

1.21 The Problem of Elastico-Plasticity; Small Deformations

Following the lines of a paper by I.I. Vorovich and Yu.P. Krasovskij [25] that was published in a sketchy form, we consider a variant of the theory of elastico-plasticity (Il'yushin [13]), and justify the so-called method of elastic solutions for corresponding boundary value problems.

The system of partial differential equations describing the behavior of an elastic-plastic body occupying a bounded volume is

$$\begin{aligned} & \left(\frac{\nu}{\nu - 2} - \frac{\omega}{3} \right) \frac{\partial \theta}{\partial x_k} + (1 - \omega) \Delta u_k - \\ & - \frac{2}{3} e_I \frac{d\omega}{de_I} \sum_{s,t=1}^3 \epsilon_{ks}^* \sum_{l=1}^3 \epsilon_{lt}^* \frac{\partial^2 u_l}{\partial x_s \partial x_t} + \frac{F_k}{G} = 0 \end{aligned} \quad (1.21.1)$$

where ν is Poisson's ratio, G is the shear modulus, $\mathbf{F} = (F_1, F_2, F_3)$ are the volume forces, $\omega(e_I)$ is a function of the variable e_I , an intensity of the tensor of strains which defines plastic properties of the material with hardening; $\omega(e_I)$ must satisfy the following condition:

$$0 \leq \omega(e_I) \leq \omega(e_I) + e_I \frac{d\omega(e_I)}{de_I} \leq \lambda < 1. \quad (1.21.2)$$

Other bits of notation are

$$\theta \equiv \theta(\mathbf{u}) = \epsilon_{11}(\mathbf{u}) + \epsilon_{22}(\mathbf{u}) + \epsilon_{33}(\mathbf{u}),$$

$$\epsilon_{ks}^* = \begin{cases} \left(\frac{\partial u_k}{\partial x_s} - \frac{\theta}{3} \right) \frac{\sqrt{2}}{e_I}, & k = s, \\ \left(\frac{\partial u_k}{\partial x_s} + \frac{\partial u_s}{\partial x_k} \right) \frac{1}{\sqrt{2}e_I}, & k \neq s, \end{cases}$$

$$e_I = \frac{\sqrt{2}}{3} [(\epsilon_{11} - \epsilon_{22})^2 + (\epsilon_{11} - \epsilon_{33})^2 + (\epsilon_{22} - \epsilon_{33})^2 + 6(\epsilon_{12}^2 + \epsilon_{13}^2 + \epsilon_{23}^2)]^{1/2},$$

and

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

If $\omega(e_I) \equiv 0$ we get the equations of linear elasticity (for an isotropic homogeneous body). By analogy with elasticity problems, to pose a boundary value problem for (1.21.1) we must supplement it with boundary conditions. We consider a mixed boundary value problem: a part S_0 of the boundary $\partial\Omega$ of a body occupying the domain Ω is fixed,

$$\mathbf{u}|_{S_0} = 0, \quad (1.21.3)$$

and the remainder $S = \partial\Omega \setminus S_0$ is subjected to surface forces $\mathbf{f}(\mathbf{x})$:

$$\begin{aligned} & \left[K\theta \cos(\mathbf{n}, \mathbf{x}_k^0) + \sqrt{2G}e_I \sum_{m=1}^3 \epsilon_{km}^* \cos(\mathbf{n}, \mathbf{x}_k^0) \right] \Big|_S \\ &= \sum_{k=1}^3 f_k \cos(\mathbf{n}, \mathbf{x}_k^0) + G\sqrt{2}\omega e_I \sum_{m=1}^3 \epsilon_{km}^* \cos(\mathbf{n}, \mathbf{x}_m^0) \end{aligned} \quad (1.21.4)$$

where K is the modulus of volume compressibility and $\cos(\mathbf{n}, \mathbf{x}_k^0)$ is the cosine of the angle between the direction of the outward normal \mathbf{n} to the boundary at a point and the direction \mathbf{x}_k^0 of the Cartesian axis OX_k .

When $\omega(e_I)$ is small (as it is if e_I is small) we have a nonlinear boundary value problem which is, in a certain way, a perturbation of a corresponding boundary value problem of linear elasticity. It leads to the idea of using an iterative procedure, the so-called method of elastic solutions, to solve the former. This procedure looks like that of the contraction mapping principle if we can make the problem take the corresponding operator form. Then it remains to show that the operator of the problem is a contraction. Now we begin to carry out the program.

