

$$\begin{aligned} \mathcal{H} &\equiv \mathcal{H}_1, & \mathcal{G} &\equiv \mathcal{H}_2 \\ \mathcal{U} &\equiv \mathcal{H}_1, & \mathcal{V} &\equiv \mathcal{H}_2 \end{aligned}$$

Proof: We first consider the operator A . Since $Au \in \mathcal{U} = \mathcal{V}' \subset \mathcal{G}'_0$ for $u \in \mathcal{G}_A$, we may write

$$B(u, v) = (v, Au)_v, \quad \forall v \in \mathcal{G}_0$$

Now pick $v \in \mathcal{G}$ and introduce the operator C_A defined by

$$(C_A u, v)_v = B(u, v) - (v, Au)_v, \quad u \in \mathcal{G}_A, \quad v \in \mathcal{G}$$

Since $(C_A u, v)_v = 0 \forall v \in \mathcal{G}_0$, C_A maps \mathcal{G}_A into the orthogonal complement $\mathcal{G}_0^\perp \subset \mathcal{G}_0$.

Next we recall that γ^* maps \mathcal{G} onto $\partial\mathcal{G}$, and, therefore, its transpose γ^{**} is a bijective map of $\partial\mathcal{G}'$ onto its closed range in \mathcal{G}' . Hence γ^{**} is invertible and the range of γ^{**} is \mathcal{G}_0^\perp ; i.e., $(\gamma^{**})^{-1}$ maps \mathcal{G}_0^\perp onto $\partial\mathcal{G}'$. Set

$$\delta = (\gamma^{**})^{-1} C_A$$

Then δ maps \mathcal{G}_A onto $\partial\mathcal{G}'$, and we can write

$$C_A u = \gamma^{**}(\gamma^{**})^{-1} C_A u = \gamma^{**} \delta u, \quad u \in \mathcal{G}_A$$

Consequently, for $C_A u \in \mathcal{U}$, we have

$$(C_A u, v)_v = (\gamma^{**} \delta u, v)_v = \langle \delta u, \gamma^{**} v \rangle_{\partial\mathcal{G}}$$

Thus, for $u \in \mathcal{G}_A$, we have

$$B(u, v) = (v, Au)_v + \langle \delta u, \gamma^{**} v \rangle_{\partial\mathcal{G}}$$

as asserted.

That δ is unique follows by contradiction: assume otherwise; i.e., let there exist another operator $\delta_1 \in \mathcal{L}(\mathcal{G}_A, \partial\mathcal{G})$ such that the above formula holds. Then, for any $u \in \mathcal{G}_A$,

$$\delta u = (\gamma^{**})^{-1} C_A u = (\gamma^{**})^{-1} \gamma^{**} \delta_1 u = \delta_1 u$$

Hence $\delta = \delta_1$.

The derivation of the second formula follows identical arguments. We introduce an operator $C_A : \mathcal{G}_A \rightarrow \mathcal{G}_0^\perp$ such that $(C_A v, u)_u = B(u, v) - (A^* v, u)_u$. Then

$$\delta^* = \gamma'^{-1} C_A \in \mathcal{L}(\mathcal{G}_A, \partial\mathcal{G})$$

is a uniquely defined linear operator such that the second formula holds. ■

A schematic diagram illustrating the various spaces and operators is given in Fig. 6-2.

Dirichlet and Neumann Operators Let $B : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{R}$ be a continuous bilinear form on Hilbert spaces \mathcal{G} and \mathcal{G} having the trace property. Let, in addition, A be the formal operator associated with $B(\cdot, \cdot)$ and let A^* be its formal adjoint. The operators $\gamma \in \mathcal{L}(\mathcal{G}, \partial\mathcal{G})$ and $\delta \in \mathcal{L}(\mathcal{G}_A, \partial\mathcal{G})$ described above are called the *Dirichlet operator* and the *Neumann operator*, respectively, corresponding to the operator A . Likewise, the operators $\gamma^* \in \mathcal{L}(\mathcal{G}, \partial\mathcal{G})$

