Foundations of Functional Analysis in Continuum Mechanics

HOMEWORK #1

1. Let $1 \leq p \leq p' \leq \infty$. Prove that

$$L^{p'}(\Omega) \subset L^p(\Omega),$$

where Ω is a bounded domain in $\mathbb{R}^n$. Prove also that the injection $i : L^{p'}(\Omega) \longrightarrow L^p(\Omega)$ is continuous, that is, there exists a constant $C > 0$ such that

$$\|u\|_p \leq C\|u\|_{p'} \quad \forall u \in L^{p'}(\Omega).$$

2. For a vector $x \in \mathbb{R}^n$ with positive components $x_1, x_2, ..., x_n$, define the following means:

$$M_p(x) = \left(\frac{x_1^p + x_2^p + ... + x_n^p}{n}\right)^{1/p},$$

for any real $p \neq 0$. In particular, $M_1(x)$ is the arithmetic mean and $M_1(x)$ is the quadratic mean.

   a. Using Hölder inequality, prove that, for any $p \geq 1$, $M_1(x) \leq M_p(x)$.
   b. Prove that, for all $0 < p \leq q$, $M_p(x) \leq M_q(x)$.
   c. Define $M_p(x)$ for $p = \pm\infty$ as follows: $M_{+\infty} = \max(x_1, x_2, ..., x_n)$, $M_{-\infty} = \min(x_1, x_2, ..., x_n)$. Prove that

$$\lim_{p\to+\infty} M_p(x) = M_{+\infty}(x), \qquad \lim_{p\to-\infty} M_p(x) = M_{-\infty}(x).$$

   d. Define $M_0(x)$ as $M_0(x) = (x_1 x_2 ... x_n)^{1/n}$, that is, $M_0(x)$ is the geometric mean. Prove that

$$\lim_{p\to 0} M_p(x) = M_0(x).$$

   e. Prove that, for all $-\infty \leq p \leq q$, $M_p(x) \leq M_q(x)$ (Extension of b).

3. Let X be a Banach space and Y a closed subspace. Introduce the following norm in the factor space X/Y: if $\xi \in X/Y$, then $\|\xi\|_{X/Y} := \inf\{\|x\| : x \in \xi\}$ (recall that X/Y consists of subsets of X which are equivalence classes associated to the relation $x_1 \sim x_2$ if $x_1 - x_2 \in Y$).

   a. Prove that $\|\cdot\|_{X/Y}$ defines a norm in X/Y.
   b. Prove that X/Y is complete with respect to the norm $\|\cdot\|_{X/Y}$ and conclude that it is a Banach space.

4. Let X be a Banach space and $A : X \longrightarrow X$ a linear operator. Suppose that A is bounded.

   a. Prove that, for any positive integer n, $\|A^n x\| \leq \|A\|^n \|x\|$. Deduce that $\|A^n\| \leq \|A\|^n$ and that, in particular, if A is bounded so is A^n.
   b. Assuming that $\|A\| < 1$, prove that the *geometric series* $x + Ax + A^2x + ... + A^n x + ...$ converges. Hence, show that the operator $B := I + A + A^2 + ... + A^n + ...$ is linear and bounded. It is called *von Neumann series*.
   c. Under the previous assumptions, prove that the identity $(I - A)B = I$. Hence, conclude that the operator $(I - A)^{-1}$ is bounded and it is precisely B.

5. Let $k(x,y)$ an integrable function on $(a,b)\times(a,b)$. Consider the operator K that acts on functions defined on (a,b) as follows:

$$(Kf)(x) := \int_a^b k(x,y)f(y)dy.$$

   a. Prove that if

$$\int_a^b \int_a^b |k(x,y)|^2 dxdy =: C < \infty$$

   then K is a bounded linear operator in $L^2(a,b)$ and $\|K\| \le \sqrt{C}$.

   b. Prove that if

$$C_1 := \sup_x \int_a^b |k(x,y)|dy < \infty, \quad C_2 := \sup_y \int_a^b |k(x,y)|dx < \infty,$$

   then K is a bounded linear operator in $L^2(a,b)$ and $\|K\| \le \sqrt{C_1 C_2}$.

