

ANNEX 2

Example 5.10.4.

The requirement that $\mathcal{R}(A)$ be closed is often difficult to meet. Let $A: \ell_1 \rightarrow \ell_1$ be defined by

$$A(\{\xi_1, \xi_2, \dots, \xi_k, \dots\}) = \left\{ \xi_1, \frac{1}{2}\xi_2, \dots, \frac{1}{k}\xi_k, \dots \right\}$$

We easily convince ourselves that $\overline{\mathcal{R}(A)} = \ell_1$. However, the sequence $\{1, 1/4, 1/9, \dots, 1/k^2, \dots\}$ is not in $\mathcal{R}(A)$. Hence $\mathcal{R}(A)$ is not closed. ■

We next come to a fundamental theorem on the existence and stability of solutions of linear operator equations.

THEOREM 5.10.9. *Let \mathfrak{U} and \mathfrak{V} be Banach spaces and consider the following linear problem: Find an element $u \in \mathfrak{U}$ such that*

$$Au = f$$

where

$$A \in \mathcal{L}(\mathfrak{U}, \mathfrak{V}) \quad \text{and} \quad f \in \mathfrak{V}$$

Moreover, let the following conditions hold:

(i) The range $\mathcal{R}(A)$ is closed.

(ii) The data f is such that

$$\langle g, f \rangle_{\mathfrak{V}} = 0 \quad \forall g \in \mathcal{R}(A')$$

Then there exists a solution $u \in \mathfrak{U}$ to the problem. Moreover, there exists a constant $C > 0$ such that

$$\|u\|_{\mathfrak{U}} \leq C \|f\|_{\mathfrak{V}} + \|u_0\|_{\mathfrak{U}}$$

where $u_0 \in \mathcal{R}(A)$.

Proof: Condition (ii) shows that $f \in \mathcal{R}(A')^\perp$. Hence, by Theorems 5.10.5 and 5.10.6,

$$f \in (\mathcal{R}(A)^\perp)^\perp = \mathcal{R}(A)$$

because $\mathcal{R}(A)$ is closed. Hence the data f is compatible with A .

Now we would like to use the fact that A is closed and its range is closed to conclude, from Theorem 5.10.8, that A is bounded below and, therefore, has a continuous inverse defined on its range. However, the conditions of that theorem require that A be injective, and in the present case we clearly have $\mathcal{R}(A) \neq \{0\}$. Thus, if u'_0 and u''_0 are two elements in $\mathcal{R}(A)$, and $u_1 = w + u'_0$ while $u_2 = w + u''_0$, w being an arbitrary element of \mathfrak{U} , we have $Au_1 = Au_2$.

To circumvent these difficulties, we observe that u_1 and u_2 share the property that $u_1 - w \in \mathcal{R}(A)$ and $u_2 - w \in \mathcal{R}(A)$. Hence we are led to the consideration of a relation R on \mathfrak{U} such that uRv implies that $u - v \in \mathcal{R}(A)$. We easily verify that R is an equivalence relation on \mathfrak{U} , and, therefore, R partitions \mathfrak{U} into equivalence classes of the type

$$[u] = \{v \in \mathfrak{U} : u - v \in \mathcal{R}(A)\}$$

The collection of all such equivalence classes is a linear vector space under

$$[\alpha u_1 + \beta u_2] = \alpha[u_1] + \beta[u_2]$$

$\forall \alpha, \beta \in \mathbb{R}$ and $\forall u_1, u_2 \in \mathfrak{U}$, and we denote this space by $\mathfrak{U}/\mathcal{R}(A)$. It is called the quotient space of \mathfrak{U} , modulo $\mathcal{R}(A)$; it is also a normed space with

$$\|[u]\|_{\mathfrak{U}/\mathcal{R}(A)} = \inf_{v \in \mathcal{R}(A)} \|u - v\|_{\mathfrak{U}}$$

Moreover, if \mathfrak{U} is a Banach space, so also is $\mathfrak{U}/\mathcal{R}(A)$ (see Exercise 5.10.5).