Let us introduce the notation

$$\begin{aligned} (\mathbf{u}, \mathbf{v}) = & \frac{2}{9} \{ [\epsilon_{11}(\mathbf{u}) - \epsilon_{22}(\mathbf{u})][\epsilon_{11}(\mathbf{v}) - \epsilon_{22}(\mathbf{v})] + \\ & + [\epsilon_{11}(\mathbf{u}) - \epsilon_{33}(\mathbf{u})][\epsilon_{11}(\mathbf{v}) - \epsilon_{33}(\mathbf{v})] + \\ & + [\epsilon_{22}(\mathbf{u}) - \epsilon_{33}(\mathbf{u})][\epsilon_{22}(\mathbf{v}) - \epsilon_{33}(\mathbf{v})] + \\ & + 6[\epsilon_{12}(\mathbf{u})\epsilon_{12}(\mathbf{v}) + \epsilon_{13}(\mathbf{u})\epsilon_{13}(\mathbf{v}) + \epsilon_{23}(\mathbf{u})\epsilon_{23}(\mathbf{v})] \} \end{aligned} \quad (1.21.5)$$

If we consider the terms on the right-hand side of (1.21.5) as coordinates of vectors $\mathbf{a} = (a_1, \dots, a_6)$, $\mathbf{b} = (b_1, \dots, b_6)$,

$$a_i = c_i(\mathbf{u}), \quad b_i = c_i(\mathbf{v}), \quad i = 1, \dots, 6,$$

then the form $\langle \mathbf{u}, \mathbf{v} \rangle$ is a scalar product of \mathbf{a} by \mathbf{b} in \mathbb{R}^6 :

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^6 a_i b_i.$$

Besides,

$$\langle \mathbf{u}, \mathbf{u} \rangle = \sum_{i=1}^6 c_i^2(\mathbf{u}) = e_I^2(\mathbf{u}) \quad (1.21.6)$$

and by the Schwarz inequality we get

$$|\langle \mathbf{u}, \mathbf{v} \rangle| = \left| \sum_{i=1}^6 c_i(\mathbf{u})c_i(\mathbf{v}) \right| \leq e_I(\mathbf{u})e_I(\mathbf{v}). \quad (1.21.7)$$

On the set C_2 of vector functions satisfying the boundary condition (1.21.3) and such that each of their components is of class $C^{(2)}(\Omega)$, let us now introduce an inner product

$$(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \left(\frac{3}{2}G\langle \mathbf{u}, \mathbf{v} \rangle + \frac{1}{2}K\theta^2(\mathbf{u})\theta(\mathbf{v}) \right) d\Omega. \quad (1.21.8)$$

This coincides with a special case of the inner product (1.10.26) of the linear theory of elasticity. So the completion of C_2 in the metric corresponding to (1.21.8) is the energy space of linear elasticity E_{EM} (M for “mixed”) if we suppose that the condition (1.21.3) provides $\mathbf{u} = 0$ if

$$\|\mathbf{u}\|^2 = \int_{\Omega} \left(\frac{3}{2}Ge_I^2(\mathbf{u}) + \frac{1}{2}K\theta^2(\mathbf{u}) \right) d\Omega = 0.$$

The norm on E_{EM} is equivalent to one of $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ (see Section 1.10 and Fichera [9]). (By $H_1 \times H_2$ we denote the so-called Cartesian product of Hilbert spaces H_1 and H_2 , the elements of which are pairs (x, y) , $x \in H_1$, $y \in H_2$. The scalar product in $H_1 \times H_2$ is defined by the relation

$$(x_1, x_2)_1 + (y_1, y_2)_2$$

where $x_1, x_2 \in H_1$ and $y_1, y_2 \in H_2$.)

By the principle of virtual displacements, the integro-differential equation of equilibrium of an elasto-plastic body is

$$\begin{aligned} (\mathbf{u}, \mathbf{v}) - \frac{3}{2}G \int_{\Omega} \omega(e_I(\mathbf{u}))(\mathbf{u}, \mathbf{v}) d\Omega - \\ - \sum_{i=1}^3 \int_{\Omega} F_i v_i d\Omega - \sum_{i=1}^3 \int_S f_i v_i dS = 0. \end{aligned} \quad (1.21.9)$$