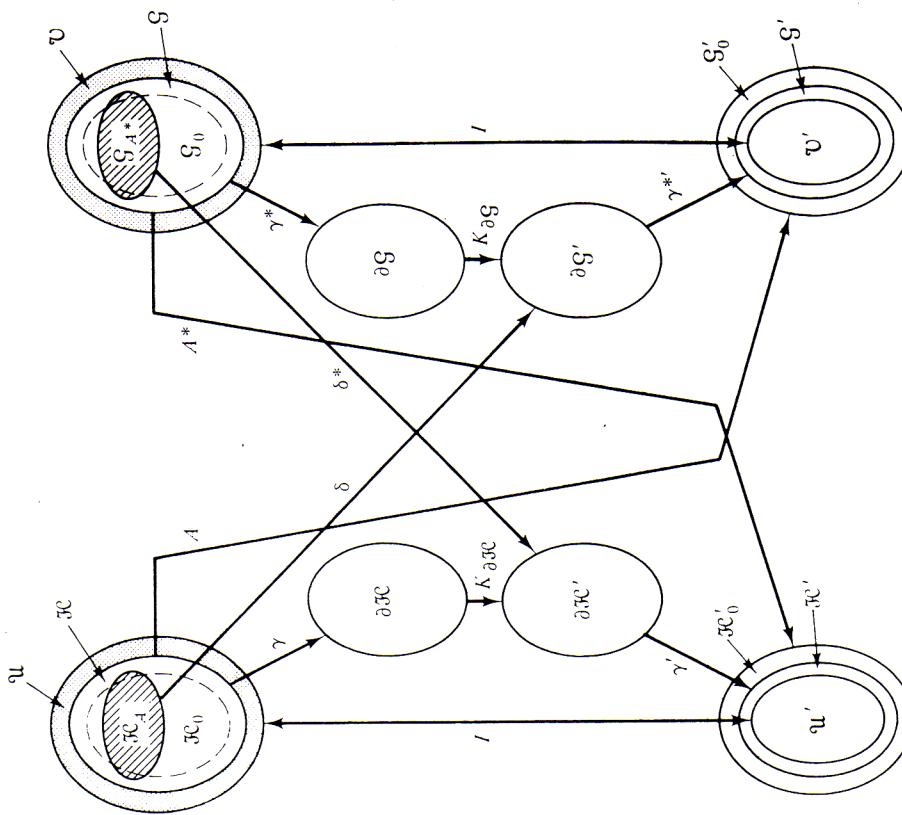


Figure 6-2 Diagram of spaces and operators in the generalized Green's formulae

and $\delta^* \in \mathcal{L}(\mathcal{G}_A, \partial\mathcal{G})$ described above are called the *Dirichlet* and *Neumann operators*, respectively, corresponding to the operator A^* . ■

Green's Formulae The relationships derived in Theorem 6.6.1 are called *Green's formulae* for the bilinear form $B(\cdot, \cdot)$:

$$B(u, v) = (v, Au)_v + \langle \delta u, \gamma^* v \rangle_{\partial\mathcal{G}}, \quad u \in \mathcal{G}_A, \quad v \in \mathcal{G}$$

$$B(u, v) = (A^* v, u)_u + \langle \delta^* v, \gamma u \rangle_{\partial\mathcal{G}}, \quad u \in \mathcal{G}, \quad v \in \mathcal{G}_A$$

If we take $u \in \mathcal{G}_A$ and $v \in \mathcal{G}_A$, we obtain the *abstract Green's formula* for

the operator $A \in \mathcal{L}(\mathfrak{J}\mathcal{C}_A, \mathfrak{G}'_0) \cap \mathcal{L}(\mathfrak{J}\mathcal{C}_A, \mathfrak{U})$:

$$(A^*v, u)_\Omega = (\mathbf{v}, Au)_0 + \langle \delta u, \gamma^* \mathbf{v} \rangle_{\partial\Omega} - \langle \delta^* \mathbf{v}, \gamma u \rangle_{\partial\Omega}$$

$$u \in \mathfrak{J}\mathcal{C}_A, \quad \mathbf{v} \in \mathfrak{G}_A.$$

The collection of boundary terms,

$$\Gamma(u, \mathbf{v}) = \langle \delta u, \gamma^* \mathbf{v} \rangle_{\partial\Omega} - \langle \delta^* \mathbf{v}, \gamma u \rangle_{\partial\Omega}$$

is called the *bilinear concomitant* of A ; $\Gamma: \mathfrak{J}\mathcal{C}_A \times \mathfrak{G}_A \rightarrow \mathbb{R}$. ■

Example 6.6.2.

