On the annihilated polynomial of Birkhoff interpolation

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Abstract

Knowing a few real zeros of a real polynomial and its derivatives, as many as the degree of the polynomial, we study here (via Birkhoff interpolation) the distribution and the variation of the other zeros of the polynomial and its derivatives when varying the known zeros.

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1 Introduction

Let $E = [e_{ij}]_{i=1,\ j=0}^{m\ \ \ n-1}$ be an $m \times n$ *incidence matrix* with entries e_{ij} equal to 0 or 1, with exactly n ones and with no rows composed only of zeros. Let D and S be the subsets of $\mathbb{R}^m$ defined by $D = \{(x_1, \dots, x_m) \ : \ x_i \neq x_k \text{ if } i \neq k\}$ and $S = \{(x_1, \dots, x_m) \ : \ x_1 < \cdots < x_m\}$. A pair E, X with $X \in D$ is said to be *regular* or *poised* if the determinant $D(E, X)$ of the matrix

$$A(E,X) = \left[\begin{array}{cccccc} \frac{x_i^{-j}}{(-j)!} & \frac{x_i^{1-j}}{(1-j)!} & \frac{x_i^{2-j}}{(2-j)!} & \cdots & \frac{x_i^{n-1-j}}{(n-1-j)!} & ; e_{ij=1} \end{array} \right],$$

is non-zero [5, 10]. Matrix E is called *order regular* when $D(E, X) \neq 0$ for all $X \in S$. If E, X is regular then there exists a unique monic polynomial P of degree n which is annihilated by E, X (see [8, 9]). That is, satisfying $P^{(j)}(x_i) = 0$ for $e_{ij} = 1$. This polynomial is given explicitly by

$$P(t) = P(E,X;t) = \frac{n!(-1)^n}{D(E,X)} \left[\frac{\begin{array}{ccccc} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^n}{n!} \end{array}}{A'(E,X)} \right] \tag{1}$$

where $A'(E, X)$ is the $n \times (n+1)$ matrix obtained from $A(E, X)$ by adding the last column $\left[\frac{x_i^{n-j}}{(n-j)!}; e_{ij} = 1 \right]$. In this way, every polynomial of degree n is determined by n numbers chosen from its zeros or the zeros of its derivatives if the pair E, X obtained from the n numbers is regular. Thus, the study of this polynomial is the study of polynomial $P(t)$, and the distribution of its zeros and the zeros of its derivatives is determined by these n numbers.

We are interested in studying the real zeros of polynomial (1) and the real zeros of its derivatives when varying X. To this end, notice that we only need to study the non-specified zeros in the pair E, X, since the specified zeros in E, X are known. It may be that a non-specified zero, in its variation, occupies the same place as a specified one, which will be detected by an increase of multiplicity in the specified zero. These specified zeros which show an increase of multiplicity will also be taken into account.

More precisely, we shall call *virtual zero* of order j of the pair E, X a real zero z of $P^{(j)}$ non-specified in E, X (i.e. with $z \neq x_i$ for all i, or $z = x_i$ for some i with $e_{ij} = 0$) or specified in E, X with $q < q'$, where q' is the multiplicity of z in polynomial $P^{(j)}$ and q is the specified multiplicity of z in matrix E. That is, if $z = x_i$, $1 \leq i \leq m$, then q is the number that verifies $e_{ij} = e_{i,j+1} = \cdots = e_{i,j+q-1} = 1$, $e_{i,j+q} = 0$. From what has been stated above, we shall deal with the study of the virtual zeros.

On the other hand, in order to follow the variation of the virtual zeros, we need their number to be independent from X. This condition is very restrictive, since even for Pólya matrices with no odd supported sequences (conservative Pólya matrices), all of them order regular, the number of virtual zeros is not usually constant when $X \in S$, as is the case for example with matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

In fact, taking $X = (0, a, 1)$ it is easy to see that if $0 < a < \frac{1}{3}$ then there are two virtual zeros of order zero, when $a = \frac{1}{3}$ there is just one virtual zero of order zero, and if $\frac{1}{3} < a < 1$ there are no virtual zeros of order zero.

The paper is structured as follows. In Section 2 we introduce some preliminary concepts about Birkhoff interpolation that we shall need later on. In Section 3, Theorem 3, we establish that for a wide range of order regular matrices, called strongly conservative Pólya matrices, the number of virtual zeros of each order is constant. That is, it does not depend on X even when X varies in D. In the results that follow we show some properties on the distribution of the zeros of the polynomial annihilated by the pair E, X. Finally, in Theorem 10 we establish the continuity of the virtual zeros as functions of X with $X \in D$, when E is a strongly conservative Pólya matrix.

In Section 4 we state and prove the main results, which deal with the behavior of the polynomial and the virtual zeros when E is a strongly conservative Pólya matrix and X tends to the boundary of D. In Theorem 12 we show that polynomial $P(E, X; t)$ has a limit when X tends to the boundary of D, and that limit polynomial is the polynomial annihilated by the coalesced pair $\bar{E}, \bar{X}$ (see Section 2). Regarding the virtual zeros we deal with the following question: *Do the virtual zeros have a limit when X tends to the boundary of D?* We obtain in Theorem 17 that the answer is affirmative, by showing that the virtual zeros, considered as functions of $X \in D$, can be continuously extended to every point of $\mathbb{R}^m$. These extensions to the whole $\mathbb{R}^m$ are obtained by means of the concept of latent zero, which allows us to obtain the relationship between the limit of the virtual zeros and the virtual zeros of the limit polynomial on the boundary.