6. Let X be the space $C^1[0,1]$ with the norm $\|f\|_X := \sup_{[0,1]} |f| + \sup_{[0,1]} |f'|$.

   a. Consider the functional $F(f) = f'(c)$, where $c \in (0,1)$ is fixed. Prove that F is a bounded linear functional in X, and find its norm.

   b. Given $a \in X$, define the operator $A : X \longrightarrow X$ by $Af(x) = a(x)f(x)$. Prove that A is bounded and $\|A\|_{L(X,X)} \le \|a\|_X$.

7. Let ℓ_p be the space of sequences $x = \{x_n\}$ of real numbers such that $\|x\|_p := (\sum_{n=1}^\infty |x_n|^p)^{1/p}$ is finite, $1 \le p < \infty$, and let ℓ_∞ be the space of bounded sequences. Let $x \in \ell_p$, $1 < p < q$, and let y be a sequence such that the series $\sum_{i=1}^\infty x_i y_i$ is convergent for all x in ℓ_p. Prove that $y \in \ell_q$, with $p + q = pq$. (Hint: Consider the operator A from ℓ_p to ℓ_∞ defined by $Ax = z$, where $z_m = \sum_{i=1}^m x_i y_i$, and use the closed graph theorem).

HOMEWORK #2

1. Let $\{u_n\}$ be a sequence of elements in a Hilbert space H. Prove that if $(u_n, u) \to (u, u)$ and $\|u_n\| \to \|u\|$, then $u_n \to u$, that is to say, $\|u_n - u\| \to 0$, where $\|\cdot\|$ is the norm induced by the inner product in H.

2. Let X be the space $C^0([-1, 1])$ endowed with the inner product $(f, g) := \int_{-1}^{1} fg dx$. Let X_1 be the subspace of X made up of even functions (i.e., if $f \in X_1$ then $f(x) = f(-x)$) and X_2 the space of odd functions (i.e., if $f \in X_2$ then $f(x) = -f(-x)$). Prove that $X = X_1 \oplus X_2$ and that X_1 and X_2 are orthogonal. Is X a Hilbert space?.

3. Let $p,\ q$ and r be real functions defined on the interval $[a, b]$, with $p(x) > 0$ and $r(x) > 0$ $\forall x \in [a, b]$, and let us consider the differential operator $\mathcal{L}(u(x)) \equiv (r(x)u'(x))' + q(x)u(x)$, where the prime denotes differentiation with respect to x. One calls *Sturm–Liouville problem* the eigenvalue problem of $\mathcal{L}$ with weight p, that is, the problem consisting of finding constants λ and functions $u(x)$ such that

$$\begin{aligned} \mathcal{L}(u(x)) = -\lambda p(x)u(x), \qquad & x \in (a, b), \\ a_1u(a) + a_2u'(a) = 0, & \\ b_1u(b) + b_2u'(b) = 0, & \end{aligned}$$

where a_1, a_2, b_1 and b_2 are constants. Let us consider the inner product

$$(u, v)_p := \int_a^b p(x)u(x)v(x)dx.$$

Prove that if λ_n and λ_m are different eigenvalues, the corresponding eigenfunctions $u_n(x)$ and $u_m(x)$ are orthogonal.

4. Let P_1 and P_2 be projectors onto closed subspaces K_1 and K_2 of a Hilbert space H.
   a. Prove that $K_1 \perp K_2$ if and only if $P_1P_2 = 0$.
   b. Prove that if $K_1 \perp K_2$ and $K_1 \oplus K_2 = H$, then $P_1 + P_2 = I$.
   c. Prove that if $f(z)$ is a polynomial, then $f(\alpha_1P_1 + \alpha_2P_2) = f(\alpha_1)P_1 + f(\alpha_2)P_2$ for all scalars α_1, α_2, provided $K_1 \perp K_2$ and $K_1 \oplus K_2 = H$.
   d. State and prove a similar result for n projectors P_1, P_2,...,P_n.