Consider the case in which Ω is a smooth open bounded subset of \mathbb{R}^2 with a smooth boundary $\partial\Omega$ and

$$\mathfrak{J}\mathcal{C} = \mathfrak{G} = H^1(\Omega) = \{u; u, u_x, u_y \in L_2(\Omega)\}$$

Let $a = a(x, y)$, $b = b(x, y)$, and $c = c(x, y)$ be sufficiently smooth functions of x and y [e.g., $a, b, c \in C^1(\Omega)$], and define the bilinear form $B: H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$ by

$$B(u, v) = \int_\Omega (a \nabla u \cdot \nabla v + buu_x + cvu_y) dx dy$$

where $\nabla u = \text{grad } u = (u_x, u_y)$. We observe that $H^1(\Omega)$ has the trace property; it is dense in $L_2(\Omega)$ and trace operator $\gamma v \equiv v|_{\partial\Omega}$ can be extended to a continuous operator from $H^1(\Omega)$ onto the space $H^{1/2}(\partial\Omega)$ of functions defined as the closure of $L_2(\partial\Omega)$ in the norm $\|y\|_{H^{1/2}(\partial\Omega)}$ (see Example 6.6.1). Thus

$$\partial\mathfrak{J}\mathcal{C} = \partial\mathfrak{G} = H^{1/2}(\partial\Omega); \quad \ker \gamma = H_0^1(\Omega) = \{u \in H^1(\Omega); u = 0 \text{ on } \partial\Omega\}$$

In this case, if u is sufficiently smooth,

$$B(u, v) = \int_\Omega v(-\nabla \cdot (a \nabla u) + bu_x + cu_y) dx dy + \oint_{\partial\Omega} a \frac{\partial u}{\partial n} v ds$$

Thus the formal operator corresponding to $B(\cdot, \cdot)$ is

$$Au = -\nabla \cdot (a \nabla u) + b \frac{\partial u}{\partial x} + c \frac{\partial u}{\partial y}$$

if we take $u \in \mathfrak{J}\mathcal{C}_A = \{u \in H^1(\Omega); Au \in L_2(\Omega)\}$

Also, we now have

$$\delta u = a \frac{\partial u}{\partial n} \Big|_{\partial\Omega}, \quad \gamma^* v = v|_{\partial\Omega}$$

Similarly, if $v \in \mathfrak{G}_A = \mathfrak{J}\mathcal{C}_A$, we have

$$B(u, v) = \int_\Omega u \left(-\nabla \cdot (a \nabla v) - \frac{\partial bv}{\partial x} - \frac{\partial cv}{\partial y} \right) dx dy$$

$$+ \oint_{\partial\Omega} \left(au \frac{\partial v}{\partial n} + bvn_x + cvn_y \right) ds$$

where n_x, n_y are the components of the unit outward normal \mathbf{n} to $\partial\Omega$. Thus

$$A^*v = -\nabla \cdot (a \nabla v) - \frac{\partial bv}{\partial x} - \frac{\partial cv}{\partial y}$$

$$\gamma u = u|_{\partial\Omega}, \quad \delta^* v = \left[a \frac{\partial v}{\partial n} + (bn_x + cn_y)v \right]_{\partial\Omega}$$

The bilinear concomitant is then

$$\Gamma(u, v) = \oint_{\partial\Omega} \left[av \frac{\partial u}{\partial n} - au \frac{\partial v}{\partial n} - (cn_y + bn_x)uv \right] ds \blacksquare$$

Example 6.6.3.