2 Preliminary concepts

Let $E = [e_{ij}]_{i=1,\ j=0}^{m \quad n-1}$ be an incidence matrix with n ones. The $(j+1)$-th column of E contains the entries e_{ij}, $1 \leq i \leq m$. The numbers $m(j) = \sum_{i=1}^{m} e_{ij}$ and $M(j) = \sum_{k=0}^{j} m(k)$, defined for $j = -1, 0, \ldots, n-1$ ($m(-1) = M(-1) = 0$) are called *Pólya constants* of E. If E, X is regular for some $X \in D$ then E is a Pólya matrix [10]. That is, it satisfies the *Pólya condition* $M(j) \geq j+1$ if $j = 0, 1, \ldots, n-1$.

A sequence of $1'$s in the i-th row of E is *supported* if when (i, j) is the position of the first 1 of the sequence, this implies that there exist two $1'$s: $e_{i_1,j_1} = e_{i_2,j_2} = 1$ with $i_1 < i < i_2, j_1 < j$, and $j_2 < j$. E is said to be *conservative* if it contains no odd supported sequences. Then, we have [1]

Theorem 1 *Each conservative Pólya matrix is order regular.*

We call *strongly conservative* matrix a matrix such that all the sequences having its first 1 in a column of E different from the first column, are even sequences. If we permute the rows of a strongly conservative matrix we obtain a strongly conservative matrix again and in particular conservative. Therefore, strongly conservative Pólya matrices have the property that E, X is regular for any $X \in D$.

To a point X in $\mathbb{R}^m$ corresponds a partition of the set $I = \{1, \ldots, m\}$ into classes of equivalence defined by the equivalence relation $i \sim k$ if $x_i = x_k$ ($i, k \in I$). We will denote the classes by letters ι, ι', ι_1, ι_2, ... Each class ι determines a real number x_ι defined by $x_\iota = x_i$ if $i \in \iota$. If we arrange the numbers x_ι in increasing order, we will obtain a point $\bar{X} = (x_{\iota_1}, \ldots, x_{\iota_q})$ with $x_{\iota_1} < \cdots < x_{\iota_q}$, called *coalescence* of X.

The partition of the set I also gives a coalescence of E to a $q \times n$ matrix $\bar{E}$ defined as follows. Let $\tilde{E}_s$ be the matrix that consists of rows i, $i \in \iota_s$, of E, we coalesce this matrix to one row, which will be the s-th row of $\bar{E}$. Let $m(j)$ and $M(j)$ be the Pólya constants of $\tilde{E}_s$, $j = -1, \ldots, n-1$, and define values $m^o(j)$, $j = 0, \ldots, n-1$, that only take values 0 and 1 by induction. We put $m^o(0) = 1$ if and only if $m(0) \geq 1$; if $m^o(k)$ and hence $M^o(k) = \sum_{r=0}^{k} m^o(r)$ are known for $k = 0, \ldots, j$, we define $m^o(j+1) = 1$ if $M(j+1) - M^o(j) \geq 1$, and $m^o(j+1) = 0$ otherwise. Values $m^o(j)$,

$j = 0, \ldots, n-1$, determine the so called *level function* of m and that satisfies $M^o(j) \le M(j)$ for $j = 0, \ldots, n-1$. The elements of the s-th row of $\bar{E}$ are $\bar{e}_{sj} = m^o(j)$, $j = 0, \ldots, n-1$.

The operation $\bar{E}$ of coalescence of E preserves the number n of ones, the Pólya condition and strongly conservative matrices. If $X \in \bar{S} = \{X\colon\ x_1 \le \cdots \le x_m\}$ then $\bar{E}$ is obtained by coalescing consecutive rows of E, and $\bar{E}$ is conservative if E does. Notice that when $X \in D$, $\bar{E}$ and $\bar{X}$ are obtained from a permutation of the rows of E and the coordinates of X according to the same permutation.

Given real numbers $a < b$, we define $S_{a,b} = \{X\colon\ a \le x_1 < \cdots < x_m \le b\}$ and $\bar{S}_{a,b} = \{X\colon\ a \le x_1 \le \cdots \le x_m \le b\}$. For a conservative Pólya matrix E and a point $X \in S_{a,b}$, the interpolating polynomial of a function $f \in C^{(n-1)}([a,b])$ is the unique polynomial $P(t)$ of degree at most $n-1$ that satisfies $P^{(j)}(x_i) = f^{(j)}(x_i)$ for $e_{ij} = 1$. We denote this polynomial by $P(f, E, X; t)$. There is a natural extension of this polynomial onto $\bar{S}_{a,b}$ given by $P(f, E, X; t) = P(f, \bar{E}, \bar{X}; t)$. We have [2, 5]

Theorem 2 *For a conservative Pólya matrix E and $f \in C^{(n-1)}([a,b])$, the natural extension of $P(f, E, X; t)$ onto $\bar{S}_{a,b}$ is continuous.*

A review on univariate Birkhoff interpolation can be found in Lorentz, et al. [5]. See also [1, 3, 4, 7].