5. For all integers l, m such that $l \geq 0$ and $0 \leq m < 2^{l-1}$, define the function $h_{l,m}$ on $(0, 1)$ as follows:

$$\text{for } l = 0, \quad h_{0,0}(x) = 1$$

$$\text{for } l \geq 1, \quad h_{l,m}(x) = \begin{cases} 1 & \text{if } \frac{2m}{2^l} < x \leq \frac{2m+1}{2^l} \\ -1 & \text{if } \frac{2m+1}{2^l} < x \leq \frac{2m+2}{2^l} \\ 0 & \text{otherwise} \end{cases}$$

These functions $h_{l,m}$ are called *Haar functions*.

   a. Prove that $\{h_{l,m}\}$ is an orthogonal system in $L^2(0, 1)$.
   b. Prove that $\{h_{l,m}\}$ is complete in $L^2(0, 1)$. Hence, conclude that $\{h_{l,m}\}$ is an orthogonal basis in $L^2(0, 1)$.

6. The deflection $v(x)$ of a beam of length ℓ and stiffness EI is the solution of the boundary value problem

$$
\begin{aligned}
EI\frac{d^4v}{dx^4} &= q(x),\\
-EI\frac{d^2v}{dx^2}(0) = M_0, &\quad EI\frac{d^3v}{dx^3}(0) = Q_0,\\
-EI\frac{d^2v}{dx^2}v(\ell) = M_1, &\quad EI\frac{d^3v}{dx^3}(\ell) = Q_1,
\end{aligned}
$$

where $q(x)$ is the load on the beam, M_0, M_1 are the bending moments and Q_0, Q_1 are the shear forces at the end of the bar. Obtain the necessary conditions that these data must satisfy in order to have a solvable problem.

7. Let us consider the boundary value problem in two dimensions

$$
\begin{aligned}
\Delta\Delta u &= f \quad \text{in } \Omega,\\
u &= 0 \quad \text{on } \partial\Omega,\\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
$$

where Δ is the Laplacian operator and $f \in L^2(\Omega)$. Set up a variational formulation of this problem, identify the appropriate functional spaces and prove that it has a unique solution.

HOMEWORK #3

1. The Navier equations for an elastic material can be written in three different ways

$$
\begin{aligned}
-2\mu\nabla\cdot(\boldsymbol{\varepsilon}(\boldsymbol{u})) - \lambda\nabla(\nabla\cdot\boldsymbol{u}) &= \rho\boldsymbol{b}\\
-\mu\Delta\boldsymbol{u} - (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) &= \rho\boldsymbol{b}\\
\mu\nabla\times(\nabla\times\boldsymbol{u}) - (\lambda+2\mu)\nabla(\nabla\cdot\boldsymbol{u}) &= \rho\boldsymbol{b}
\end{aligned}
$$

   where $\boldsymbol{u}$ is the displacement field, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ the symmetric part of $\nabla\boldsymbol{u}$, λ and μ the Lamé coefficients, ρ the density of the material and $\boldsymbol{b}$ the body forces. Let us assume that $\boldsymbol{u} = \boldsymbol{0}$ in a part of the boundary of the domain Ω occupied by the solid. Write down the weak form of the previous equations in the appropriate functional spaces, discuss which are the associated natural boundary conditions and establish and prove existence and uniqueness theorems for each case.

2. Choose one of the following exercises:

   a) The problem of finding the deformation of a membrane placed on a ground that behaves as a Winkler material can be written as

$$
\begin{aligned}
\Delta u - \alpha u + f &= \frac{\partial^2 u}{\partial t^2} && \text{in } \Omega,\ t>0\\
u &= g && \text{on } \Gamma_1,\ t>0\\
\frac{\partial u}{\partial n} &= h && \text{on } \Gamma_2,\ t>0\\
u &= u_0 && \text{in } \Omega,\ t=0\\
\frac{\partial u}{\partial t} &= v_0 && \text{in } \Omega,\ t=0
\end{aligned}
$$

   where Ω is the domain of the problem, Γ_1 and Γ_2 are a partition of the boundary $\partial\Omega$, $\alpha > 0$ is a constant, u is the unknown and $f,\ g,\ h,\ u_0$ and v_0 are given data. Write this problem as an abstract evolution problem of the form $\dot{x} + Ax = F$ in the adequate functional spaces. Prove that the operator A associated to this problem generates a quasi-contractive semigroup in this space (Hint: use Theorem 3.1 of Chapter 6 in the book *Mathematical Foundations of Elasticity*, by J.E. Marsden and T.J.R. Hughes).

   b) Consider the Von Kármán equations of a plate given by equations (3.4.1) and (3.4.2) of the book *Functional Analysis in Mechanics*, by L.P. Lebedev and I.I. Vorovich. Prove that their weak form (with homogeneous Dirichlet conditions) is given by equations (3.4.5) and (3.4.6). Determine the appropriate functional spaces for the problem and prove that the bilinear form a and the trilinear form B are continuous in the adequate norms.