Let Ω be as in Example 6.6.2 and define

$$B(u, \mathbf{v}) = \int_\Omega \text{grad } u \cdot \mathbf{v} dx dy$$

as a bilinear form on $\mathfrak{J}\mathcal{C} \times \mathfrak{G}$, where

$$\mathfrak{J}\mathcal{C} = H^1(\Omega) = \{u; u, u_x, u_y \in L_2(\Omega)\}$$

$$\mathfrak{G} = H^1(\Omega) = \{\mathbf{v} = (v_1, v_2); v_1, v_{1x}, v_{1y}, v_2, v_{2x}, v_{2y} \in L_2(\Omega)\}$$

$H^1(\Omega)$ is dense in $\mathfrak{U} = L_2(\Omega)$ and $\mathbf{H}^1(\Omega)$ is dense in $\mathfrak{V} = \mathbf{L}_2(\Omega) = L_2(\Omega) \times L_2(\Omega)$. In this case, we may take $\gamma^* \mathbf{v} = \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega}$ and $\ker \gamma^* = \mathbf{H}_0^1(\Omega)$, but $\delta \equiv 0$. Indeed, if the formal operator associated with the given bilinear form is

$$Au = \text{grad } u$$

$$\mathfrak{J}\mathcal{C}_A = \mathfrak{J}\mathcal{C} = H^1(\Omega)$$

$C_A: \mathfrak{J}\mathcal{C}_A \rightarrow \mathfrak{G}_0^1$ is identically zero. We next note that

$$B(u, \mathbf{v}) = \int_\Omega u(-\text{div } \mathbf{v}) dx dy + \oint_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} u ds$$

Thus $\mathfrak{G}_A = \mathbf{H}^1(\Omega)$ and

$$A^* \mathbf{v} = -\text{div } \mathbf{v}; \quad \gamma u = u|_{\partial\Omega}; \quad \delta^* \mathbf{v} = \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega}$$

The Green's formula is the classical relation

$$\int_\Omega (\text{grad } u \cdot \mathbf{v} + u \text{div } \mathbf{v}) dx dy = \oint_{\partial\Omega} uv \cdot \mathbf{n} ds \blacksquare$$

6.7 VARIATIONAL BOUNDARY-VALUE PROBLEMS

The Green's formulae and various properties of the bilinear forms $B(\cdot, \cdot)$ described in the previous section provide the basis for a theory of boundary-value problems involving linear operators on Hilbert spaces. We shall continue to use the notations and conventions of the previous section: $\mathfrak{J}\mathcal{C}$ and \mathfrak{G} are Hilbert spaces with the trace property, densely imbedded in pivot spaces

\mathfrak{U} and \mathfrak{V} , respectively, and $\gamma^*: \mathcal{G} \rightarrow \partial\mathcal{G}_0$, $\ker \gamma^* = \mathcal{G}_0$, $\gamma: \mathcal{J}\mathcal{C} \rightarrow \partial\mathcal{J}\mathcal{C}$, $\ker \gamma = \mathcal{J}\mathcal{C}_0$, etc.

Let $B: \mathcal{J}\mathcal{C} \times \mathcal{G} \rightarrow \mathbb{R}$ be a continuous bilinear form and let A be the formal operator associated with $B(u, v)$. We consider two types of boundary-value problems associated with A :

The Dirichlet Problem for A Given data $f \in \mathfrak{U}$ and $g \in \partial\mathcal{J}\mathcal{C}$, the problem of finding $u \in \mathcal{J}\mathcal{C}_A$ such that

$$Au = f$$

$$u = g$$

is called the *Dirichlet problem* for the operator A . ■

The Neumann Problem for A Given data $f \in \mathfrak{U}$ and $s \in \partial\mathcal{G}'$, the problem of finding $u \in \mathcal{J}\mathcal{C}_A$ such that

$$Au = f$$

$\delta u = s$

is called the *Neumann problem* for the operator A . ■

It is also possible to describe other boundary-value problems for A . For example, the *mixed problem* for A is described in Exercises 6.7.1 and 6.7.3.

