Some Additive Counterparts of CBS Inequality

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Abstract

In this paper elementary numerical inequalities are used to obtain some additive inequalities related to the classical Cauchy-Bunyakowsky-Schwarz inequality.

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1 Introduction

Cauchy-Bunyakowsky-Schwarz inequality, for short CBS inequality, plays a very important role in some branches of Mathematics such as Real and Complex Analysis, Probability and Statistics, Hilbert Spaces Theory, Numerical Analysis and Differential Equations. Many discrete inequalities are connected in some way with CBS inequality as it has been extensively documented by Mitrinovic ([1], [2]) and more recently by Dragomir [3] among others. In this paper we derive some real additive inequalities, related to classical CBS, using elementary numerical inequalities similar the ones obtained in ([4],[5]). Furthermore, their complex companions are also given.

2 Main results

In the sequel we present some additive counterparts to CBS inequality that will be derived using elementary numerical inequalities. We begin with a generalization of CBS inequality extending the one appeared in [4].

Theorem 1 *Let* $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$; $c_1, c_2, \ldots, c_n$ *and* $d_1, d_2, \ldots, d_n$ *be positive real numbers and let* $r_1, r_2, \ldots, r_n$ *and* $s_1, s_2, \ldots, s_n$ *be nonnegative numbers. Then, for all integer* p*, holds:*

$$\frac{1}{2}\left(\sum_{k=1}^{n} r_k a_k^p \sum_{k=1}^{n} s_k b_k^p + \sum_{k=1}^{n} r_k c_k^p \sum_{k=1}^{n} s_k d_k^p\right)$$

$$\geq \left(\sum_{k=1}^{n} r_k a_k^{p/2} c_k^{p/2}\right)\left(\sum_{k=1}^{n} s_k b_k^{p/2} d_k^{p/2}\right)$$

Proof. Applying mean inequalities to positive numbers a and b, we have

$$a^p + b^p \geq 2a^{p/2}b^{p/2}$$

valid for all integer p. Therefore, for $1 \leq i, j \leq n$, we have

$$a_i^p b_j^p + c_i^p d_j^p \geq 2a_i^{p/2} b_j^{p/2} c_i^{p/2} d_j^{p/2}$$

Multiplying up by $r_i s_j \geq 0, (1 \leq i, j \leq n)$, both sides of the preceding inequalities yields

$$r_i s_j a_i^p b_j^p + r_i s_j c_i^p d_j^p \geq 2 r_i s_j a_i^{p/2} b_j^{p/2} c_i^{p/2} d_j^{p/2}$$

Adding up the above inequalities, we obtain:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(r_i s_j a_i^p b_j^p + r_i s_j c_i^p d_j^p\right) = \sum_{k=1}^{n} r_k a_k^p \sum_{k=1}^{n} s_k b_k^p + \sum_{k=1}^{n} r_k c_k^p \sum_{k=1}^{n} s_k d_k^p$$

$$\geq \sum_{i=1}^{n}\sum_{j=1}^{n}\left(2 r_i s_j a_i^{p/2} b_j^{p/2} c_i^{p/2} d_j^{p/2}\right) = 2\left(\sum_{k=1}^{n} r_k a_k^{p/2} c_k^{p/2}\right)\left(\sum_{k=1}^{n} s_k b_k^{p/2} d_k^{p/2}\right)$$

and this completes the proof.

□

Notice that when $p = 2$, $r_k = s_k = 1$ and $c_k = b_k, d_k = a_k, (1 \leq k \leq n)$, we get CBS inequality.

In what follows the same key idea is used to obtain some related results to CBS inequality. We start with

Theorem 2 *Let* $a_1, a_2, \ldots, a_n$ *and* $b_1, b_2, \ldots, b_n$ *be positive numbers and let* $c_1, c_2, \ldots, c_n$ *and* $d_1, d_2, \ldots, d_n$ *be nonnegative numbers. Then, for all integer* p*, holds:*

$$\frac{1}{2}\left(\sum_{k=1}^{n} d_k \sum_{k=1}^{n} c_k a_k^{p/2} + \sum_{k=1}^{n} c_k \sum_{k=1}^{n} d_k b_k^{p/2}\right) \geq \left(\sum_{k=1}^{n} c_k a_k^{p/2}\right)\left(\sum_{k=1}^{n} d_k b_k^{p/2}\right)$$

