Suppose that X is a vector space over the complex numbers. We write

$$f(z) = \operatorname{Re} f(z) + i \operatorname{Im} f(z), \qquad z \in M,$$

where $\operatorname{Re} f(z)$ and $\operatorname{Im} f(z)$ denote the real and imaginary parts of $f(z)$, respectively. Since $\operatorname{Im} f(z) = -\operatorname{Re} f(iz)$, we have

$$(8) \qquad f(z) = \operatorname{Re} f(z) - i \operatorname{Re} f(iz).$$

Let X_r be X considered as a vector space over the reals. Now $\operatorname{Re} f$ is a linear functional on M, considered as a subspace of X_r. Moreover,

$$|\operatorname{Re} f(z)| \le |f(z)| \le p(z), \qquad z \in M.$$

Hence, by the result we proved for real vector spaces, there exists a linear extension G of $\operatorname{Re} f$ to all of X_r such that $|G(z)| \le p(x)$, $x \in X_r$. Guided by (8), we define F on X by

$$F(x) = G(x) - iG(ix).$$

Now G is a linear extension of f to all of X. Writing $F(x) = |F(x)|e^{i\theta}$, we get

$$|F(x)| = F(e^{-i\theta}x) = \operatorname{Re} F(e^{-i\theta}x) = G(e^{-i\theta}x)$$
$$\le p(e^{-i\theta}x) = p(x).$$

This completes the proof of the theorem.

HAHN-BANACH THEOREM. *If f is a bounded linear functional which is defined on a subspace M of a normed linear space X, then f can be extended to a bounded linear functional F defined on X such that $\|f\| = \|F\|$.*

PROOF. Define p on X by $p(x) = \|f\|\|x\|$ and apply the preceding theorem to f and p.

ANNEX 1

APPENDIX 4

PROOF OF THE CLOSED GRAPH THEOREM

For convenience we introduce the following notation. Given x_0 in a normed linear space X and $r > 0$,

$$S(x_0,r) = \{x : \|x-x_0\| < r\}$$
$$\overline{S}(x_0,r) = \{x : \|x-x_0\| \le r\}.$$

For $Z \subset X$ and $\alpha \in C$,

$$\alpha Z = \{\alpha z : z \in Z\}.$$

A vector x_0 is called an *interior point* of a set $Z \subset X$ if $S(x_0,r) \subset Z$ for some $r > 0$.

The following important result is used to prove the closed graph theorem.

BAIRE CATEGORY THEOREM. *If a Banach space is the union of a countable number of closed sets, then at least one of the closed sets has an interior point.*

PROOF. Let the Banach space $X = \bigcup_n C_n$, where each C_n is a closed set. Suppose that none of the C_n has an interior point. Choose $x_1 \in C_1$. Since $S(x_1,1) \not\subset C_1$ and C_1 is closed, there exist x_2 and r_2, $0 < r_2 < \frac{1}{2}$, such that

$$\overline{S}(x_2,r_2) \subset S(x_1,r_1) \quad \text{and} \quad \overline{S}(x_2,r_2) \cap C_1 = \emptyset.$$

Since $S(x_2,r_2) \not\subset C_2$ and C_2 is closed, there exist x_3

and r_3, $0 < r_3 < \frac{1}{3}$, such that

$$\overline{S}(x_3,r_3) \subset S(x_2,r_2) \quad \text{and} \quad \overline{S}(x_3,r_3) \cap C_2 = \emptyset.$$

Continuing in this manner, we obtain sequences $\{x_n\}$ and $\{r_n\}$, $0 < r_n < \frac{1}{n}$, such that

(1) $\overline{S}(x_{n+1},r_{n+1}) \subset S(x_n,r_n)$ and $\overline{S}(x_{n+1},r_{n+1}) \cap C_n = \emptyset$.