3 First results

In this section some results concerning virtual zeros in the case when E is a strongly conservative Pólya matrix are presented. We begin with

Theorem 3 *Let E be a strongly conservative Pólya matrix with n ones and with Pólya constants $m(j)$ and $M(j)$, and let X belong to D. Then $v_j = M(j-1) - j$ for $j = 0, \ldots, n-1$, where v_j is the number of virtual zeros of order j of the pair E, X. In particular v_j does not depend on X.*

Let P be the monic polynomial of degree n annihilated by E, X. Before giving the proof of the theorem we state and prove the following lemma.

Lemma 4 *A real number, z, is a virtual zero of order j of the pair E, X, $1 \le j \le n$, if one of the following conditions is satisfied: (i) $P^{(j-1)}(z) = 0$ and z is a specified virtual zero of order $j-1$ of E, X, (ii) $P^{(j-1)}(z) \ne 0$, $P^{(j)}(z) = 0$ and z has odd multiplicity in $P^{(j)}$.*

Proof. If (i) holds, from the definition of virtual zero it follows that z is a virtual zero of order j of E, X. Suppose that (ii) holds. If z is non-specified in E, X as zero of order j then z is a virtual zero and we are done. Assume that $z = x_i$ for some i with $e_{ij} = 1$. It follows from $P^{(j-1)}(z) \neq 0$ that $e_{i,j-1} = 0$. Moreover, we have $e_{ij} = \cdots = e_{i,j+p-1} = 1$ and $e_{i,j+p} = 0$ being p an even number, since E is strongly conservative. However, z has odd multiplicity in $P^{(j)}$, and so $P^{(j+p)}(z) = 0$. Hence, z is a virtual zero of order j and lemma follows. $\square$

Proof of Theorem 3. Let $1 \leq j \leq n$. We claim that between two adjacent zeros of polynomial $P^{(j-1)}$, there is a virtual zero of order j. Indeed, it is well known that between two adjacent zeros of $P^{(j-1)}$ there is (counting multiplicities) an odd number of zeros of $P^{(j)}$. We can select one of these zeros having odd multiplicity as a zero of $P^{(j)}$. By Lemma 4, this zero is a virtual zero of order j and the claim is proved. By applying the claim to each couple of adjacent zeros of $P^{(j-1)}$, we obtain at least $u-1$ virtual zeros of order j where u is the number of distinct real zeros of $P^{(j-1)}$.

Moreover, by Lemma 4 we have that if z is a specified virtual zero of order $j-1$ then z is also a virtual zero of order j. Hence, we have s new virtual zeros of order j, where s is the number of specified virtual zeros of order $j-1$. Thus, $v_j \geq u+s-1$. Since $u = r + v_{j-1}$ and $r + s = m(j-1)$, where r is the number of distinct real zeros of $P^{(j-1)}$ which are not virtual zeros of order $j-1$, we get that $v_j \geq r + v_{j-1} + s - 1 = v_{j-1} + m(j-1) - 1$. That is, $v_j = v_{j-1} + m(j-1) - 1 + w_j$ for some $w_j \geq 0$. By applying successively this equality for $j = 1, \ldots, j_0$, where $1 \leq j_0 \leq n$, we get $v_{j_0} = v_0 + M(j_0-1) - j_0 + w_1 + \cdots + w_{j_0}$, and from $M(n-1) = n$ and $v_n = 0$ we obtain that $0 = v_0 + w_1 + \cdots + w_n$. Hence $v_0 = 0$ and $w_j = 0$, $j = 1, \ldots, n$. It follows that $v_{j_0} = M(j_0-1) - j_0$ for $j_0 = 0, \ldots, n-1$, and this completes the proof. $\square$

Notice that the preceding theorem implies $v_0 = 0$. In the next results we show some properties on the distribution of the zeros of the polynomial P annihilated by a pair E, X, where E is a strongly conservative Pólya matrix and $X \in D$. We begin with a corollary that follows from the proof of Theorem 3.

Corollary 5 *Let $z_1 < z_2 < \cdots < z_u$ be the distinct real zeros of $P^{(j-1)}$, being $1 \leq j \leq n-1$. Between each couple of adjacent zeros $z_i < z_{i+1}$, we select a zero a_i of $P^{(j)}$ with odd multiplicity. Let $z_{i_1} < \cdots < z_{i_s}$ be the specified virtual zeros of order $j-1$ of the pair E, X. Then the virtual zeros of order j of the pair E, X are the zeros $a_1, \ldots, a_{u-1}, z_{i_1}, \ldots, z_{i_s}$.*

The next two corollaries are a straightforward consequence of the previous one. We point out that the second one is the converse of Lemma 4.

Corollary 6 *Between two adjacent zeros of $P^{(j-1)}$ there is a unique zero of $P^{(j)}$ with odd multiplicity. However, if z is a zero of $P^{(j)}$ located on the right* *of the greatest real zero (or on the left of the smallest real zero) of $P^{(j-1)}$, then the multiplicity of z in $P^{(j)}$ is even.*

Corollary 7 *Let z be a virtual zero of order j of E, X, $1 \leq j \leq n-1$. If $P^{(j-1)}(z) = 0$ then z is a specified virtual zero of order $j-1$ of E, X, and if $P^{(j-1)}(z) \neq 0$ the multiplicity of z as a zero of $P^{(j)}$ is odd; furthermore, z is located between two adjacent zeros of $P^{(j-1)}$.*

Now we state and prove a result that we shall need further on.