Proof. Applying mean inequalities to positive numbers a and b, we have

$$a^p + b^p \geq 2a^{p/2}b^{p/2}$$

valid for all positive integer p. Therefore, for $1 \leq i, j \leq n$, we have

$$a_i^p + b_j^p \geq 2a_i^{p/2}b_j^{p/2}$$

Multiplying both sides by $c_i d_j \geq 0$, $(1 \leq i, j \leq n)$, we obtain

$$c_i d_j a_i^p + c_i d_j b_j^p \geq 2c_i d_j a_i^{p/2} b_j^{p/2}$$

Adding up the preceding inequalities, yields

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(c_i d_j a_i^p + c_i d_j b_j^p\right) = \sum_{k=1}^{n} d_k \sum_{k=1}^{n} c_k a_k^p + \sum_{k=1}^{n} c_k \sum_{k=1}^{n} d_k b_k^p$$

$$\geq 2\sum_{i=1}^{n}\sum_{j=1}^{n}\left(c_i d_j a_i^{p/2} b_j^{p/2}\right) = 2\left(\sum_{k=1}^{n} c_k a_k^{p/2}\right)\left(\sum_{k=1}^{n} d_k b_k^{p/2}\right)$$

and this completes the proof. □

The complex version of the preceding result is stated in the following

Corollary 1 *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be complex numbers and let $c_1, c_2, \ldots, c_n$ and $d_1, d_2, \ldots, d_n$ be nonnegative numbers. Then, for all integer p, holds:*

$$\frac{1}{2}\left(\sum_{k=1}^{n} d_k \sum_{k=1}^{n} c_k |a_k|^{p/2} + \sum_{k=1}^{n} c_k \sum_{k=1}^{n} d_k |b_k|^{p/2}\right)$$

$$\geq \left(\sum_{k=1}^{n} c_k |a_k|^{p/2}\right)\left(\sum_{k=1}^{n} d_k |b_k|^{p/2}\right)$$

Now, we state and proof our second main result.

Theorem 3 *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be positive numbers and let $c_1, c_2, \ldots, c_n$ and $d_1, d_2, \ldots, d_n$ be nonnegative numbers. Then, for all integer $p \geq 1$, holds:*

$$\sum_{k=1}^{n} c_k \sum_{k=1}^{n} d_k b_k^p + \sum_{k=1}^{n} d_k \sum_{k=1}^{n} c_k a_k^p$$

$$\geq \left(\sum_{k=1}^{n} c_k a_k^{p-1} \sum_{k=1}^{n} d_k b_k\right) + \left(\sum_{k=1}^{n} c_k a_k \sum_{k=1}^{n} d_k b_k^{p-1}\right)$$

Proof. To prove the preceding inequality we need the following

Lemma 1 *Let a, b be positive real numbers. Then, for every integer $p \geq 1$, holds:*

$$a^p + b^p \geq a^{p-1}b + ab^{p-1}$$

Proof. We will argue by mathematical induction. The cases when $p = 1$ and $p = 2$ trivially hold. Suppose that the given inequality holds for $p - 1$, that is, it holds that $a^{p-1} + b^{p-1} \geq a^{p-2}b + ab^{p-2}$. Writing now

$$a^p + b^p = a\left(a^{p-1} + b^{p-1}\right) + b^p - ab^{p-1}$$

and taking into account the inductive hypotheses, we get

$$a^p + b^p \geq a\left(a^{p-2}b + ab^{p-2}\right) + b^p - ab^{p-1} = a^{p-1}b + a^2b^{p-2} + b^p - ab^{p-1}$$

Since $a^2b^{p-2} + b^p - ab^{p-1} = b^{p-2}\left(a^2 + b^2 - ab\right) \geq b^{p-2}(ab) = ab^{p-1}$, then $a^p + b^p \geq a^{p-1}b + ab^{p-1}$ as desired. We observe that equality holds if, and only if, $a = b$ and the proof is complete.

□

From the previous lemma, we have for $1 \leq i, j \leq n$,

$$a_i^p + b_j^p \geq a_i^{p-1}b_j + a_i b_j^{p-1}$$

Multiplying both sides by $c_i d_j \geq 0$, $(1 \leq i, j \leq n)$, we obtain

$$c_i d_j a_i^p + c_i d_j b_j^p \geq c_i d_j a_i^{p-1} b_j + c_i d_j a_i b_j^{p-1}$$

Adding up those inequalities yields:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(c_i d_j a_i^p + c_i d_j b_j^p\right) = \sum_{k=1}^{n} d_k \sum_{k=1}^{n} c_k a_k^p + \sum_{k=1}^{n} c_k \sum_{k=1}^{n} d_k b_k^p$$

$$\geq \sum_{i=1}^{n}\sum_{j=1}^{n}\left(c_i d_j a_i^{p-1} b_j + c a_i d_j a_i b_j^{p-1}\right)$$

$$= \left(\sum_{k=1}^{n} c_k a_k^{p-1} \sum_{k=1}^{n} d_k b_k\right) + \left(\sum_{k=1}^{n} c_k a_k \sum_{k=1}^{n} d_k b_k^{p-1}\right)$$