This equation can be obtained using the equations (1.21.1) and the boundary conditions (1.21.3) and (1.21.4). Conversely, using the technique of the

$$\begin{aligned} c_1(\mathbf{w}) &= \frac{\sqrt{2}}{3} [\epsilon_{11}(\mathbf{w}) - \epsilon_{22}(\mathbf{w})], & c_2(\mathbf{w}) &= \frac{\sqrt{2}}{3} [\epsilon_{11}(\mathbf{w}) - \epsilon_{33}(\mathbf{w})], \\ c_3(\mathbf{w}) &= \frac{\sqrt{2}}{3} [\epsilon_{22}(\mathbf{w}) - \epsilon_{33}(\mathbf{w})], & c_4(\mathbf{w}) &= \frac{2}{\sqrt{3}} \epsilon_{12}(\mathbf{w}), \\ c_5(\mathbf{w}) &= \frac{2}{\sqrt{3}} \epsilon_{13}(\mathbf{w}), & c_6(\mathbf{w}) &= \frac{2}{\sqrt{3}} \epsilon_{23}(\mathbf{w}), \end{aligned}$$

classical calculus of variations we can get (1.21.1) and the natural boundary conditions (1.21.4). Thus, in a certain way, (1.21.9) is equivalent to the above statement of the problem. So we can introduce

Definition 1.21.1. A vector function $\mathbf{u} \in E_{EM}$ is called the generalized solution of a problem of elasto-plasticity if it satisfies (1.21.9) for every $\mathbf{v} \in E_{EM}$.

For correctness of this definition we must impose some restrictions on external forces. It is evident that they coincide with those for linear elasticity. So we assume that

$$F_i(x_1, x_2, x_3) \in L^{6/5}(\Omega), \quad f_i(x_1, x_2, x_3) \in L^{4/3}(S). \quad (1.21.10)$$

Consider the form

$$A[\mathbf{u}, \mathbf{v}] = \frac{3}{2}G \int_{\Omega} \omega(e_I(\mathbf{u})) \langle \mathbf{u}, \mathbf{v} \rangle d\Omega + \sum_{i=1}^3 \int_{\Omega} F_i v_i d\Omega + \sum_{i=1}^3 \int_S f_i v_i dS$$

as a functional in E_{EM} with respect to $\mathbf{v}(x_1, x_2, x_3)$ when $\mathbf{u}(x_1, x_2, x_3) \in E_{EM}$ is fixed. As in linear elasticity, the load terms, thanks to (1.21.10), are continuous linear functionals with respect to $\mathbf{v} \in E_{EM}$. Finally, in accordance with (1.21.5) and (1.21.2), we get

$$\left| \frac{3}{2}G \int_{\Omega} \omega(e_I(\mathbf{u})) \langle \mathbf{u}, \mathbf{v} \rangle d\Omega \right| \leq \lambda \frac{3}{2}G \int_{\Omega} |\langle \mathbf{u}, \mathbf{v} \rangle| d\Omega \leq \lambda \|\mathbf{u}\| \|\mathbf{v}\|,$$

so this part of the functional is also continuous.

Therefore we can apply the Riesz representation theorem to $A[\mathbf{u}, \mathbf{v}]$, which gives

$$A[\mathbf{u}, \mathbf{v}] = (\mathbf{v}, \mathbf{f}) \equiv (\mathbf{f}, \mathbf{v}).$$

This representation uniquely defines a correspondence $\mathbf{u} \mapsto \mathbf{f}$, where $\mathbf{u}, \mathbf{f} \in E_{EM}$, we obtain an operator $\mathbf{f} = A(\mathbf{u})$ acting in E_{EM} . Equation (1.21.9) is now equivalent to

$$(\mathbf{u}, \mathbf{v}) - (A(\mathbf{u}), \mathbf{v}) = 0 \quad (1.21.11)$$

or, since $\mathbf{v} \in E_{EM}$ is arbitrary,

$$\mathbf{u} = A(\mathbf{u}). \quad (1.21.12)$$

The operator A is nonlinear. We shall show that it is a contraction operator. For this, take arbitrary elements $\mathbf{u}, \mathbf{v}, \mathbf{w} \in E_{EM}$ and consider

$$(A(\mathbf{u}) - A(\mathbf{v}), \mathbf{w}) = \frac{3}{2}G \int_{\Omega} [\omega(e_I(\mathbf{u})) \langle \mathbf{u}, \mathbf{w} \rangle - \omega(e_I(\mathbf{v})) \langle \mathbf{v}, \mathbf{w} \rangle] d\Omega. \quad (1.21.13)$$

First, let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be in C_2 . At every point of Ω , by (1.21.7), we can estimate the integrand from (1.21.13) as follows. We have

$$\begin{aligned} \text{Int} &= |\omega(e_I(\mathbf{u})) \langle \mathbf{u}, \mathbf{w} \rangle - \omega(e_I(\mathbf{v})) \langle \mathbf{v}, \mathbf{w} \rangle| \\ &= \left| \omega(e_I(\mathbf{u})) \sum_{i=1}^6 c_i(\mathbf{u}) c_i(\mathbf{w}) - \omega(e_I(\mathbf{v})) \sum_{i=1}^6 c_i(\mathbf{v}) c_i(\mathbf{w}) \right|. \end{aligned}$$