Lemma 8 *The following statements hold.*

(i) If $e_{ij} = e_{i,j+1} = \cdots = e_{i,j+p-1} = 1$ and $e_{i,j-1} = e_{i,j+p} = 0$, then x_i can not be a zero of $P^{(j-1)}$ and of $P^{(j+p)}$ simultaneously.

(ii) The multiplicity of each real zero of $P^{(j)}$ non-specified in E, X is odd.

Proof. (i) Suppose, on the contrary, that we have $P^{(j-1)}(x_i) = P^{(j+p)}(x_i) = 0$. By applying Corollary 7 we have that x_i is a specified virtual zero of order $j-1$ of E, X, and this contradicts the fact that $e_{i,j-1} = 0$.

(ii) Let z be a real zero of $P^{(j)}$ non-specified in E, X. If $P^{(j+1)}(z) \neq 0$ then the multiplicity of z in $P^{(j)}$ is 1 and we are done. If not, from Corollary 7 it follows that z is not a virtual zero of order $j+1$ and in particular it is specified in E, X as zero of order $j+1$. By statement (i) of this lemma and the fact that E is strongly conservative, the multiplicity of z in $P^{(j+1)}$ is even, and hence the multiplicity of z in $P^{(j)}$ is odd. $\square$

Next result gives an interval where the virtual zeros can be located.

Proposition 9 *If z is a virtual zero of order j, then the inequalities $\min_{i \in A_j}\{x_i\} < z < \max_{i \in A_j}\{x_i\}$ hold, where A_j is the set of all indices $i = 1, \ldots, m$ for which $e_{is} = 1$ for some $s < j$.*

Proof. We argue by induction in j. When $j = 0$ the inequalities are true. Let the inequalities be true for any j with $0 \leq j < j_0$. We will see that they are true for $j = j_0$. Let $k \leq j_0$ satisfy $P^{(k-1)}(z) \neq 0$ and $P^{(k)}(z) = P^{(k+1)}(z) = \cdots = P^{(j_0)}(z) = 0$. From Corollary 7 we get that z is a virtual zero of order k located between two adjacent zeros $u < v$ of $P^{(k-1)}$. If u is a virtual zero of order $k-1$ then we apply the hypothesis of induction with $j = k-1$; if not, u is a zero of $P^{(k-1)}$ specified in E, X. In both cases we have $\min_{i \in A_k}\{x_i\} \leq u < z$. Proceeding in the same way we will obtain that $z < v \leq \max_{i \in A_k}\{x_i\}$ and since $A_k \subseteq A_{j_0}$, then $\min_{i \in A_{j_0}}\{x_i\} < z < \max_{i \in A_{j_0}}\{x_i\}$. This completes the proof. $\square$

Let $a_{j1} < a_{j2} < \cdots < a_{j,v_j}$ be the virtual zeros of order j of the pair E, X. Then each a_{jr} depends on X, say $a_{jr} = a_{jr}(X)$. We obtain in this way functions

$$a_{jr} : D \to \mathbb{R};\ r = 1, \ldots, v_j;\ j = 0, \ldots, n-1$$

that will be called *functions of virtual zeros.* We have,

Theorem 10 *For a strongly conservative Pólya matrix, the functions of virtual zeros are continuous on D.*

Theorem 10 is a straightforward consequence of Lemma 11 stated below, and which follows from the continuity of the zeros as functions of its coefficients [6] when it is applied to all successive derivatives of a real polynomial. By $\|\cdot\|$ we denote any norm in the space $\mathbb{P}_n$ of real polynomials of degree at most n.

Lemma 11 *Let P be a real monic polynomial of degree n and let $\varepsilon > 0$. There exists a real number $r > 0$ such that every real monic polynomial $\tilde{P}$ of degree n with $||\tilde{P} - P|| < r$, satisfies the following properties for any $j = 0, \ldots, n-1$: (i) All the real zeros of $\tilde{P}^{(j)}$ belong to intervals of the form $(z - \varepsilon, z + \varepsilon)$ being z a real zero of $P^{(j)}$, and (ii) If z is a real zero of $P^{(j)}$ of multiplicity m, the number of real zeros of $\tilde{P}^{(j)}$ in the interval $(z - \varepsilon, z + \varepsilon)$ counting its multiplicities is $m - 2\sigma$ where $\sigma \geq 0$ is a whole number.*

If P is a real polynomial of degree n, we denote by ε_P the number $\varepsilon_P = \min \frac{|z_1 - z_2|}{2}$ being z_1 and z_2 distinct real numbers such that $P^{(j_1)}(z_1) = P^{(j_2)}(z_2) = 0$ for some j_1 and j_2 with $0 \leq j_1 \leq j_2 \leq n-1$. In order to do more readable the paper, now we give a proof of Theorem 10.

Proof of Theorem 10. From (1) it follows that polynomial $P(E, X; t)$ is a continuous function of $X \in D$. Let $X = (x_1, \ldots, x_m) \in D$ and let ε be real with $0 < \varepsilon < \varepsilon_P$, being $P(t) = P(E, X; t)$. From the continuity of the polynomial and the previous lemma, we get that there exists $\delta > 0$ such that if $\tilde{X} \in B(X, \delta)$ (ball of center X and radius δ with the infinite norm) then the perturbed polynomial $\tilde{P}(t) = P(E, \tilde{X}; t)$ satisfies conclusions (i) and (ii) of the previous lemma. We can assume that $0 < \delta < \varepsilon$.

