseen" by the Galerkin method. This is what VMS-type stabilized FE methods provide. What is mandatory is that the additional control obtained *should not be lost in the asymptotic limits of the physical parameters*.

With variants and adaptations to each particular problems, the ideas presented here have been applied to the flow problems considered in this article in the references where the numerical analysis of the methods is carried out. The reader is referred to these works for details.

Even if the structure of the numerical analysis of the methods described is well understood, there are still some questions that remain open. Apart from adequate stability and error estimates for time dependent problems, which we have not touched here, an important issue not satisfactorily answered is the *hp* analysis, for example of CDR equations. The techniques in this case are also well understood, but there is a gap in the range of physical parameters for which optimal error estimates have not been obtained (see (Lube and Rapin, 2006; Houston and Süli, 2001)).

Another open question is how to design the stabilization parameters and obtain the corresponding stability and error estimates in the case of anisotropic mesh refinement. Some results in this direction can be found in (Apel and Lube, 1996).

Finally, in the case of the incompressible Navier-Stokes equations the combination of stabilized FE methods to deal with convection-dominated flows with elements that satisfy the inf-sup condition is a natural option. However, even if it is possible to obtain stability and error estimates in this case, the results obtained so far are not robust with respect to the viscosity (see e.g. (Gelhard, Lube, Olshanskii, and Starcke, 2005)).

7.2. A posteriori error analysis

The possibility of using the VMS method in order to build a posteriori error estimators was already pointed out in (Hughes, Feijóo, Mazzei, and Quincy, 1998) and it is also explained in some detail in (Hughes, Scovazzi, and Franca, 2004). Since the subscales are a model for the unresolved part of the solution, it seems natural to use them as an explicit a posteriori estimator for the error. A number of works have exploited this feature in adaptive refinement algorithms in the recent years.

A first approach to use the VMS framework in error estimation is presented in (Hauke, Doweidar, and Miana, 2006c). The exact subscales can be considered the error of the FE solution, and they can be computed by using the Green's function associated to the fine scale equation. If the subscales do no vanish on the interelement boundaries, but are zero on Γ, (34) has to be generalized to

$$u'(\boldsymbol{x}) = \sum_K \int_K g_K(\boldsymbol{x},\boldsymbol{y}) \mathcal{R} u_h(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} + \sum_{E \in \mathcal{E}_h^0} \int_E g_E(\boldsymbol{x},\boldsymbol{y}) [\![\mathcal{F}_n u_h]\!](\boldsymbol{y}) \mathrm{d}\boldsymbol{y}$$

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for appropriate Green's functions g_K and g_E. As their exact expression cannot be computed in general, the approximations described in the previous sections can be employed.

For example, in (Hauke, Doweidar, and Miana, 2006c) the error $L^2(K)$-norm for each element is computed for the 1D CDR equation neglecting the contribution of the subscales on the element boundaries as

$$\|u'\|^2_{L^2(K)} \approx \tau_K^2 \int_K (\mathcal{R}u_h)^2$$

with τ_K is computed in each element using (36).

When applied to the high Péclet number CDR equation, this approach results in a robust error estimator with an effectivity index close to 1. On the other hand, it is observed that for diffusion dominated problems the relative importance of the subscales on the element boundaries is higher, and the error estimator without accounting for the interelement subscales has a poorer performance. The methodology is extended to higher order elements in (Hauke, Doweidar, Fuster, Gómez, and Sayas, 2006a). Other possibilities for the definition of τ_K in the error estimator are explored in (Hauke, Doweidar, and Miana, 2006b).

The previous strategy was extended to two-dimensional domains in (Hauke, Fuster, and Doweidar, 2008), where also the contribution of the error on the element boundaries is taken into account in the error estimator. The expression used is:

$$\|u'\|^2_{L^2(K)} \approx \tau_K^2 \int_K (\mathcal{R}u_h)^2 + \frac{1}{2}\frac{|\partial K|}{|K|}\tau_K^2 \int_{\partial K} [\![\mathcal{F}_n u_h]\!]^2$$

The same ideas are applied to the incompressible Navier-Stokes equations in (Rossi, Cotela, Lafontaine, Dadvand, and Idelsohn, 2013; Hauke, Fuster, and Lizarraga, 2015) and to linear elasticity in (Hauke and Irisarri, 2015).

A different approach can be found in (Larson and Malqvist, 2007), where following (Hou and Wu, 1997) the equation for the subscales is solved in a finer mesh instead of using an analytical expression for them. The basic idea in (Larson and Malqvist, 2007) is to use a localized partition of unity interpolation which allows one to solve the equations for the subscales in a decoupled way on patches with homogeneous Dirichlet boundary conditions. The error of the fine scale problem is then a function of the fine mesh size and of the size of the element patches used to build the subgrid scale meshes. This approach can be easily linked with adaptive refinement strategies by changing the size and patches of each subgrid scale subdomain; it was also pursued in (Larson and Malqvist, 2009) applied to the CDR equation, and it was applied to nearly-incompressible elasticity in (Masud, Truster, and Bergman, 2011), and to elliptic problems with highly varying physical coefficients in (Malqvist and Peterseim, 2014).

Obviously, the literature about a posteriori error estimation using *standard* analytical tools and applied to stabilized FE methods is richer, but here our objective has been to highlight those works that make explicit use of the VMS concept to derive a posteriori error estimations, in spite of the fact that a rigorous numerical analysis of the methods described has not been done, yet.

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