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3.8 Steady-State Flow of Viscous Liquid

Following I.I. Vorovich and V.I. Yudovich [27], we consider the steady-state flow of a viscous incompressible liquid described by the Navier-Stokes equations

$$\nu \Delta \mathbf{v} = (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p + \mathbf{f}, \quad (3.8.1)$$

$$\nabla \cdot \mathbf{v} = 0. \quad (3.8.2)$$

Let $\nu > 0$. We are treating a problem with boundary condition

$$\mathbf{v}|_{\partial\Omega} = \alpha. \quad (3.8.3)$$

From now on, we assume:

- (i) Ω is a bounded domain in \mathbb{R}^2 or \mathbb{R}^3 whose boundary $\partial\Omega$ consists of r closed curves or surfaces S_k , $k = 1, \dots, r$ with continuous curvature.
- (ii) There is a continuously differentiable vector-function

$$\mathbf{a}(\mathbf{x}) = (a_1(\mathbf{x}), a_2(\mathbf{x}), a_3(\mathbf{x}))$$

such that

$$a_k(\mathbf{x}) \in C^{(1)}(\bar{\Omega}), \quad \nabla \cdot \mathbf{a} = 0 \text{ in } \Omega, \quad \mathbf{a}|_{\partial\Omega} = \alpha.$$

- (iii) On each S_k , $k = 1, \dots, r$, we have

$$\int_{S_k} \alpha \cdot \mathbf{n} dS = 0 \quad (3.8.4)$$

where \mathbf{n} is the unit outward normal at a point of S_k .

We note that the condition

$$\sum_{k=1}^r \int_{S_k} \alpha \cdot \mathbf{n} dS = 0$$

is necessary for solvability of the problem.

Let $H(\Omega)$ be the completion of the set $S^0(\Omega)$ of all smooth solenoidal vector-functions $\mathbf{u}(\mathbf{x})$ satisfying the boundary condition, in the norm induced by the scalar product

$$(\mathbf{u}, \mathbf{v})_{H(\Omega)} = \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} d\Omega \equiv \int_{\Omega} \operatorname{rot} \mathbf{u} \cdot \operatorname{rot} \mathbf{v} d\Omega$$

and so each of the components of $\mathbf{u}(\mathbf{x}) \in H(\Omega)$ is of $W^{1,2}(\Omega)$. Thus in the three dimensional case, the imbedding operator of $H(\Omega)$ into $(L^p(\Omega))^3$ is continuous when $1 \leq p \leq 6$ and compact when $1 \leq p < 6$; in the two dimensional case, the imbedding operator is compact into $(L^p(\Omega))^2$ for any $1 \leq p < \infty$.

We assume

- (iv) $f_k(\mathbf{x}) \in L^p(\Omega)$, $p \geq 6/5$ in the three dimensional case ($k = 1, 2, 3$),
 $p > 1$ in the two dimensional case ($k = 1, 2$).

Definition 3.8.1. $\mathbf{v}(\mathbf{x}) = \mathbf{a}(\mathbf{x}) + \mathbf{u}(\mathbf{x})$ is called a generalized solution to the problem (3.8.1)–(3.8.3) if $\mathbf{u}(\mathbf{x}) \in H(\Omega)$ and satisfies the integro-differential equation

$$\begin{aligned} \nu(\mathbf{u}, \Phi)_{H(\Omega)} &= - \int_{\Omega} [(\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \Phi + (\mathbf{u} \cdot \nabla) \mathbf{a} \cdot \Phi + (\mathbf{a} \cdot \nabla) \mathbf{u} \cdot \Phi + \\ &\quad + (\mathbf{a} \cdot \nabla) \mathbf{a} \cdot \Phi + \nu \operatorname{rot} \mathbf{a} \cdot \operatorname{rot} \Phi + \mathbf{f} \cdot \Phi] d\Omega \end{aligned} \quad (3.8.5)$$

for any $\Phi \in H(\Omega)$.

It is easily seen that if $\mathbf{a}(\mathbf{x})$ and $\mathbf{u}(\mathbf{x})$ belong to $C^{(2)}(\bar{\Omega})$ then $\mathbf{v}(\mathbf{x})$ is a classical solution to the problem (3.8.1)–(3.8.3).

Note that there are infinitely many vectors $\mathbf{a}(\mathbf{x})$ satisfying the assumption (ii) if there is one, but the set of generalized solutions does not depend on the choice of $\mathbf{a}(\mathbf{x})$.