Now the bilinear form described in Theorem 6.6.1 can be used to construct boundary-value problems analogous to those for A . Indeed, if

$$B(u, v) = (Au, v)_0 + \langle \delta u, \gamma^* v \rangle_{\partial\mathcal{G}}$$

we may construct the following *variational boundary-value problems*:

The Variational Dirichlet Problem for A Given data $f \in \mathfrak{U}$ and $g \in \partial\mathcal{J}\mathcal{C}$, find $u \in \mathcal{J}\mathcal{C}_0 = \ker \gamma$ such that

$$B(u, v) = (f, v)_0 - B(\gamma^{-1}g, v), \quad \forall v \in \mathcal{G}_0. \blacksquare$$

The Variational Neumann Problem for A Given data $f \in \mathfrak{U}$ and $s \in \partial\mathcal{G}'$, find $u \in \mathcal{J}\mathcal{C}$ such that

$$B(u, v) = (f, v)_0 + \langle s, \gamma^* v \rangle_{\partial\mathcal{G}}, \quad \forall v \in \mathcal{G}. \blacksquare$$

That these two seemingly different classes of boundary-value problems actually coincide is established in the following.

THEOREM 6.7.1.

The Dirichlet and Neumann problems for A are equivalent to the respective variational Dirichlet and Neumann problems for A in the following sense: if u is the solution of the Dirichlet problem, $w = u - \gamma^{-1}g$ is the solution of the variational Dirichlet problem; conversely, if w is the solution of the variational Dirichlet problem, $u = w + \gamma^{-1}g$ is the solution of the Dirichlet problem. Con-

tinuing, if u is the solution of the Neumann problem, it is also a solution of the variational Neumann problem, and if w is a solution of the variational Neumann problem, it is a solution of the Neumann problem.

Proof: First we consider the Dirichlet problem. Let w be a solution of the variational Dirichlet problem:

$$B(w + \gamma^{-1}g, v) = (f, v)_0, \quad \forall v \in \mathcal{G}_0$$

Let $u = w + \gamma^{-1}g$. Then $\gamma u = \gamma w + g = g$ since $w \in \mathcal{J}\mathcal{C}_0$. From Green's formula,

$$B(u, v) = (Au, v)_0 = (f, v)_0, \quad \forall v \in \mathcal{G}_0$$

But $(Au - f, v)_0 = 0 \forall v \in \mathcal{G}_0 \implies Au = f$, and since $f \in \mathfrak{U}$, $u \in \mathcal{J}\mathcal{C}_A$. Hence $u = w + \gamma^{-1}g$ is a solution of the Dirichlet problem.

Conversely, let u be a solution of the Dirichlet problem. Then $w = u - \gamma^{-1}g$ is such that $\gamma w = 0$; i.e., $w \in \mathcal{J}\mathcal{C}_0$. Then for $v \in \mathcal{G}_0$, $B(u, v) = B(w, v) + B(\gamma^{-1}g, v) = (Au, v)_0 = (f, v)_0$. Hence w is a solution to the variational Dirichlet problem. ■

Turning next to the Neumann problem, let u be a solution of the variational Neumann problem. Pick $v \in \mathcal{G}_0$. Then $B(u, v) = (Au, v)_0 = (f, v)_0$, so that $Au = f$ and $u \in \mathcal{J}\mathcal{C}_A$. From Green's formula, we then deduce that $(Au - f, v)_0 = \langle s - \delta u, \gamma^* v \rangle_{\partial\mathcal{G}} = 0 \forall v \in \mathcal{G}$. Hence $\delta u = s$.

Conversely, if u is a solution of the Neumann problem, we have immediately from Green's formula, $B(u, v) = (Au, v)_0 + \langle \delta u, v \rangle_{\partial\mathcal{G}} = (f, v)_0 + \langle s, v \rangle_{\partial\mathcal{G}}, \forall v \in \mathcal{G}$. ■

A similar equivalence holds for other variational boundary-value problems for A , including the mixed problem (see Exercise 6.7.1).

We observe that boundary conditions enter the statement of a variational boundary-value problem in two distinct ways. The *essential* or *stable* boundary conditions enter by simply defining the spaces $\mathcal{J}\mathcal{C}_0$ and \mathcal{G}_0 on which the problem is posed. The *natural* or *unstable* boundary conditions are introduced in the definition of the bilinear form $B(u, v)$ and are defined on the spaces $\mathcal{J}\mathcal{C}_A$ and \mathcal{G}_A .