□

The complex counterpart of the previous result is given in the next

Corollary 2 *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be complex numbers and let $c_1, c_2, \ldots, c_n$ and $d_1, d_2, \ldots, d_n$ be nonnegative numbers. Then, for all integer $p \geq 1$, holds:*

$$\begin{aligned}\sum_{k=1}^{n} c_k \sum_{k=1}^{n} d_k |b_k|^p + \sum_{k=1}^{n} d_k \sum_{k=1}^{n} c_k |a_k|^p \quad &\geq \quad \left(\sum_{k=1}^{n} c_k |a_k|^{p-1} \sum_{k=1}^{n} d_k |b_k| \right) \\ &+ \quad \left(\sum_{k=1}^{n} c_k |a_k| \sum_{k=1}^{n} d_k |b_k|^{p-1} \right)\end{aligned}$$

Finally, we will use a constrained elementary inequality to obtain the following result.

Theorem 4 *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be positive numbers and let $c_1, c_2, \ldots, c_n$ and $d_1, d_2, \ldots, d_n$ be nonnegative numbers. If α, β are positive numbers such that $\alpha = 1 + \beta$, then*

$$\frac{1}{\alpha} \left(\sum_{k=1}^{n} c_k \sum_{k=1}^{n} c_k a_k^{\alpha} + \beta \sum_{k=1}^{n} c_k \sum_{k=1}^{n} d_k b_k \right) \geq \left(\sum_{k=1}^{n} c_k a_k \right) \left(\sum_{k=1}^{n} d_k b_k^{\beta} \right)$$

Proof. We begin with a Lemma.

Lemma 2 *Let a, b, α and β be real numbers such that $a \geq 0$, $b, \alpha, \beta > 0$ and $\alpha = 1 + \beta$. Then,*

$$a^{\alpha} + \beta b^{\alpha} \geq \alpha a b^{\beta}$$

with equality if, and only if, $a = b$.

Proof. The inequality claimed can be written in the equivalent form

$$b^{\beta}(\alpha a - \beta b) \leq a^{\alpha}$$

When $a = 0$ the inequality is strict, and when $a = b$ the inequality becomes equality. Hence, we can assume that $a > 0$ and $a \neq b$. Set $\lambda = a/b$. Then, the inequality is equivalent to $\alpha\lambda - \beta < \lambda^{\alpha}$ for $\lambda \neq 1$. Therefore, we have to prove that holds $\lambda^{\alpha} - \alpha\lambda + \alpha - 1 > 0$ for any $0 < \lambda \neq 1$. Indeed, let f be the function defined by $f(\lambda) = \lambda^{\alpha} - \alpha\lambda + \alpha - 1$. It is easy to see that $f'(1) = 0$, $f'(\lambda) < 0$ on $(0, 1)$ and $f'(\lambda) > 0$ on $(1, +\infty)$. This implies that $f(\lambda) > f(1) = 0$ if $\lambda \neq 1$ and this completes the proof. □

Now carrying out the same procedure as in the previous results, we can write for $1 \leq i, j \leq n$,

$$a_i^{\alpha} + \beta b_j^{\alpha} \geq \alpha a_i b_j^{\beta}$$

Multiplying up both sides for $c_i d_j > 0, 1 \leq i, j \leq n$, yields

$$c_i d_j a_i^{\alpha} + \beta c_i d_j b_j^{\alpha} \geq \alpha c_i d_j a_i b_j^{\beta}$$

Adding up those inequalities, we get

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(c_i d_j a_i^{\alpha}+\beta c_i d_j b_j^{\alpha}\right)\geq \alpha\sum_{i=1}^{n}\sum_{j=1}^{n}c_i d_j a_i b_j^{\beta}$$

from which the result immediately follows and the proof is complete.

□

Likewise, the complex version of the about inequality is presented in

Corollary 3 *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be complex numbers and let $c_1, c_2, \ldots, c_n$ and $d_1, d_2, \ldots, d_n$ be nonnegative numbers. If α, β are positive numbers such that $\alpha = 1 + \beta$, then*

$$\frac{1}{\alpha}\left(\sum_{k=1}^{n}c_k\sum_{k=1}^{n}c_k|a_k|^{\alpha}+\beta\sum_{k=1}^{n}c_k\sum_{k=1}^{n}d_k|b_k|\right)\geq\left(\sum_{k=1}^{n}c_k|a_k|\right)\left(\sum_{k=1}^{n}d_k|b_k|^{\beta}\right)$$

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