To use Lemma 3.7.1, we reduce equation (3.8.5) to the operator form $\mathbf{u} + F(\mathbf{u}) = 0$, defining F with use of the Riesz representation theorem from the equality

$$\begin{aligned} \nu(F(\mathbf{u}), \Phi)_{H(\Omega)} &= \int_{\Omega} [(\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \Phi + (\mathbf{u} \cdot \nabla) \mathbf{a} \cdot \Phi + (\mathbf{a} \cdot \nabla) \mathbf{u} \cdot \Phi + \\ &\quad + (\mathbf{a} \cdot \nabla) \mathbf{a} \cdot \Phi + \nu \operatorname{rot} \mathbf{a} \cdot \operatorname{rot} \Phi + \mathbf{f} \cdot \Phi] d\Omega. \end{aligned} \quad (3.8.6)$$

The estimates needed to prove that the right-hand side of (3.8.6) is a continuous linear functional in $H(\Omega)$ with respect to Φ follow from traditional estimates of the terms using the Hölder inequality. But we now show a sharper result; namely,

Lemma 3.8.1. F is a completely continuous operator in $H(\Omega)$.

Proof. Let $\{\mathbf{u}_n(x)\}$ be a weakly convergent sequence in $H(\Omega)$. Then it converges strongly in $(L^4(\Omega))^k$ ($k = 2$ or 3). From (3.8.6), we get

$$\begin{aligned} \nu|F(\mathbf{u}_m) - F(\mathbf{u}_n)| &= \\ &= \left| \int_{\Omega} \{[(\mathbf{u}_m - \mathbf{u}_n) \cdot \nabla] \mathbf{u}_m \cdot \Phi - (\mathbf{u}_m - \mathbf{u}_n) \cdot \Phi + \right. \\ &\quad \left. + [(\mathbf{u}_m - \mathbf{u}_n) \cdot \nabla] \mathbf{a} \cdot \Phi + (\mathbf{a} \cdot \nabla)(\mathbf{u}_m - \mathbf{u}_n) \cdot \Phi\} d\Omega \right| \\ &\leq M \|\mathbf{u}_m - \mathbf{u}_n\|_{L^4(\Omega)} \|\Phi\|_{H(\Omega)} \end{aligned}$$

with a constant M which does not depend on m, n , or Φ . Setting

$$\Phi = F(\mathbf{u}_m) - F(\mathbf{u}_n)$$

in the inequality, we obtain

$$\nu|F(\mathbf{u}_m) - F(\mathbf{u}_n)|_{H(\Omega)} \leq M \|\mathbf{u}_m - \mathbf{u}_n\|_{(L^4(\Omega))^k} \rightarrow 0$$

when $m, n \rightarrow \infty$, and so F is completely continuous. \square

From Definition 3.8.1 it follows that

Lemma 3.8.2. A generalized solution of the problem under consideration in the sense of Definition 3.8.1 satisfies the operator equation

$$\mathbf{u} + F(\mathbf{u}) = 0; \quad (3.8.7)$$

conversely, a solution to (3.8.7) is a generalized solution of the problem.

By Lemma 3.7.1, it now suffices to show that all solutions of the equation $\mathbf{u} + tF(\mathbf{u}) = 0$, for all $t \in [0, 1]$, lie in a sphere $\|\mathbf{u}\|_{H(\Omega)} \leq R$ for some $R < \infty$. First we show this in the simpler case of homogeneous boundary condition (3.8.3). Here $\alpha = 0$ and thus $\mathbf{a}(\mathbf{x}) = 0$. \square

Theorem 3.8.1. The problem (3.8.1)–(3.8.3) with $\alpha = 0$ has at least one generalized solution in the sense of Definition 3.8.1. Each generalized solution $\mathbf{u}(\mathbf{x})$ is bounded, $\|\mathbf{u}\|_{H(\Omega)} < R$ for some $R < \infty$ and the degree of $I + F$ with respect to 0 and $D = \{\mathbf{u} \in H(\Omega) \mid \|\mathbf{u}\| < R\}$ is +1.