Example 6.7.1.

Let $\mathcal{J}\mathcal{C} = \mathcal{G} = H^1(\Omega)$, $\mathfrak{U} = \mathfrak{V} = L_2(\Omega)$, $\mathcal{J}\mathcal{C}_0 = \mathcal{G}_0 = H_0^1(\Omega)$, and $\partial\mathcal{J}\mathcal{C} = \partial\mathcal{G} = H^{1/2}(\partial\Omega)$, where Ω is a smooth open bounded domain in \mathbb{R}^2 with a smooth boundary $\partial\Omega$. These are precisely the spaces described in Example 6.6.2. Consider the bilinear form

$$B(u, v) = \int_{\Omega} (\nabla u \cdot \nabla v + \alpha uv) dx dy$$

where α is a nonnegative constant. Clearly, $B(\cdot, \cdot)$ is a continuous bilinear form from $H^1(\Omega) \times H^1(\Omega)$ into \mathbb{R} . The formal operator associated with $B(\cdot, \cdot)$ is

$$A = -\Delta + \alpha$$

Dirichlet problem

for the operator A : Find $u \in \mathcal{J}\mathcal{C}_A$ such that

$$Au = f, \quad f \in \mathfrak{U}$$

$$\gamma u = g, \quad g \in \partial\mathfrak{U}$$

Let $A^* \in \mathcal{L}(\mathfrak{G}, \mathcal{J}\mathcal{C}'_0)$ denote the formal adjoint of A and let $\gamma^* \in \mathcal{L}(\mathfrak{G}, \partial\mathfrak{G})$, $\ker \gamma^* = \mathfrak{G}_0$. Then the *adjoint Dirichlet problem* corresponding to A is the problem of finding $\mathbf{v} \in \mathfrak{G}_A$, such that

$$\begin{aligned} A^*\mathbf{v} &= f^*, & f^* &\in \mathfrak{U} \\ \gamma^*\mathbf{v} &= g^*, & g^* &\in \partial\mathfrak{G} \end{aligned}$$

We also introduce the null spaces,

$$\mathfrak{N}(A, \gamma) = \{u \in \mathcal{J}\mathcal{C}_A : Au = 0, \gamma u = 0\}$$

$$\mathfrak{N}(A^*, \gamma^*) = \{\mathbf{v} \in \mathfrak{G}_A : A^*\mathbf{v} = 0, \gamma^*\mathbf{v} = 0\}$$

We shall assume that these spaces are finite-dimensional.

Now if $\mathfrak{N}(A, \gamma)$ is finite-dimensional, it is closed in \mathfrak{U} , and we have

$$\mathfrak{U} = \mathfrak{N}(A, \gamma) \oplus \mathfrak{N}(A, \gamma)^\perp$$

where $\mathfrak{N}(A, \gamma)^\perp = \{u \in \mathfrak{U} : (u, v)_\mathfrak{U} = 0 \quad \forall v \in \mathfrak{N}(A, \gamma)\}$

Then A can be regarded as a continuous linear operator from $\mathfrak{N}(A, \gamma)^\perp$ onto its range $\mathfrak{R}(A) \subset \mathfrak{U}$. Suppose A is bounded below. Then, by the Banach theorem, a continuous inverse A^{-1} exists from $\mathfrak{R}(A)$ onto $\mathfrak{N}(A, \gamma)^\perp$, and $\mathfrak{R}(A)$ is closed in \mathfrak{U} . Thus $\mathfrak{U} = \mathfrak{R}(A) \oplus \mathfrak{R}(A)^\perp$, where $\mathfrak{R}(A)^\perp = \{\mathbf{v} \in \mathfrak{U} : (\mathbf{v}, \mathbf{w})_0 = 0 \quad \forall \mathbf{w} \in \mathfrak{R}(A)\}$. The Dirichlet problem for A clearly has at least one solution whenever the data $f \in \mathfrak{R}(A)$ and $g \in \mathfrak{R}(\gamma)$. Data satisfying these requirements is said to be *compatible* with the operators (A, γ) .