The device of introducing the quotient space makes it possible to define a “restriction” \hat{A} of A to $\mathfrak{U}/\mathcal{R}(A)$ that is injective as a map into \mathfrak{U} . Indeed, if $\hat{A}: \mathfrak{U}/\mathcal{R}(A) \rightarrow \mathfrak{U}/\mathcal{R}(A)$ is defined so that $\hat{A}[u] = Au$, then \hat{A} is bijective. Since $\mathcal{R}(A)$ is closed, $\mathcal{R}(A)$ is a Banach space. Hence we have shown that \hat{A} is a continuous bijective map of one Banach space \mathfrak{U} onto another Banach space $\mathfrak{U}/\mathcal{R}(A)$. Therefore, it follows from the Banach theorem, Theorem 5.10.2 (or from Theorem 5.10.8), that A has a continuous inverse on its range.

Now suppose that $f \in \mathcal{R}(\hat{A})$. Then there is an equivalence class $[u] \in \mathfrak{U}/\mathcal{R}(A)$ such that $[u] = \hat{A}^{-1}f$ and

$$\|[u]\|_{\mathfrak{U}/\mathcal{R}(A)} = \|\hat{A}^{-1}f\|_{\mathfrak{U}} \leq \|\hat{A}^{-1}\| \|\hat{A}\| \|f\|_{\mathfrak{V}}$$

Next let w be a representative of this class such that $\|w\|_{\mathfrak{U}} \leq C \|u\|_{\mathfrak{U}}$, where C is an appropriately large constant. Then $\|w\|_{\mathfrak{U}} \leq \hat{C} \|\hat{A}^{-1}\| \|\hat{A}\| \|f\|_{\mathfrak{V}} = C \|f\|_{\mathfrak{V}}$.

Next let $u_0 \in \mathcal{R}(A)$ and let $u = w + u_0$. Then

$$\|u\|_{\mathfrak{U}} \leq \|w\|_{\mathfrak{U}} + \|u_0\|_{\mathfrak{U}} \leq C \|f\|_{\mathfrak{V}} + \|u_0\|_{\mathfrak{U}}$$

as asserted. ■

Condition (ii) of this theorem is called the *condition for compatibility of the data*. It provides an important a priori test on the data f of an abstract linear problem to determine if it is “compatible” with the operator A (i.e., $f \in \mathcal{R}(A)$). The final inequality provides for two important observations. First, under the stated conditions, the solution is *stable*; i.e., it depends continuously on the data f . Second, the solution is not necessarily unique. Any two solutions may differ by a different choice of a representative u_0 from $\mathcal{R}(A)$. Uniqueness occurs when $\mathcal{R}(A) = \{0\}$.

THEOREM 5.10.10.

Let \mathfrak{U} and \mathfrak{V} be Banach spaces and let $A \in \mathcal{L}(\mathfrak{U}, \mathfrak{V})$. Let $\mathcal{R}(A)$ be closed. Then $\mathcal{R}(A') = \mathcal{R}(A)^\perp$

Proof: If $f \in \mathcal{R}(A')$, then $f = A'g$ for some $g \in \mathfrak{V}'$. Hence, for any $u \in \mathcal{R}(A)$, $\langle f, u \rangle_{\mathfrak{U}} = \langle A'g, u \rangle_{\mathfrak{U}} = \langle g, Au \rangle_{\mathfrak{U}} = 0$, so that $f \in \mathcal{R}(A)^\perp$; i.e., $\mathcal{R}(A') \subset \mathcal{R}(A)^\perp$.

Next let $f \in \mathcal{R}(A)^\perp$. For $v \in \mathcal{R}(A)$ and each $u \in \mathfrak{U}$ such that $v = Au$, $\langle f, u \rangle_{\mathfrak{U}}$ has the same value. Let f_0 be a functional defined on $\mathcal{R}(A)$ by

$$f_0(v) = \langle f, A^{-1}v \rangle_{\mathfrak{U}}$$