Proof. As was said, it suffices to show an a priori estimate for solutions to the equation $\mathbf{u} + tF(\mathbf{u}) = 0$ for $t \in [0, 1]$. For a solution, there holds the identity

$$(\mathbf{u} + tF(\mathbf{u}), \mathbf{u})_{H(\Omega)} = 0$$

or, the same,

$$\nu(\mathbf{u}, \mathbf{u})_{H(\Omega)} + t \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{u} d\Omega = -t \int_{\Omega} \mathbf{f} \cdot \mathbf{u} d\Omega.$$

Integration by parts gives

$$\begin{aligned} \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{u} d\Omega &= \frac{1}{2} \int_{\Omega} \sum_k u_k \frac{\partial}{\partial x_k} (\mathbf{u} \cdot \mathbf{u}) d\Omega \\ &= -\frac{1}{2} \int_{\Omega} (\mathbf{u} \cdot \mathbf{u})(\nabla \cdot \mathbf{u}) d\Omega = 0 \end{aligned} \quad (3.8.8)$$

since $\nabla \cdot \mathbf{u} = 0$ and thus, for a solution \mathbf{u} , we get

$$|\nu(\mathbf{u}, \mathbf{u})_{H(\Omega)}| = \left| t \int_{\Omega} \mathbf{f} \cdot \mathbf{u} d\Omega \right| \leq \frac{\nu R}{2} \|\mathbf{f}\|_{L^p(\Omega)} \|\mathbf{u}\|_{H(\Omega)}$$

with some constant R , or

$$\|\mathbf{u}\|_{H(\Omega)} < R.$$

This completes the proof. \square

Now we consider the more complicated case of nonhomogeneous boundary conditions (3.8.3). We need some auxiliary results.

Let ω_{ε} be a domain in $\bar{\Omega}$ which consists of points covered by all inward normals to $\partial\Omega$ of the length ε . For sufficiently small $\varepsilon > 0$, these normals

do not intersect and thus in ω_{ε} we can use a coordinate system pointing out for a $\mathbf{x} \in \omega_{\varepsilon}$ a point Q on $\partial\Omega$ and a number s , the distance from Q to \mathbf{x} along the corresponding normal. So for a function $g(\mathbf{x})$ given on ω_{ε} , we write down: $g(s, Q)$.

Lemma 3.8.3. There is a solenoidal vector function $\mathbf{a}_{\varepsilon}(\mathbf{x}) \in (C^{(1)}(\bar{\Omega}))^k$ such that $\mathbf{a}_{\varepsilon}(\mathbf{x}) = 0$ in $\Omega \setminus \omega_{\varepsilon}$,

$$\mathbf{a}_{\varepsilon}(\mathbf{x})|_{\partial\Omega} = \boldsymbol{\alpha}, \quad \text{and} \quad |\mathbf{a}_{\varepsilon}(\mathbf{x})| \leq M_1/\varepsilon \text{ in } \bar{\Omega} \quad (3.8.9)$$

with a constant M_1 not depending on ε .

Proof. Let us introduce a function $q(\mathbf{x})$ by

$$q(s, Q) = \begin{cases} (\varepsilon^2 - s^2)^2/\varepsilon^4, & 0 \leq s \leq \varepsilon, \\ 0, & s > \varepsilon. \end{cases}$$

Let $\mathbf{a}(\mathbf{x})$ be a solenoidal vector-function satisfying the assumption (ii) of the beginning of the section. Under the taken assumptions, there is a vector-function $\mathbf{p}(\mathbf{x})$ such that

$$\mathbf{a}(\mathbf{x}) = \operatorname{rot} \mathbf{p}(\mathbf{x}).$$

It is seen that the vector function $\mathbf{a}_{\varepsilon}(\mathbf{x}) = \operatorname{rot}(q\mathbf{p})$ is needed.

Note that in the plane case, this is a vector $(0, 0, q\psi)$ where $\psi(x_1, x_2)$ is the flow function of $\mathbf{a}(\mathbf{x})$. \square

Lemma 3.8.4. For $\mathbf{u} \in H(\Omega)$, we have

$$\int_{\omega_{\varepsilon}} |\mathbf{u}|^2 d\Omega \leq M_2^2 \varepsilon^2 \int_{\omega_{\varepsilon}} \sum_{i,j} \left| \frac{\partial u_i}{\partial x_j} \right|^2 d\Omega \quad (3.8.10)$$

with a constant M_2 not depending on \mathbf{u} or ε .