Let us introduce a real-valued function $f(t)$ of a real variable t by the relation

$$f(t) = \sum_{i=1}^6 \omega(e_I(t\mathbf{u} + (1-t)\mathbf{v})) c_i(t\mathbf{u} + (1-t)\mathbf{v}) c_i(\mathbf{w}).$$

It is seen that

$$\text{Int} = |f(1) - f(0)|.$$

As $f(t)$ is continuously differentiable, the classical mean value theorem gives

$$f(1) - f(0) = f'(z)(1-0) = f'(z) \quad \text{for some } z \in [0, 1],$$

or, in the above terms, we get

$$\begin{aligned} \text{Int} &= \left| \frac{d}{dt} \left\{ \sum_{i=1}^6 \omega(e_I(t\mathbf{u} + (1-t)\mathbf{v})) c_i(t\mathbf{u} + (1-t)\mathbf{v}) c_i(\mathbf{w}) \right\} \right|_{t=0} \\ &= \left| \left\{ \frac{d\omega(e_I(t\mathbf{u} + (1-t)\mathbf{v}))}{dt} de_I(t\mathbf{u} + (1-t)\mathbf{v}) \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^6 c_i(t\mathbf{u} + (1-t)\mathbf{v}) c_i(\mathbf{w}) + \omega \sum_{i=1}^6 c_i(\mathbf{u} - \mathbf{v}) c_i(\mathbf{w}) \right\} \right|_{t=0}. \end{aligned}$$

(Here we have used the linearity of $c_i(\mathbf{u})$ in \mathbf{u} and, thus, in t .) Let us consider the term

$$\begin{aligned} T &= \sum_{i=1}^6 \frac{de_I(t\mathbf{u} + (1-t)\mathbf{v})}{dt} c_i(t\mathbf{u} + (1-t)\mathbf{v}) c_i(\mathbf{w}) \\ &= \sum_{i=1}^6 \frac{d}{dt} \left(\sum_{j=1}^6 c_j^2(t\mathbf{u} + (1-t)\mathbf{v}) \right) c_i(t\mathbf{u} + (1-t)\mathbf{v}) c_i(\mathbf{w}) \\ &= \sum_{i=1}^6 \frac{2}{2} \sum_{j=1}^6 c_j(t\mathbf{u} + (1-t)\mathbf{v}) c_j(\mathbf{u} - \mathbf{v}) \\ &\quad \times \frac{1}{2} \left(\sum_{j=1}^6 c_j^2(t\mathbf{u} + (1-t)\mathbf{v}) \right)^{1/2} c_i(t\mathbf{u} + (1-t)\mathbf{v}) c_i(\mathbf{w}). \end{aligned}$$

Applying the Schwarz inequality, we obtain

$$|T| \leq \sum_{i=1}^6 \frac{\left(\sum_{j=1}^6 c_j^2(t\mathbf{u} + (1-t)\mathbf{v}) \right)^{1/2} \left(\sum_{j=1}^6 c_j^2(\mathbf{u} - \mathbf{v}) \right)^{1/2}}{\left(\sum_{j=1}^6 c_j^2(t\mathbf{u} + (1-t)\mathbf{v}) \right)^{1/2}} \cdot |c_i(t\mathbf{u} + (1-t)\mathbf{v})| |c_i(\mathbf{w})|$$

$$= \left(\sum_{j=1}^6 c_j^2(\mathbf{u} - \mathbf{v}) \right)^{1/2} \sum_{i=1}^6 |c_i(t\mathbf{u} + (1-t)\mathbf{v})| |c_i(\mathbf{w})|$$

$$\leq e_I(\mathbf{u} - \mathbf{v}) \left(\sum_{i=1}^6 c_i^2(t\mathbf{u} + (1-t)\mathbf{v}) \right)^{1/2} \left(\sum_{i=1}^6 c_i^2(\mathbf{w}) \right)^{1/2}$$

$$= e_I(\mathbf{u} - \mathbf{v}) e_I(t\mathbf{u} + (1-t)\mathbf{v}) e_I(\mathbf{w})$$