The sequence $\{x_n\}$ is a Cauchy sequence. For if $n > m$, then we have from (1) that $x_n \in S(x_m,r_m)$, i.e.,

(2) $$\|x_n - x_m\| < r_m < \frac{1}{m}.$$

Hence $\{x_n\}$ converges to some $x \in X$. Fix m and let $n \to \infty$ in (2). Then $\|x-x_m\| \le r_m$, i.e., $x \in \overline{S}(x_m,r_m)$ which is disjoint from C_{m-1}, $m = 2,\dots$. But this is impossible since $X = \cup_n C_n$.

LEMMA. *Suppose* C *is a convex set in* X *and* $C = (-1)C$. *If* C *has an interior point, then zero is also an interior point of* C.

PROOF. Suppose $S(x_0,r) \subset C$. If $\|x\| < 2r$, then

$$x = (x_0 + \frac{x}{2}) - (x_0 - \frac{x}{2}) \in S(x_0,r) + (-1)S(x_0,r) \subset C + C.$$

But $C + C = 2C$. Indeed, given u and v in C,

$$u + v = 2(\tfrac{1}{2}u + \tfrac{1}{2}v) \in 2C$$

since C is convex. We have shown that $S(0,2r) \subset 2C$, which implies that $S(0,r) \subset C$.

CLOSED GRAPH THEOREM. *A closed linear operator which maps a Banach space into a Banach space is continuous.*

PROOF. Let T be a closed linear operator which maps the Banach space X into the Banach space Y. Define $Z = \{x : \|Tx\| < 1\}$. First we prove that the closure $\overline{Z}$ of Z has an interior point.

Since $D(T) = X$ and T is linear, $X = \cup_{n=1}^{\infty} nZ$. It follows from the Baire Category Theorem that there exists a positive integer k such that $k\overline{Z} = \overline{kZ}$ has an interior point. Therefore, $\overline{Z}$ has an interior point. It is easy to verify that $\overline{Z}$ is convex and $\overline{Z} = (-1)\overline{Z}$. By the lemma, $S(0,r) \subset \overline{Z}$ for some $r > 0$, which implies

(1) $$S(0,\alpha r) \subset \alpha\overline{Z} = \overline{\alpha Z}, \qquad \alpha > 0.$$

Given $0 < \varepsilon < 1$ and $\|x\| < r$, we have from (1) that x is in $\overline{Z}$. Therefore, there exists an $x_1 \subset Z$ such that $\|x-x_1\| < \varepsilon r$. Since $x - x_1 \in S(0,r) \subset \overline{\varepsilon Z}$, there exists an $x_2 \in \varepsilon Z$ such that

$$\|x-x_1-x_2\| < \varepsilon^2 r.$$

Inductively, there exists a sequence $\{x_n\}$ such that

(2) $$\left\|x - \sum_{k=1}^{n} x_k\right\| < \varepsilon^n r, \qquad x_n \in \varepsilon^{n-1}Z.$$

Let $s_n = \sum_{k=1}^{n} x_k$. From (2) and the definition of Z we get

(3) $$s_n \to x \quad \text{and} \quad \|Tx_n\| < \varepsilon^{n-1}.$$

Now $\{Ts_n\}$ is a Cauchy sequence since (3) implies that for $n > m$,

$$\|Ts_n - Ts_m\| \le \sum_{k=m+1}^{\infty} \|Tx_k\| \le \sum_{k=m}^{\infty} \varepsilon^k = \frac{\varepsilon^m}{1-\varepsilon} \to 0$$

as $m \to \infty$. Hence, by the completeness of Y, $Ts_n \to y$ for some $y \in Y$. So, we have $s_n \to x$ and $Ts_n \to y$. Since T is closed, $Tx = y$. Thus

$$\|Tx\| = \|y\| = \lim_{n\to\infty} \|Ts_n\| \le \sum_{k=1}^{\infty} \|Ts_k\| \le \frac{1}{1-\varepsilon},$$

whenever $\|x\| < r$. In particular, if $\|v\| = 1$, then

$$\left\|T(\tfrac{r}{2}v)\right\| < \frac{1}{1-\varepsilon}.$$

Thus

$$\|Tv\| < \frac{2}{r(1-\varepsilon)}, \qquad \|v\| = 1,$$

which shows that $\|T\| \le \dfrac{2}{r(1-\varepsilon)}$.