Proof. We show (3.8.10) for a smooth function. The limit passage will prove the general case. So for points of ω_{ε} we have

$$\mathbf{u}(s, Q) = \int_0^s \frac{\partial \mathbf{u}(t, Q)}{\partial t} dt.$$

By the Cauchy inequality

$$\begin{aligned} \int_0^{\varepsilon} |\mathbf{u}(t, Q)|^2 dt &= \int_0^{\varepsilon} \left| \int_0^s \frac{\partial \mathbf{u}(t, Q)}{\partial t} dt \right|^2 ds \\ &\leq \int_0^{\varepsilon} s \int_0^s \left| \frac{\partial \mathbf{u}(t, Q)}{\partial t} \right|^2 dt ds \\ &\leq \frac{\varepsilon^2}{2} \int_0^{\varepsilon} \left| \frac{\partial \mathbf{u}(t, Q)}{\partial t} \right|^2 dt. \end{aligned}$$

It is easily seen that for any $g(\mathbf{x})$

$$m_1 \int_0^\varepsilon \int_{\partial\Omega} g^2(s, Q) ds dS \leq \int_{\omega_\varepsilon} g^2 d\Omega \leq m_2 \int_0^\varepsilon \int_{\partial\Omega} g^2(s, Q) ds dS$$

and so

$$\begin{aligned} \int_{\omega_\varepsilon} |\mathbf{u}|^2 d\Omega &\leq m_2 \int_{\partial\Omega} \int_0^\varepsilon |\mathbf{u}(s, Q)|^2 ds dS \\ &\leq m_2 \int_{\partial\Omega} \frac{\varepsilon^2}{2} \int_0^\varepsilon \left| \frac{\partial \mathbf{u}}{\partial t} \right|^2 dt dS \\ &\leq \frac{m_2}{2m_1} \varepsilon^2 \int_{\omega_\varepsilon} \sum_{i,j} \left| \frac{\partial u_i}{\partial x_j} \right|^2 d\Omega. \end{aligned}$$

To apply degree theory to the problem under consideration, it remains to establish \square

Lemma 3.8.5. All solutions of the equation

$$\mathbf{u} + tF(\mathbf{u}) = 0 \quad (3.8.11)$$

for all $t \in [0, 1]$, are in a ball $\|\mathbf{u}\|_{H(\Omega)} < R$ whose radius R depends only on $\mathbf{f}, \partial\Omega, \mathbf{a}$, and ν .

Proof. Suppose that the set of solutions to (3.8.11) is unbounded. This means there is a sequence $\{t_k\} \subset [0, 1]$ and a corresponding sequence $\{\mathbf{u}_k\}$ such that $\mathbf{u}_k + t_k F(\mathbf{u}_k) = 0$ and

$$\|\mathbf{u}_k\|_{H(\Omega)} \rightarrow \infty \text{ as } k \rightarrow \infty. \quad (3.8.12)$$

Without loss of generality, we can consider $\{t_k\}$ to be convergent to $t_0 \in [0, 1]$ and, moreover, the sequence $\{\mathbf{u}_k^*\}$, $\mathbf{u}_k^* = \mathbf{u}_k / \|\mathbf{u}_k\|_{H(\Omega)}$, to be weakly convergent to an element $\mathbf{u}_0 \in H(\Omega)$ since $\{\mathbf{u}_k^*\}$ is bounded.

Let us consider the identity $(\mathbf{u}_k + t_k F(\mathbf{u}_k), \mathbf{u}_k) = 0$, namely,

$$\begin{aligned} -\nu \|\mathbf{u}_k\|_{H(\Omega)}^2 &= t_k \int_{\omega_\varepsilon} (\mathbf{a}_\varepsilon \cdot \nabla) \mathbf{a}_\varepsilon \cdot \mathbf{u}_k + \nu \operatorname{rot} \mathbf{a}_\varepsilon \cdot \operatorname{rot} \mathbf{u}_k + \mathbf{f} \cdot \mathbf{u}_k d\Omega + \\ &\quad + t_k \int_{\Omega} [(\mathbf{a}_\varepsilon \cdot \nabla) \mathbf{a}_\varepsilon \cdot \mathbf{u}_k + \nu \operatorname{rot} \mathbf{a}_\varepsilon \cdot \operatorname{rot} \mathbf{u}_k + \mathbf{f} \cdot \mathbf{u}_k] d\Omega \end{aligned} \quad (3.8.13)$$

which is valid because of (3.8.8) and a similar equality

$$\int_{\omega_\varepsilon} (\mathbf{u}_k \cdot \nabla) \mathbf{a}_\varepsilon \cdot \mathbf{u}_k d\Omega = 0.$$