(here we also used (1.21.6)). Similarly,

$$\left| \sum_{i=1}^6 c_i(\mathbf{u} - \mathbf{v}) c_i(\mathbf{w}) \right| \leq \left(\sum_{i=1}^6 c_i^2(\mathbf{u} - \mathbf{v}) \right)^{1/2} \left(\sum_{i=1}^6 c_i^2(\mathbf{w}) \right)^{1/2}$$

$$= e_I(\mathbf{u} - \mathbf{v}) e_I(\mathbf{w}).$$

Combining all these, we get

$$\begin{aligned} \text{Int} &\leq \left\{ \frac{d\omega(e_I(t\mathbf{u} + (1-t)\mathbf{v}))}{de_I} e_I(t\mathbf{u} + (1-t)\mathbf{v}) e_I(\mathbf{u} - \mathbf{v}) e_I(\mathbf{w}) + \right. \\ &\quad \left. + \omega(e_I(t\mathbf{u} + (1-t)\mathbf{v})) e_I(\mathbf{u} - \mathbf{v}) e_I(\mathbf{w}) \right\}_{t=z} \\ &= \left\{ \omega(e_I(t\mathbf{u} + (1-t)\mathbf{v})) + \frac{d\omega(e_I(t\mathbf{u} + (1-t)\mathbf{v}))}{de_I} \right. \\ &\quad \left. \cdot e_I(t\mathbf{u} + (1-t)\mathbf{v}) \right\}_{t=z} e_I(\mathbf{u} - \mathbf{v}) e_I(\mathbf{w}). \end{aligned}$$

By the condition (1.21.2), we have

$$\text{Int} \leq \lambda e_I(\mathbf{u} - \mathbf{v}) e_I(\mathbf{w}) \quad (1.21.14)$$

at every point of Ω .

Returning to (1.21.13) we have, using (1.21.14),

$$|(A(\mathbf{u}) - A(\mathbf{v}), \mathbf{w})| \leq \lambda \int_{\Omega} \frac{3}{2} G e_I(\mathbf{u} - \mathbf{v}) e_I(\mathbf{w}) d\Omega.$$

In accordance with the norm of E_{EM} it follows that

$$|(A(\mathbf{u}) - A(\mathbf{v}), \mathbf{w})| \leq \lambda \|\mathbf{u} - \mathbf{v}\| \|\mathbf{w}\|$$

or, putting $\mathbf{w} = A(\mathbf{u}) - A(\mathbf{v})$, we get

$$\|A(\mathbf{u}) - A(\mathbf{v})\| \leq \lambda \|\mathbf{u} - \mathbf{v}\|, \quad \lambda = \text{const} < 1. \quad (1.21.15)$$

Being obtained for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in C_2$, this inequality holds for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in E_{EM}$ since in this inequality we can pass to the limit for corresponding Cauchy sequences in E_{EM} .

Inequality (1.21.15) states that A is a contraction operator in E_{EM} ; hence we can apply the contraction mapping principle, which states that (1.21.12) has a unique solution that can be found using the iterative procedure

$$\mathbf{u}_{k+1} = A(\mathbf{u}_k), \quad k = 0, 1, 2, \dots$$

This procedure begins with an arbitrary element $\mathbf{u}_0 \in E_{EM}$; when $\mathbf{u}_0 = 0$, the procedure is called the method of elastic solutions since at each step we must solve a problem of linear elasticity with given load terms. In practical terms, the method works best when the constant λ is small.

So we can formulate

Theorem 1.21.1. Assume S_0 is a piecewise smooth surface of nonzero area and that the conditions (1.21.2) and (1.21.10) are fulfilled. Then a mixed boundary value problem of elasto-plasticity has a unique generalized solution in the sense of Definition 1.21.1; the iterative procedure (1.21.15) defines a sequence of successive approximations $\mathbf{u}_k \in E_{EM}$ which converges to the solution $\mathbf{u} \in E_{EM}$ and

$$\|\mathbf{u}_k - \mathbf{u}\| \leq \frac{\lambda^k}{1-\lambda} \|\mathbf{u}_0 - \mathbf{u}_1\|. \quad (1.21.16)$$

It is clear that we cannot apply this theorem when, say, $S = \partial\Omega$. In such a case, we must add the self-balance conditions (1.14.16). These guarantee that we can repeat the above method for a free elastic-plastic body, and so can formulate

Theorem 1.21.2. Assume that all the requirements of Theorem 1.21.1 and the self-balance conditions (1.14.16) are met. Then there is a unique generalized solution of the boundary value problem for a bounded elastic-plastic body, and it can be found by an iterative procedure of the form (1.21.15).

Problem 1.21.1. Is an estimate of the type (1.21.16) valid in Theorem 1.21.2?

We recommend that the reader prove Theorem 1.21.2 in detail, in order to gain experience with the technique.

Remark 1.21.1. We would like to call attention to the way in which we obtained the main inequality of this section. First it was proved for smooth functions and then was extended to the general case. This is a standard technique in the treatment of nonlinear problems of mechanics.