We claim that if $\tilde{X} = (\tilde{x}_1, \ldots, \tilde{x}_m) \in B(X, \delta)$ and $a_{jr}(X)$ is a virtual zero of order j of E, X, then there is a virtual zero of order j of $E, \tilde{X}$ that belongs to the interval $(a_{jr}(X) - \varepsilon, a_{jr}(X) + \varepsilon)$. Indeed, if $a_{jr}(X)$ is non-specified in E, X as zero of order j, the multiplicity of $a_{jr}(X)$ in $P^{(j)}$ is odd. From Lemma 11 it follows that $\tilde{P}^{(j)}$ has at least one zero $\tilde{z}$ in $(a_{jr}(X) - \varepsilon, a_{jr}(X) + \varepsilon)$. Such zero can not be specified in $E, \tilde{X}$ as zero of order j, since otherwise we would arrive to a contradiction with the fact that $\delta < \varepsilon < \varepsilon_P$. Hence, $\tilde{z}$ is a virtual zero of order j of $E, \tilde{X}$ and we are done.

Now, we assume that $a_{jr}(X)$ is specified in E, X as zero of order j. We have $a_{jr}(X) = x_i$ for some i with $e_{ij} = e_{i,j+1} = \cdots = e_{i,j+p-1} = 1$, $e_{i,j+p} = 0$, and $P^{(j+p)}(x_i) = 0$. Since x_i is a non-specified virtual zero of order $j+p$ of E, X, and in particular x_i has odd multiplicity in $P^{(j+p)}$, the multiplicity of x_i in $P^{(j)}$ is $p+q$ with q an odd number. For some $\sigma \geq 0$ polynomial $\tilde{P}^{(j)}$ has $p+q-2\sigma$ zeros in $(x_i - \varepsilon, x_i + \varepsilon)$ counting its multiplicities. Since p of these zeros are equal to $\tilde{x}_i$, q is odd and 2σ is even, we get that $\tilde{P}^{(j)}$ must have a new zero, say $\tilde{z}$, which has not been counted in the p zeros equal to $\tilde{x}_i$ and belongs to $(x_i - \varepsilon, x_i + \varepsilon)$. If $\tilde{z} = \tilde{x}_i$ then $\tilde{P}^{(j+p)}(\tilde{x}_i) = 0$ and $\tilde{z}$ is a virtual zero of order j of $E, \tilde{X}$. Hence, when $\tilde{z} = \tilde{x}_i$ we are done. If $\tilde{z} \neq \tilde{x}_i$ then $\tilde{z} \neq \tilde{x}_k$ for all $k \neq i$ (this fact follows from $\delta < \varepsilon < \varepsilon_P$) and so $\tilde{z}$ is also a virtual zero of order j of $E, \tilde{X}$. The claim is proved.

The claim implies that $|a_{jr}(\tilde{X}) - a_{jr}(X)| < \varepsilon$ for $r = 1, \ldots, v_j$, and this completes the proof. $\square$

4 Main results

In this section we deal with the behavior of the virtual zeros on the boundary of D. Especially, we are interested in the following fundamental question: *Have the virtual zeros a limit when $X \to X_0 \in \partial D$?* or, in other words, *Can the functions of virtual zeros be continuously extended to the whole $\mathbb{R}^m$?* We will give affirmative answers to these questions in Theorem 17.

First, we show an interesting consequence of Theorem 2 that we will need further on. Let E be a conservative Pólya matrix. Then polynomial $P(E, X; t)$, defined for $X \in S$, can be extended to all points of $\bar{S}$ by $P(E, X; t) = P(\bar{E}, \bar{X}; t)$. We have,

Theorem 12 *If E is a conservative Pólya matrix, the natural extension of $P(E, X; t)$ onto $\bar{S}$ is continuous.*

Proof. Let $a < b$ be given and let $S_{a,b} = \{X\colon\ a \leq x_1 < \cdots < x_m \leq b\}$. Consider the matrix E' obtained from E by adding a last column of zeros and the last row $[0, \ldots, 0, 1]$. Note that E' has $n+1$ ones and it is Pólya and conservative. If $X = (x_1, \ldots, x_m) \in S_{a,b}$ we also define $X' = (x_1, \ldots, x_m, c)$, where c is a fixed number with $c > b$. Clearly, the operations X' and E' preserve coalescences. That is, $\overline{X'} = \overline{X}'$ and $\overline{E'} = \overline{E}'$. Let $f \in C^{(n)}(\mathbb{R})$ be such that $f^{(j)} \equiv 0$ in $[a, b]$, $j = 0, \ldots, n-1$, and $f^{(n)}(c) = n!$. When $X \in S_{a,b}$ the interpolating polynomial $P = P(f, E', X'; t)$ is a monic polynomial of degree n annihilated by E, X, since it satisfies $P^{(j)}(x_i) = f^{(j)}(x_i) = 0$ for $e_{ij} = 1$ and $P^{(n)}(c) = n!$. This implies that $P(E, X; t) = P(f, E', X'; t)$. This fact remains valid to all points of $\bar{S}_{a,b}$, since if $X \in \partial S_{a,b}$ then $P(E, X; t) = P(\bar{E}, \bar{X}; t) = P(f, \overline{E}', \overline{X}'; t) = P(f, \overline{E'}, \overline{X'}; t) = P(f, E', X'; t)$. Therefore $P(E, X; t) = P(f, E', X'; t)$ for all $X \in \bar{S}_{a,b}$, and from Theorem 2 it follows the continuity of $P(E, X; t)$ as function of $X \in \bar{S}_{a,b}$, for any $a < b$. We conclude that $P(E, X; t)$ is a continuous function of $X \in \bar{S}$, and this completes the proof. $\square$