The first integral on the right-hand side of (3.8.13) is a weakly continuous functional with respect to \mathbf{u}_k , and for the second integral we have

$$\left| \int_{\Omega} [(\mathbf{a}_\varepsilon \cdot \nabla) \mathbf{a}_\varepsilon \cdot \mathbf{u}_k + \nu \operatorname{rot} \mathbf{a}_\varepsilon \cdot \operatorname{rot} \mathbf{u}_k + \mathbf{f} \cdot \mathbf{u}_k] d\Omega \right| \leq M_3 \|\mathbf{u}_k\|_{H(\Omega)}$$

where $M_3 \in \mathbb{R}$ does not depend on \mathbf{u}_k . Dividing both sides of (3.8.13) by $\|\mathbf{u}_k\|_{H(\Omega)}^2$, it follows that

$$-\nu = t_0 \int_{\omega_\varepsilon} (\mathbf{a}_\varepsilon \cdot \nabla) \mathbf{u}_0 \cdot \mathbf{u}_0 d\Omega. \quad (3.8.14)$$

We note that this holds for any small positive $\varepsilon < \varepsilon_0$ with a fixed ε_0 for which the above construction of the frame for ω_{ε_0} is valid. To prove it, take $\varepsilon = \eta < \varepsilon_0$

$$\mathbf{w}_k = \mathbf{u}_k + \mathbf{a}_{\varepsilon_0} - \mathbf{a}_\eta$$

and consider the identity

$$(\mathbf{u}_k + t_k F(\mathbf{u}_k), \mathbf{w}_k)_{H(\Omega)} = 0$$

which takes the form

$$\begin{aligned} -\nu \|\mathbf{w}_k\|_{H(\Omega)}^2 &= t_k \int_{\omega_\eta} (\mathbf{a}_\eta \cdot \nabla) \mathbf{w}_k \cdot \mathbf{w}_k d\Omega + \\ &\quad + t_k \int_{\Omega} [(\mathbf{a}_\eta \cdot \nabla) \mathbf{a}_\eta \cdot \mathbf{w}_k + \nu \operatorname{rot} \mathbf{a}_\eta \cdot \operatorname{rot} \mathbf{w}_k + \mathbf{f} \cdot \mathbf{w}_k] d\Omega. \end{aligned}$$

Divide this equality by $\|\mathbf{u}_k\|_{H(\Omega)}^2$ term by term. Consider the sequence

$$\mathbf{w}_k^* = \mathbf{u}_k^* + (\mathbf{a}_\varepsilon - \mathbf{a}_\eta) / \|\mathbf{u}_k\|_{H(\Omega)}.$$

Since $\|\mathbf{u}_k\|_{H(\Omega)} \rightarrow \infty$, we have $(\mathbf{a}_\varepsilon - \mathbf{a}_\eta) / \|\mathbf{u}_k\|_{H(\Omega)} \rightarrow 0$ strongly. Since $\|\mathbf{u}_k^*\|_{H(\Omega)} = 1$, we have that $\|\mathbf{w}_k^*\|_{H(\Omega)} \rightarrow 1$. Besides, it is clear that $\mathbf{w}_k^* \rightarrow \mathbf{u}_0$ weakly and thus we get the needed equality (3.8.14) again.

Now we show that the limit of the integral on the right-hand side of (3.8.14) is zero. Thanks to (3.8.9) and (3.8.10), we obtain

$$\left| \int_{\omega_\varepsilon} (\mathbf{a}_\varepsilon \cdot \nabla) \mathbf{u}_0 \cdot \mathbf{u}_0 d\Omega \right| \leq M_1 M_2 \int_{\omega_\varepsilon} |\operatorname{rot} \mathbf{u}_0|^2 d\Omega \rightarrow 0$$

as $\varepsilon \rightarrow 0$. Since $\nu > 0$, we have a contradiction which completes the proof. \square

Now we can formulate

Theorem 3.8.2. Under assumptions (i)–(iv), there exists at least one generalized solution of the problem (3.8.1)–(3.8.3) in the sense of Definition 3.8.1. All generalized solutions of the problem are bounded in the energy space and the degree of the operator $I + F$ of the problem with respect to zero and a ball about zero with sufficiently large radius is +1.