3. Choose one of the following exercises:

   a) Read and comment Section 6.4 of the book *Mathematical Foundations of Elasticity*, by J.E. Marsden and T.J.R. Hughes. In particular, comment which restrictions have the main results of this chapter (from the practical point of view) and put examples that do not satisfy the assumptions of the main theorems. Explain which results should be proven to complete the basic theory.

   b) Consider the nonlinear problem of equilibrium of the theory of elastic shallow shells as described in the book *Functional Analysis in Mechanics*, by L.P. Lebedev and I.I. Vorovich. Prove that problem (3.6.1)-(3.6.2), supplied with the appropriate boundary conditions, is equivalent to problem (3.6.8) in the adequate functional spaces, and also to the minimization of the functional given by (3.6.9). Fill up the details of the proof of Theorem 3.6.1.

HOMEWORK #4

1. When the Navier-Stokes equations are written in a rotating frame of reference two additional terms appear, namely, the Coriolis force $2\boldsymbol{\omega}\times\boldsymbol{u}$ and the centrifugal force $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})$ ($\boldsymbol{\omega}$: angular velocity of the reference frame, $\boldsymbol{u}$ velocity of the fluid, $\boldsymbol{r}$: position vector). If the convective term can be assumed to be negligible, the problem to be studied is

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + 2\boldsymbol{\omega}\times\boldsymbol{u} + \nabla p = \boldsymbol{f}, &\quad \text{in } \Omega \\ \nabla\cdot\boldsymbol{u} = 0, &\quad \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{0}, &\quad \text{on } \partial\Omega \end{aligned}$$

where Ω is the domain of the problem. Write down the variational formulation of this problem and prove and existence and uniqueness theorem.

2. In some models that take into account the resistance to the flow due to the porosity of the medium, a term of the form $\alpha\boldsymbol{u}$, with $\alpha > 0$, is added to the Navier-Stokes equations. The resulting continuous problem is

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \alpha\boldsymbol{u} + \nabla p = \boldsymbol{f}, &\quad \text{in } \Omega \\ \nabla\cdot\boldsymbol{u} = 0, &\quad \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{0}, &\quad \text{on } \partial\Omega \end{aligned}$$

Set up a variational formulation of the problem and prove that it has a solution (Hint: adapt the proof of Theorem II.1.2 of the book *Navier-Stokes equations*, by R. Temam).

3. One of the possible ways to linearize the steady-state incompressible Navier-Stokes equations is the so called fixed point or Picard's method. At each iteration, the problem to be solved is:

$$\begin{aligned} -\nu\Delta\boldsymbol{u}_i + (\boldsymbol{u}_{i-1}\cdot\nabla)\boldsymbol{u}_i + \nabla p_i = \boldsymbol{f}, &\quad \text{in } \Omega \\ \nabla\cdot\boldsymbol{u}_i = 0, &\quad \text{in } \Omega \\ \boldsymbol{u}_i = \boldsymbol{0}, &\quad \text{on } \partial\Omega \end{aligned}$$

where the subscript is the iteration counter. Let us assume that the process starts with the initial guess $\boldsymbol{u}_0 = \boldsymbol{0}$.

   a) Set up the variational formulation of the problem.
   b) Prove that for each iteration i the problem has a unique solution, regardless of the value of ν.
   c) Obtain an a priori estimate for the norm of $\boldsymbol{u}_i$ in terms of $\boldsymbol{f}$ in the appropriate functional spaces.
   d) Prove that the iterates $\boldsymbol{u}_i$ converge to the solution of the problem if the viscosity ν is large enough so as to guarantee uniqueness (Hint: consult Theorem II.1.3 of the book *Navier-Stokes equations,* by R. Temam).