For a strongly conservative Pólya matrix we can also extend polynomial $P(E, X; t)$, $X \in D$, to all points of $\mathbb{R}^m$ by $P(E, X; t) = P(\bar{E}, \bar{X}; t)$. By applying the previous theorem to each one of the matrices obtained by rearranging the rows of E, we obtain the following

Corollary 13 *If E is a strongly conservative Pólya matrix, the natural extension of $P(E, X; t)$ onto $\mathbb{R}^m$ is continuous.*

From now on, $E = [e_{ij}]_{i=1,\ j=0}^{m\quad n-1}$ is a strongly conservative Pólya matrix with Pólya constants $m(j)$ and $M(j)$, and v_j is defined by $v_j = M(j-1)-j$, $j = 0, \dots, n-1$. Let $X = (x_1, \dots, x_m) \in \partial D$. In accordance with Theorem 3, the number of virtual zeros of order j of the coalesced pair $\bar{E}, \bar{X}$ is $\bar{v}_j = \bar{M}(j-1) - j$ where $\bar{m}(j)$ and $\bar{M}(j)$ are the Pólya constants of $\bar{E}$. If $x_{\iota_1} < x_{\iota_2} < \cdots < x_{\iota_q}$ are the coordinates of $\bar{X}$, then $\bar{M}(j-1) = \sum_{s=1}^{q} M_s^o(j-1)$ where $m_s(j)$ and $M_s(j)$ are the Pólya constants of the matrix $\tilde{E}_s$ consisting of rows i of E with $i \in \iota_s$. In particular, $\bar{M}(j-1) \leq \sum_{s=1}^{q} M_s(j-1) = M(j-1)$, and $\bar{v}_j \leq v_j$ for $j = 0, \dots, n-1$. Notice that if X is such that $m_s(j) \geq 2$ for some j and s, then $M_s^o(j) < M_s(j)$ and so $\bar{M}(j-1) < M(j-1)$. It follows that for this j we have $\bar{v}_j < v_j$, and hence the number of virtual zeros of order j of $\bar{E}, \bar{X}$ is smaller than v_j. Thus, the process of coalescence generates, in general, losses of virtual zeros.

Notice that the number of lost virtual zeros of order j is

$$v_j - \bar{v}_j = M(j-1) - \bar{M}(j-1) = \sum_{s=1}^{q} [M_s(j-1) - M_s^o(j-1)] \tag{2}$$

This leads us to the following definition.

Definition 14 *Let $X \in \partial D$ and let P be the monic polynomial of degree n annihilated by $\bar{E}, \bar{X}$. For each $j = 0, \dots, n-1$ and each real zero z of $P^{(j)}$ we define a number $l_j(z)$ as follows. If z is non-specified in $\bar{E}, \bar{X}$ as zero of order j, we put $l_j(z) = 1$; otherwise, if z is specified in $\bar{E}, \bar{X}$ and $z = x_{\iota_s}$, $1 \leq s \leq q$, we define $l_j(z) = \Delta + M_s(j-1) - M_s^o(j-1)$ where $\Delta = 1$ if z is a virtual zero of order j of $\bar{E}, \bar{X}$, and $\Delta = 0$ otherwise.*

Lemma 15 *If $z_1 < \cdots < z_u$ are the distinct real zeros of $P^{(j)}$, then the sum $l_j(z_1) + \cdots + l_j(z_u)$ coincides with v_j.*

Proof. We have

$$l_j(z_1) + \cdots + l_j(z_u) = \bar{v}_j + \sum_{s \in J} [M_s(j-1) - M_s^o(j-1)] \tag{3}$$

where J is the set of all indices $s = 1, \ldots, q$ for which x_{ι_s} is specified in $\bar{E}, \bar{X}$ as zero of order j. From the definition of $\bar{E}$, we have that $s \in J$ if and only if $m_s^o(j) = 1$. For the other s's we have $m_s^o(j) = 0$, and hence $M_s(j-1) - M_s^o(j-1) = 0$. This implies that we can replace the set J in (3) by the set $J' = \{1, \ldots, q\}$, and from (2) lemma follows. $\square$

Under the same notations of the previous lemma, vector

$$R_j = (\underbrace{z_1, \ldots, z_1}_{l_j(z_1)}, \underbrace{z_2, \ldots, z_2}_{l_j(z_2)}, \ldots\ldots, \underbrace{z_u, \ldots, z_u}_{l_j(z_u)})$$

has exactly v_j coordinates. We will denote the r-th coordinate of R_j by $\bar{a}_{jr}$. That is, we put $R_j = (\bar{a}_{j1}, \bar{a}_{j2}, \ldots, \bar{a}_{j,v_j})$. Observe that $\bar{a}_{j1} \leq \bar{a}_{j2} \leq \cdots \leq \bar{a}_{j,v_j}$ and $P^{(j)}(\bar{a}_{jr}) = 0$ if $1 \leq r \leq v_j$.

