converges to a vector in S.

3.2 THEOREM. *Let S be a compact subset of a normed linear space and let T map S into S. If*

$$(*) \quad \|Tx-Ty\| < \|x-y\| \qquad \text{for all } x,y \text{ in } S, \quad x \neq y,$$

then T has a unique fixed point in S.

PROOF. Let $m = \inf_{x \in S} \|Tx-x\|$. There exists a sequence $\{x_n\}$ in S such that $\|Tx_n-x_n\| \to m$. Since S is compact, $\{x_n\}$ has a subsequence $\{x_{n'}\}$ which converges to some $x \in S$. Hence

$$\|x-Tx\| = \lim_{n'\to\infty} \|x_{n'}-Tx_{n'}\| = m.$$

Therefore $Tx = x$, otherwise

$$m \le \|T^2x-Tx\| < \|Tx-x\| = m.$$

It is obvious from (*) that T cannot have more than one fixed point.

The proof shows that the theorem can be extended to compact metric spaces.

If the set S in the Theorem 3.2 is also convex, then we can weaken the inequality (*) and obtain the following result.

3.3 THEOREM. *Let S be a compact convex subset of a normed linear space and let T map S into S. If*

$$\|Tx-Ty\| \le \|x-y\| \qquad \text{for all } x,y \text{ in } S,$$

then T has a fixed point.

PROOF. First we assume that $0 \in S$. For $n = 1,2,\ldots$, define T_n on S by $T_nx = (1-\frac{1}{n})x$. Since S is convex and $0 \in S$, $T_nx = (1-\frac{1}{n})x + \frac{1}{n}\cdot 0 \in S$. Clearly, T_n is a contraction. Hence by Theorem 1.1, there exists an $x_n \in S$ such that

$$(1) \quad (1-\tfrac{1}{n})Tx_n = T_nx_n = x_n, \qquad n = 1,2,\ldots .$$

Since S in compact, $\{x_n\}$ has a subsequence $\{x_{n'}\}$ which converges to some $x \in S$. Thus $Tx_{n'} \to Tx$ and (1) implies

$$Tx = \lim_{n'\to\infty} Tx_{n'} = \lim_{n\to\infty} x_{n'} = x.$$

If $0 \notin S$, choose any $x_0 \in S$ and let $S_0 = -x_0 + S$. The theorem follows from the result we just proved applied to S_0 and the operator $T_0 : S_0 \to S_0$ defined by

$$T_0(-x_0+x) = -x_0 + Tx, \qquad x \in S.$$

TEOREMES DE PUNT FIX