For Neumann problems

$$\begin{aligned} Au &= f, & A^*\mathbf{v} &= f^* \\ \delta u &= s, & \delta^*\mathbf{v} &= s^* \end{aligned}$$

we define

$$\mathfrak{N}(A, \delta) = \{u \in \mathcal{J}\mathcal{C}_A : Au = 0, \delta u = 0\}$$

$$\mathfrak{N}(A^*, \delta^*) = \{\mathbf{v} \in \mathfrak{G}_A : A^*\mathbf{v} = 0, \delta^*\mathbf{v} = 0\}$$

Collecting these notions, we arrive at the following tests for compatibility of the data.

THEOREM 6.7.2 (Compatibility of the Data).

(i) Let $f \in \mathfrak{U}$ and $g \in \partial\mathfrak{U}$ be data in a Dirichlet problem for an operator A . Then a necessary condition that there exist a solution to the Dirichlet problem is that

$$(f, \mathbf{v})_\mathfrak{U} - \langle \delta^*\mathbf{v}, g \rangle_{\partial\mathfrak{U}} = 0, \quad \forall \mathbf{v} \in \mathfrak{N}(A^*, \gamma^*)$$

(ii) Let $f \in \mathfrak{U}$ and $s \in \partial\mathfrak{G}$ be data in a Neumann problem for an operator A . Then a necessary condition that there exist a solution to the Neumann

problem is that

$$(f, \mathbf{v})_\mathfrak{U} + \langle s, \gamma^*\mathbf{v} \rangle_{\partial\mathfrak{U}} = 0, \quad \forall \mathbf{v} \in \mathfrak{N}(A^*, \delta^*)$$

Proof: The proof of this theorem is similar to that of Theorems 5.10.9 and 6.5.2 and is left as an exercise. ■

A similar compatibility condition can be developed for mixed boundary-value problems (see Exercise 6.7.4).

Whenever the compatibility conditions hold, a solution to the Dirichlet or Neumann problem may exist, but it will not necessarily be unique. The solution u of the Dirichlet problem is unique, of course, whenever $\mathfrak{N}(A, \gamma) = \{0\}$, and the Neumann problem has a unique solution whenever $\mathfrak{N}(A, \delta) = \{0\}$. However, these conditions often do not hold. We can, however, force the solutions to either class of problems to be unique by imposing an additional condition.

THEOREM 6.7.3.

Let $u \in \mathcal{J}\mathcal{C}_A$ be a solution to the Dirichlet problem for the operator A . Then u is the only solution to this problem if

$$(u, w)_\mathfrak{U} = 0, \quad \forall w \in \mathfrak{N}(A, \gamma)$$

Likewise, a solution u of the Neumann problem for A is unique if $(u, w)_\mathfrak{U} = 0 \quad \forall w \in \mathfrak{N}(A, \delta)$.

Proof: If $(u, w)_\mathfrak{U} = 0 \quad \forall w \in \mathfrak{N}(A, \gamma)$, then $u \in \mathfrak{N}(A, \gamma)^\perp$. Let u_1 and u_2 be two such solutions. Then $Au_1 = f = Au_2$, so that $A(u_1 - u_2) = 0$. Hence $u_1 - u_2 \in \mathfrak{N}(A, \gamma)$, a contradiction. A similar argument applies to the Neumann problem. ■

Example 6.7.3.

Consider the Neumann problem for the Laplacian operator in $\Omega \subset \mathbb{R}^2$:

$$\begin{aligned} -\Delta u &= f, & f &\in L_2(\Omega) \\ \frac{\partial u}{\partial n} &= s, & s &\in (H^{1/2}(\partial\Omega))' \end{aligned}$$

This is a self-adjoint problem (recall Example 6.7.1) so that the elements of the space $\mathfrak{N}(A^*, \delta^*)$ are those functions in $H^1(\Delta, \Omega)$ such that

$$-\Delta v = 0$$

$$\frac{\partial v}{\partial n} = 0$$

The solutions of these homogeneous equations are simply constants:

$$v \in \mathfrak{N}\left(\Delta, \frac{\partial}{\partial n}\right) \implies v = \text{constant}$$