Zeros $\bar{a}_{jr}$, $r = 1, \ldots, v_j$ will be called *latent zeros* of order j of the pair E, X, whenever $X \in \partial D$. Notice that the latent zeros of order j of E, X contain the virtual zeros of order j of the pair $\bar{E}, \bar{X}$. In particular, if the number of virtual zeros of order j of $\bar{E}, \bar{X}$ is equal to v_j, then the latent zeros of order j of E, X coincide with the virtual zeros of order j of $\bar{E}, \bar{X}$. From Proposition 9 it is easily seen that $\min_{1 \leq i \leq m}\{x_i\} \leq \bar{a}_{jr} \leq \max_{1 \leq i \leq m}\{x_i\}$ for all j and r.

Definition 16 *Functions $\bar{a}_{jr} : \mathbb{R}^m \to \mathbb{R}$ ($1 \leq r \leq v_j$, $0 \leq j \leq n-1$) defined by $\bar{a}_{jr}(X) = a_{jr}(X)$ if $X \in D$, and $\bar{a}_{jr}(X) = \bar{a}_{jr}$ if $X \in \partial D$, will be called functions of latent zeros.*

Observe that the previous functions are indeed extensions onto $\mathbb{R}^m$ of the functions of virtual zeros. We have,

Theorem 17 *For a strongly conservative Pólya matrix E, the functions of latent zeros are continuous functions on $\mathbb{R}^m$.*

In order to prove the theorem it suffices to prove that

$$\lim_{D \ni \tilde{X} \to X} a_{jr}(\tilde{X}) = \bar{a}_{jr}(X) \tag{4}$$

for $X \in \partial D$, $0 \leq j \leq n-1$ and $1 \leq r \leq v_j$, since D is an open dense subset of $\mathbb{R}^m$. In what follows we fix $X = (x_1, \ldots, x_m) \in \partial D$ and P is defined to be the monic polynomial of degree n annihilated by $\bar{E}, \bar{X}$. In the next lemma, $B(X, \delta)$ denotes the ball of center X and radius δ with the infinite norm, and the numbers $l_j(z)$ are given in Definition 14.

Lemma 18 *For each $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 \leq j \leq n-1$, z is a real zero of $P^{(j)}$ and $\tilde{X}$ belongs to $B(X, \delta) \cap D$, then the number of virtual zeros of order j of $E, \tilde{X}$ in the interval $(z - \varepsilon, z + \varepsilon)$ is at least $l_j(z)$.*

Proof. We can assume that $0 < \varepsilon < \varepsilon_P$. Consider the number r obtained from Lemma 11 when it is applied to P and ε. Corollary 13 guarantees the existence of $\delta > 0$ such that if $\tilde{X} \in B(X, \delta)$ then $||\tilde{P} - P|| < r$, where $\tilde{P}(t) = P(E, \tilde{X}; t)$. In particular $\tilde{P}$ satisfies conclusions (i) and (ii) of Lemma 11. Number δ can be assumed to be with $0 < \delta < \varepsilon$. We will see that for this δ lemma holds. Let $\tilde{X} = (\tilde{x}_1, \ldots, \tilde{x}_m) \in B(X, \delta) \cap D$ and consider a real zero z of $P^{(j)}$. In what follows $\tilde{P}$ is the monic polynomial of degree n annihilated by $E, \tilde{X}$. If z is non-specified in $\bar{E}, \bar{X}$ as zero of order j, then $l_j(z) = 1$ and we have to see that there exists some virtual zero of order j of $E, \tilde{X}$ in $(z - \varepsilon, z + \varepsilon)$. Proceeding in the same way as in the proof of Theorem 10 (when zero $a_{jr}(X)$ was non-specified in E, X), we obtain a zero $\tilde{z}$ of $\tilde{P}^{(j)}$ in $(z - \varepsilon, z + \varepsilon)$ and non-specified in $E, \tilde{X}$. Hence $\tilde{z}$ is a virtual zero and we are done.

Suppose that z is specified in $\bar{E}, \bar{X}$ as zero of order j. In this case we have $z = x_\iota$ for some class $\iota = \iota_s$ of the set $I = \{1, 2, \ldots, m\}$. Moreover, $\bar{e}_{sj} = m_s^o(j) = 1$ where $m_s(j)$ and $M_s(j)$ are the Pólya constants of the matrix $\tilde{E}_s$ consisting of rows i of E with $i \in \iota_s$. We have to see that the number of virtual zeros of order j of $E, \tilde{X}$ in $(x_\iota - \varepsilon, x_\iota + \varepsilon)$ is at least $l_j(x_\iota)$. Recall that $l_j(x_\iota) = \Delta + M_s(j-1) - M_s^o(j-1)$ being $\Delta = 1$ if x_ι is a virtual zero of order j of $\bar{E}, \bar{X}$ and $\Delta = 0$ otherwise. We put $m_s^o(-1) = M_s^o(-1) = 0$. Let $j_0 \geq 0$ be such that $m_s^o(j_0 - 1) = 0$ and $m_s^o(j_0) = m_s^o(j_0+1) = \cdots = m_s^o(j) = 1$. From $m_s^o(j_0 - 1) = 0$ it follows that $M_s(j_0 - 1) - M_s^o(j_0 - 1) = 0$. Therefore

$$l_j(x_\iota) = \Delta + \sum\nolimits_{k=j_0}^{j-1} m_s(k) - \sum\nolimits_{k=j_0}^{j-1} m_s^o(k) \tag{5}$$

$$= \Delta + \sum\nolimits_{k=j_0}^{j-1} m_s(k) - (j - j_0)$$

We claim that a coordinate of $\tilde{X}$, say $\tilde{x}_i$, $1 \leq i \leq m$, belongs to $(x_\iota - \varepsilon, x_\iota + \varepsilon)$ if and only if $i \in \iota$. Indeed, if $i \in \iota$ then $\tilde{x}_i - x_\iota = \tilde{x}_i - x_i \in (-\delta, \delta) \subseteq (-\varepsilon, \varepsilon)$, and if $i \notin \iota$ and we consider the class ι' such that $i \in \iota'$, then $|\tilde{x}_i - x_\iota| \geq |x_{\iota'} - x_\iota| - |\tilde{x}_i - x_{\iota'}| \geq 2\varepsilon_P - \delta > \varepsilon$.

For $k = 0, \ldots, n-1$ let v_k^* be the number of virtual zeros of order k of $E, \tilde{X}$ in $(x_\iota - \varepsilon, x_\iota + \varepsilon)$. If we repeat the proof of Theorem 3 to the pair $E, \tilde{X}$, but only considering those zeros located in the interval $(x_\iota - \varepsilon, x_\iota + \varepsilon)$, we will obtain that $v_k^* \geq v_{k-1}^* + m_s(k-1) - 1$, $k = 1, \ldots, n$. By applying successively this inequality for $k = j_0 + 1, \ldots, j$, we get

$$v_j^* = v_{j_0}^* + \sum\nolimits_{k=j_0}^{j-1} m_s(k) - (j - j_0) \tag{6}$$

By (5) and (6), in order to see the lemma it suffices to prove that $v_{j_0}^* \geq \Delta$. When $\Delta = 0$ there is nothing to prove and we are done. Suppose $\Delta = 1$, that is, x_ι is a virtual zero of order j of $\bar{E}, \bar{X}$, and we have to prove that $v_{j_0}^* \geq 1$. This fact follows from the following considerations. Since x_ι is a virtual zero of order j of $\bar{E}, \bar{X}$, by applying Corollary 7 several times we obtain that x_ι is also a virtual zero of order j_0 of $\bar{E}, \bar{X}$ (in particular $j_0 \geq 1$) and $P^{(j_0-1)}(x_\iota) \neq 0$. By applying Corollary 7 again, we get that x_ι has odd multiplicity in $P^{(j_0)}$. By Lemma 11 we know that there exists some real zero $\tilde{z}$ of $\tilde{P}^{(j_0)}$ with odd multiplicity and located in the interval $(x_\iota - \varepsilon, x_\iota + \varepsilon)$. If $\tilde{z}$ is non-specified in $E, \tilde{X}$ as zero of order j_0 then $\tilde{z}$ is a virtual zero and we are done, since $v^*_{j_0} \geq 1$. Otherwise, we have $\tilde{z} = \tilde{x}_i$ for some i with $e_{ij_0} = 1$. Note that $i \in \iota$ since $\tilde{x}_i \in (x_\iota - \varepsilon, x_\iota + \varepsilon)$. This implies that the i-th row of E is contained in $\tilde{E}_s$, and from $m^o_s(j_0 - 1) = 0$ (in particular $m_s(j_0 - 1) = 0$) it follows that $e_{i,j_0-1} = 0$. Since E is a strongly conservative matrix and $j_0 \geq 1$, the specified multiplicity in E of the entry $e_{ij_0} = 1$ is even, but however $\tilde{x}_i$ has odd multiplicity in $\tilde{P}^{(j_0)}$. This implies that $\tilde{z} = \tilde{x}_i$ is a virtual zero of order j_0 of $E, \tilde{X}$, and hence $v^*_{j_0} \geq 1$. This completes the proof. □

Proof of Theorem 17. Let ε be real with $0 < \varepsilon < \varepsilon_P$ and consider the number $\delta > 0$ obtained from Lemma 18. In order to see (4) it suffices to prove that if $\tilde{X} \in B(X, \delta) \cap D$ then $|a_{jr}(\tilde{X}) - \bar{a}_{jr}(X)| < \varepsilon$ for all j and r. Let $\tilde{X}$ and j be fixed, and let $z_1 < \cdots < z_u$ be the real zeros of $P^{(j)}$. By lemmas 15 and 18, the number of virtual zeros of order j of $E, \tilde{X}$ in $(z_i - \varepsilon, z_i + \varepsilon)$ is exactly $l_j(z_i)$, $i = 1, \ldots, u$. Thus, in the sequence of virtual zeros $a_{j1}(\tilde{X}), a_{j2}(\tilde{X}), \ldots, a_{j,v_j}(\tilde{X})$, the $l_j(z_1)$ first of them belong to $(z_i - \varepsilon, z_i + \varepsilon)$, and since $\bar{a}_{jr}(X) = z_1$ for $1 \leq r \leq l_j(z_1)$ then $|a_{jr}(\tilde{X}) - \bar{a}_{jr}(X)| < \varepsilon$ for $1 \leq r \leq l_j(z_1)$. Similarly we get $|a_{jr}(\tilde{X}) - \bar{a}_{jr}(X)| < \varepsilon$ for $l_j(z_1) + 1 \leq r \leq l_j(z_1) + l_j(z_2)$, and so on. In this way we obtain $|a_{jr}(\tilde{X}) - \bar{a}_{jr}(X)| < \varepsilon$ for any r, and theorem follows. □

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